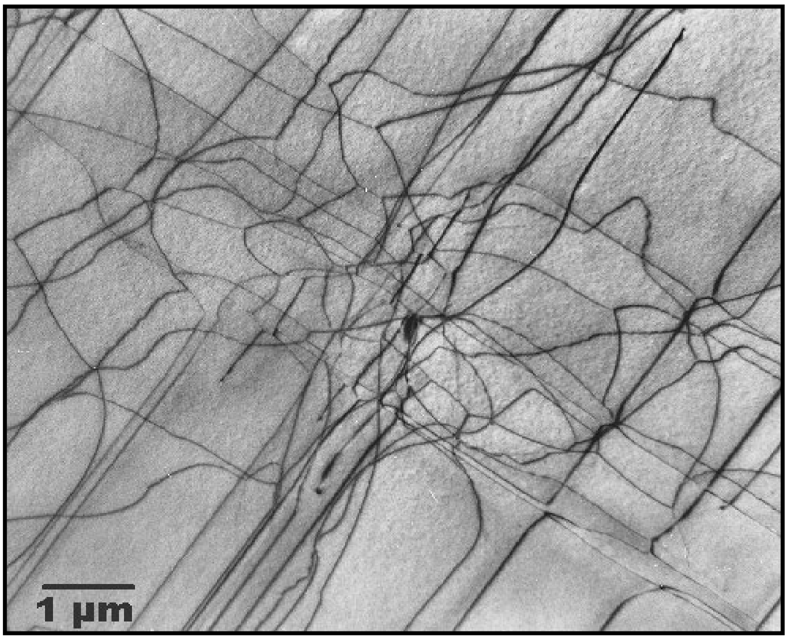
Contributions to the existence, uniqueness, and contraction of the solutions to some evolutionary partial differential equations

Maryam Al Zohbi

Par **Maryam AL ZOHBI**

Contributions to the existence, uniqueness, and contraction of the solutions to some evolutionary partial differential equations

Thèse présentée en cotutelle pour l'obtention du grade de Docteur de l'UTC



Soutenue le 10 décembre 2021

Spécialité : Mathématiques Appliquées : Laboratoire de Mathématiques Appliquées de Compiègne (Unité de recherche EA-2222)

D2646

Université Libanaise
École Doctorale
Sciences et Technologies
Doyen

ALLIANCE SORBONNE UNIVERSITÉ utc

THESE de doctorat en Cotutelle

Pour obtenir le grade de Docteur délivré par

L'Université Libanaise, L'Ecole Doctorale des Sciences et Technologie

et

L'Université de Technologie de Compiègne, L'École Doctorale des Sciences pour l'Ingénieur

Spécialité : Mathématiques Appliquées

Présentée et soutenue publiquement par

AL ZOHBI Maryam

10 Décembre 2021

Contributions to the existence, uniqueness, and contraction of the solutions to some evolutionary partial differential equations

Directeurs de thèse : **JAZAR Mustapha** et **EL HAJJ Ahmad**

Membres du Jury:

M. Nicolas Forcadel	professeur, Université de Rouen Normandie	Rapporteur
M. Youcef Mammeri	maître de conférence HDR, Université de Picardie Jules Verne	Rapporteur
M. Florian De Vuyst	professeur, Université de Technologie de Compiègne	Examinateur
M. Stéphane Junca	maître de conférence HDR, Université de Nice Sophia Antipolis	Examinateur
Mme Elizabetta Carlini	maître de conférence, Université Sapienza de Rome	Examinateur
Mme Jana Khayal	maître de conférence, Université Libanaise	Examinateur
M. Mustapha Jazar	professeur, Université Libanaise	Directeur
M. Ahmad El Hajj	professeur, Université de Technologie de Compiègne	Directeur

Abstract

In this thesis, we are mainly interested in the theoretical and numerical study of certain equations that describe the dynamics of dislocation densities. Dislocations are microscopic defects in materials, which move under the effect of an external stress.
As a first work, we prove a global in time existence result of a discontinuous solution to a diagonal hyperbolic system, which is not necessarily strictly hyperbolic, in one space dimension. Then in another work, we broaden our scope by proving a similar result to a non-linear eikonal system, which is in fact a generalization of the hyperbolic system studied first. We also prove the existence and uniqueness of a continuous solution to the eikonal system. After that, we study this system numerically in a third work through proposing a finite difference scheme approximating it, of which we prove the convergence to the continuous problem, strengthening our outcomes with some numerical simulations.
On a different direction, we were enthused by the theory of differential contraction to evolutionary equations. By introducing a new distance, we create a new family of contracting positive solutions to the evolutionary p-Laplacian equation.

Résumé

Dans cette thèse, nous nous sommes principalement intéressés à l'étude théorique et numérique de quelques équations qui décrivent la dynamique des densités des dislocations. Les dislocations sont des défauts microscopiques qui se déplacent dans les matériaux sous l'effet des contraintes extérieures.

Dans un premier travail, nous démontrons un résultat d'existence globale en temps des solutions discontinues pour un système hyperbolique diagonal qui n'est pas nécessairement strictement hyperbolique, dans un espace unidimensionnel. Ainsi dans un deuxième travail, nous élargissons notre portée en démontrant un résultat similaire pour un système d'èquations de type eikonal non-linéaire qui est en fait une généralisation du système hyperbolique déjà étudié. En effet, nous prouvons aussi l'existence et l'unicité d'une solution continue pour le système eikonal. Ensuite, nous nous sommes intéressés à l'analyse numérique de ce système en proposant un schéma aux différences finies, par lequel nous montrons la convergence vers le problème continu et nous consolidons nos résultats avec quelques simulations numériques.

Dans une autre direction, nous nous sommes intéressés à la théorie de contraction différentielle pour les équations d'évolutions. Après avoir introduit une nouvelle distance, nous construisons une nouvelle famille des solutions contractantes positives pour l'équation d'évolution p-Laplace.

List of articles

Accepted articles

- (with Ahmad El Hajj and Mustapha Jazar), *Global existence to a diagonal hyperbolic system for any BV initial data*, Nonlinearity, 34 (2021), pp. 54-85 (cf. Chapter 3).

Submitted articles

- (with Ahmad El Hajj and Mustapha Jazar), *Existence and uniqueness results to a system of Hamilton-Jacobi equations*, submitted (cf. Chapter 4).
- (with Ahmad El Hajj and Mustapha Jazar), *Convergent semi-explicit scheme to a non-linear eikonal system*, submitted (cf. Chapter 5).

Articles at the end of preparation

- (with Ghada Chmaycem and Mustapha Jazar), *New contraction result to the positive solutions of the evolutionary p-Laplacian equation*, (cf. Chapter 6).

Contents

1 General introduction **1**
- 1 Introduction to the dynamics of dislocations . . . 1
  - 1.1 Physical motivation . . . 1
    - 1.1.1 Historical background . . . 2
  - 1.2 Properties of dislocations . . . 3
    - 1.2.1 Burgers vector . . . 3
    - 1.2.2 Types of moving dislocations . . . 3
    - 1.2.3 Movement of dislocations . . . 4
  - 1.3 Equations considered . . . 6
    - 1.3.1 Eikonal system . . . 6
    - 1.3.2 Diagonal hyperbolic system . . . 7
  - 1.4 Hamilton-Jacobi equations and viscosity solutions . . . 7
    - 1.4.1 Origin of viscosity solutions . . . 8
    - 1.4.2 Vanishing viscosity solutions . . . 9
- 2 Existence result of a discontinuous viscosity solution . . . 12
- 3 Existence and uniqueness result of a continuous viscosity solution . . . 16
- 4 Convergence result of a finite difference scheme . . . 20
- 5 New contraction result to the evolutionary p-Laplacian equation . . . 24
  - 5.1 Physical interpretation . . . 25
  - 5.2 Some previous results . . . 26
  - 5.3 New contraction result . . . 27

2 Modeling **31**
- 1 The bidimensional model . . . 31
- 2 The unidimensional model . . . 34
- 3 Derivation of the non-periodic model . . . 37

3 Global existence to a diagonal hyperbolic system 39
1 Introduction and main result . . . 40
1.1 Setting of the problem . . . 40
1.2 Main results . . . 44
1.3 Organization of the paper . . . 47
2 Local solution for parabolic regularized equation . . . 47
3 *A priori* uniform estimates on the smooth solution . . . 53
4 Global existence of a solution to (3.7) . . . 59
4.1 Proof of Theorem 3.1 (i) . . . 60
5 Discontinuous viscosity sub- and super- solutions . . . 61
5.1 Some useful results . . . 61
5.2 Existence of sub and super solutions of (3.1) . . . 64
5.2.1 Definitions of viscosity solutions . . . 65
5.2.2 Proof of Theorem 3.1 (*ii*) . . . 67
6 Existence of a viscosity solution for non-decreasing initial data . . . 69
6.1 Preliminary results . . . 70
6.2 Proof of Theorem 3.2 . . . 71
6.2.1 Passing to the limit as ε and η tend to zero . . . 71
6.2.2 Existence of a discontinuous viscosity solution . . . 72
7 Link between sub and super solutions in the general case . . . 75

4 Existence and uniqueness to a Hamilton-Jacobi system 81
1 Introduction and main results . . . 82
1.1 Setting of the problem . . . 82
1.2 A brief review of some related literature . . . 84
1.3 Main results . . . 86
1.3.1 Unique continuous solution for dislocations' dynamics . . . 88
1.4 Organization of the paper . . . 89
2 Existence of a discontinuous solution . . . 89
2.1 Some useful results . . . 89
2.2 Proof of Theorem 4.1 . . . 91
3 Unique continuous solution . . . 95
4 Application to dislocations dynamics . . . 102

5 Convergent scheme approximating an eikonal system **105**
1 Introduction and main results 106
1.1 The continuous problem 106
1.1.1 Recall of previous results 108
1.2 The discrete problem 108
1.3 Main results 110
1.4 A brief review of some related literature 112
1.5 Organization of the paper 113
2 Existence of BV discrete solution to (5.9) 113
2.1 The Q^1-extension u^ε 116
3 Discontinuous viscosity sub and super solutions 118
3.1 Some useful results 118
3.2 Existence of sub and super solutions to (5.1) 121
4 Link between sub and super solutions 125
4.1 Preliminary results 125
4.2 Proof of Theorem 5.4-(iii) 127
5 Numerical simulations 132

6 New contraction to the evolutionary p-Laplacian equation **135**
1 Introduction and main result 136
2 Construction of a distance 138
3 Contraction of the distance 140

Conclusion and perspectives **143**

1 General introduction

This thesis dissertation is mainly concerned with the modeling, theoretical and numerical analysis of certain non-linear equations of Hamilton-Jacobi type that appear naturally in the modeling of the dynamics of dislocations, where the latter are microscopic defects in materials.
On a different direction at the end, we were interested in the theory of differential contraction to the solutions of the evolutionary p-Laplacian equation.

This introduction is organized as follows: In Section 1, we introduce the reader into the theory of dislocations, where we discuss elaborately some of their properties and their historical background, along with the equations we have considered to model their movement, and the type of solutions these equations have. We then mention our contributions in this theory in Sections 2, 3, and 4. In Section 5, we talk briefly about the theory of differential contraction, and we mention the work we have done in order to create a new family of contracting solutions to the evolutionary p-Laplacian equation.

1 Introduction to the dynamics of dislocations

1.1 Physical motivation

A dislocation is a linear crystallographic defect or irregularity within a crystal structure that contains an abrupt change in the arrangement of its atoms. The movement of dislocations allows closely packed crystal planes to slide over each other at low stress level. This phenomenon is known as glide or slip, as we can see in Figure 1.1. Thus, a dislocation defines the boundary between slipped and unslipped regions of a material and as a result, must either form a complete loop, intersect other dislocations or defects, or extend to the edges of the crystal. For example, the black curves in Figure 1.2 represent dislocations present in a thin Silicon sample at a scale of 1 $\mu m = 10^{-9}\ m$.
Some types of dislocations can be visualized as being caused by the termination of a plane

Figure 1.1: Slip mechanism illustration in plastic deformation.

of atoms in the middle of a crystal. In such a case, the surrounding planes are not straight, but instead bend around the edge of the terminating plane so that the crystal structure is perfectly ordered on either side. The analogy with a stack of paper can clarify more the idea: if a half piece of paper is inserted in a stack of papers, the defect in the stack is only noticeable at the edge of the half sheet. We refer to the work of Hirth, Lothe, Mura [56], Hull, Bacon [57], and Nabarro [74] for a complete introduction into the theory of dislocations.

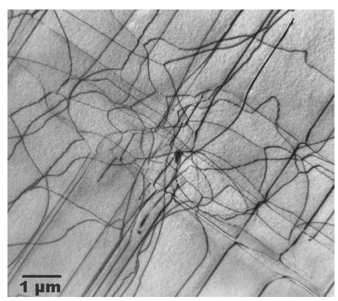


Figure 1.2: Dislocations in a thin Silicon specimen.

The concentration of dislocations in a crystal is represented by its *density*, which is defined as the number of dislocation lines traversing a certain unitary section. In this dissertation, we are interested in the dynamics of dislocation densities.

1.1.1 Historical background

The first to analyze theoretically the linear defects in materials was Volterra [84] in 1907. From a mechanical point of view, in 1934, Polanyi [78], Taylor [82] and Orowan [75], independently, proposed that these defects were the principle explanation of plastic deformations in materials, at the microscopic level. With the evolution of Transmission Electron Microscopy technique in 1956 (TEM for short), where thin foils of a certain material are prepared to be rendered transparent under the electron beam of a microscope, the first direct observations of dislocations were done by Bollman [18] and Hirsh, Horne,

Whelan [55]. These observations allowed us to test and verify some theoretical predictions of dislocations, such as their length, thickness, and speed. It is important to mention that the number and arrangement of dislocations influences many properties of materials such as hardness, yield strength, and ductility.
The theoretical study of dislocations, along with the development of means to investigate and record them, allowed us to better understand the elementary mechanisms at the origin of the plastic deformation in materials.

1.2 Properties of dislocations

There are two primary types of dislocations: the *glissile* and the *sessile*, which are mobile and immobile dislocations respectively. We are mainly concerned with some properties of the moving ones in this work.

1.2.1 Burgers vector

Under an external stress, the dislocations move in a crystallographic plane called the *slip plane*. The displacement of a dislocation is characterized by a vector $\overrightarrow{b}$ called *Burgers vector* [24]. Physically, the Burgers vector represents the magnitude and the direction of the strain carried by a dislocation.

1.2.2 Types of moving dislocations

There are two main types of mobile dislocations: the *edge* and the *screw* dislocations. The dislocations found in real materials are of *mixed* type, meaning that they have the characteristics of both edge and screw dislocations.

• **Edge dislocations:** This type of dislocations occurs when an extra half plane of atoms is introduced between certain crystal planes of the material, distorting their organization. When an external force is applied on one side of the material, this half plane starts to pass through the material, breaking and rejoining crystal planes, until it reaches the boundary of the material. In this case, the line direction of the half plane (dislocation line) is perpendicular to the Burgers vector. As we can see in Figure 1.3, the red planes represent the atomic crystal planes in a material, and the half plane of atoms can be seen in the middle with the dislocation line (in blue) at its end being perpendicular to the Burgers vector $\overrightarrow{b}$.

• **Screw dislocations:** In order to get a clear idea of screw dislocations, imagine you cut along a plane through a cube, and you slip one half a bit (creating a step), leaving

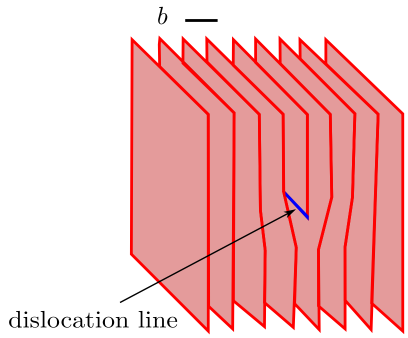


Figure 1.3: Illustration of an edge dislocation.

the two halves to reattach together. Now, if you repeat the same process, but you do not cut all the way through the cube, the boundary of the cut is a screw dislocation. In this case, the dislocation line is parallel to the Burgers vector. This is reflected in Figure 1.4.

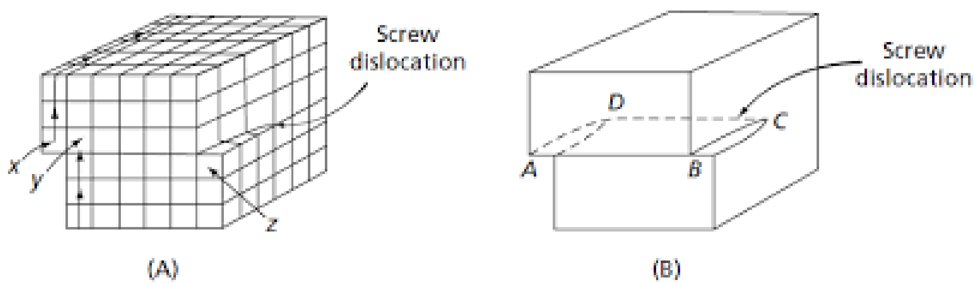


Figure 1.4: Illustration of a screw dislocation.

•**Mixed dislocations**: In many materials, dislocations are found where the line direction and Burgers vector are neither perpendicular nor parallel and these dislocations are called mixed dislocations, consisting of both screw and edge character, as we can see in Figure 1.5. They are characterized by φ, the angle between the line direction and Burgers vector, where $\varphi = \pi/2$ for pure edge dislocations and $\varphi = 0$ for screw dislocations.

1.2.3 Movement of dislocations

A dislocation can move in a plane that contains the dislocation line and the Burgers vector. This is known as *slipping* or *sliding* (See Figure 1.6). In case of a screw dislocation, there are many planes the dislocation can slip in, as the dislocation line and the Burgers vector are parallel. However, there is only one plane an edge dislocation can slip in, since in this case the dislocation line and the Burgers vector are perpendicular.

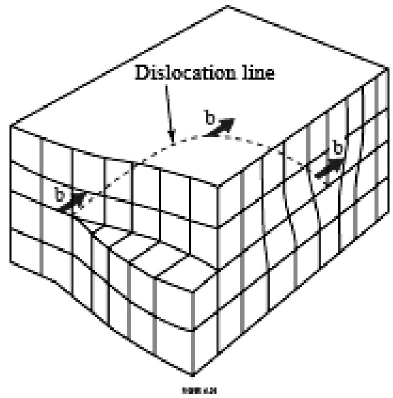


Figure 1.5: Illustration of a mixed dislocation.

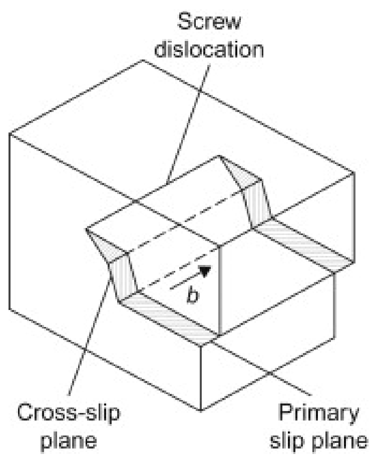


Figure 1.6: Illustration of dislocation sliding.

Another type of movement for edge dislocations is *climbing*. This takes place when a vacancy occurs between the atoms surrounding the edge dislocation. The atom in the dislocation line closest to the vacancy can either jump up or down, depending on the position of the vacancy, in the half plane of the edge dislocation (See Figure 1.7). The movement of the vacany would make the entire dislocation shift up or down in location. What distinguishes sliding from climbing is that the first occurs under a shear stress (external stress), whereas the second is caused from the motion of the atoms within the material (internal stress). Another difference is that climbing happens much more rapidly in high temperatures, due to the increase of vacancies between the atoms of materials, while sliding is not much affected by temperature changes.

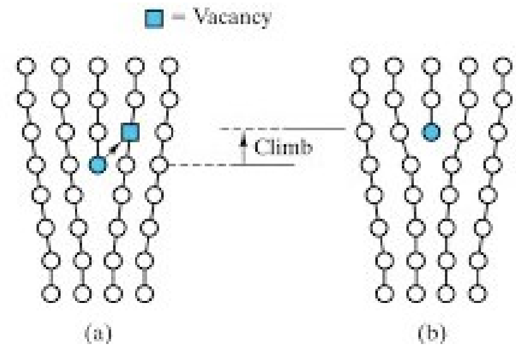


Figure 1.7: Illustration of the displacement of atoms.

In our work, we are interested in a model that describes the dynamics of a finite number of edge dislocations each propagating in a different slipping plane following a Burgers vector. We announce in the following subsection the main systems we study.

1.3 Equations considered

We will mention later on some works that have been done considering the systems listed in the following two subsections.

1.3.1 Eikonal system

The main system of equations we study in this part of the thesis is of the form

$$\begin{cases} \partial_t u^i(t,x) = \lambda^i(t,x,u(t,x))\,|\partial_x u^i(t,x)| & \text{in } \ (0,T)\times\mathbb{R}, \\ u^i(0,x) = u_0^i(x) & \text{in } \ \mathbb{R}, \end{cases} \tag{1.1}$$

for $T>0$ and $i=1,\dots,d$, where $d\in\mathbb{N}^*$. The functions u^i are real valued, $\partial_t u^i$ and $\partial_x u^i$ represent the time and spatial derivatives of u^i respectively, and the term λ^i represents the velocity of u^i. This system was initially proposed in 2 dimensions. We will show the complete modeling of this bidimensional model and how we can retrieve system (1.1) from it in Chapter 2.

System (1.1) can be seen as the “level-set approach” system associated to the motion of the front $\Gamma_t^i \coloneqq \{x : u^i(t,x)=0\}$ with a normal velocity $\lambda^i(t,x,u(t,x))$ depending on the solution u and affected by $\lambda^j(t,x,u(t,x))$ for $j\neq i$. Osher and Sethian [76] introduced the level set method (numerically) to study such problems. The rigorous treatment was developed later by Evans, Spruck [51] and Chen, Giga, Goto [31], independently.

For system (1.1) we prove an existence and uniqueness result of a continuous solution (see

Chapter 4), and we propose a convergent finite difference scheme approximating it (see Chapter 5).

1.3.2 Diagonal hyperbolic system

If we omit the absolute value in system (1.1), it becomes a diagonal hyperbolic system of transport equations. For such kind of systems, we have proven an existence result of a discontinuous solution assuming the system is not necessarily strictly hyperbolic, where the notion of *strictly hyperbolic system* here means that

$$\lambda^1 < \cdots < \lambda^d.$$

In other words, the $(d \times d)$ diagonal matrix with $(\lambda^i(t,x,u(t,x)))_{i=1,\ldots,d}$ as the diagonal has d distinct eigenvalues. Physically speaking, transport equations can intervene in the modeling of several natural phenomena, such as road traffic, gas dynamics, and of course the dynamics of dislocations.

The previously mentioned equations are first order non-linear *Hamilton-Jacobi equations*. The question now is: *what are the kinds of solutions we construct for such equations?* The answer is in the following subsection.

1.4 Hamilton-Jacobi equations and viscosity solutions

Consider the following first order Hamilton-Jacobi system

$$\partial_t u^i(t,x) + H^i(t,x,u(t,x),\partial_x u^i(t,x)) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}, \tag{1.2}$$

where $i = 1,\ldots,d$ with $d \in \mathbb{N}^*$, the Hamiltonians $H^i : (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R} \longrightarrow \mathbb{R}$ are given for every $i = 1,\ldots,d$, and they are in general non-linear. It is clear that the eikonal and diagonal hyperbolic systems mentioned in the previous section are in the form of (1.2).

Usually in the study of partial differential equations, when one can not find a classical solution, we directly think of *weak* solutions or *variational* solutions, where we multiply by a smooth test function, and we integrate by parts in order to throw the derivatives that we can not easily handle onto the test function. In the case of Hamilton-Jacobi equations, due to the non-linearity of the Hamiltonians, we can not apply this technique. Nonetheless, this reasoning of "*transferring the derivatives onto a test function*" , along with the Maximum Principle, gave a glimpse of a new kind of solutions. At this point, the notion of *viscosity solutions* came to light, and it was the foundation of the *vanishing viscosity solutions* to first order Hamilton-Jacobi equations in the sense introduced in Subsection 1.4.2.

The idea of viscosity solutions for Hamilton-Jacobi equations was first introduced and set forth in the 1980's by Crandall, Lions and Evans [34, 35, 36, 50, 69]. Actually, the first to introduce the idea of weak viscosity solutions was Evans [50], then Crandall and Lions [36] proved the uniqueness of such solutions, which established the firm foundation to the theory of viscosity solutions to first order equations. Many great textbooks have been written in the framework of viscosity solutions to Hamilton-Jacobi equations, such as Bardi and Capuzzo-Dolcetta [7], Barles [8], Koike [62], Chapter 10 in Evans [49], and Cannarsa and Sinestrari [28].

The definitions and ideas presented in this section are mainly adapted from the books of Evans [49, Chapter 10], Barles [9], and Tran [83]. We remark that the simplest presentation of viscosity solutions is in the case where $d = 1$, we are, however, presenting this theory in the case of a system of equations in order to remain in coherence with our results later on.

1.4.1 Origin of viscosity solutions

Consider the following system of equations

$$\partial_t u^i(t,x) + H^i(t,x,u(t,x),\partial_x u^i(t,x),\partial^2_{xx} u^i(t,x)) = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}, \tag{1.3}$$

where $H^i : (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$. We say that the Hamiltonians H^i are *elliptic* if and only if the following ellipticity condition is satisfied for every $i = 1,\dots,d$

$$H^i(t,x,u,p,r_1) \le H^i(t,x,u,p,r_2) \quad \text{if } r_2 \le r_1, \tag{1.4}$$

for every $t \in (0,+\infty)$, $x \in \mathbb{R}$, $u \in \mathbb{R}^d$, $p \in \mathbb{R}$, and $r_1, r_2 \in \mathbb{R}$.
The notion of viscosity solutions was originally inspired by the *Maximum Principle* of elliptic equations. In the following theorem, we recall this principle for *parabolic* equations. We note that parabolic equations are a particular case of elliptic ones.

Theorem 1.1 (**Maximum Principle**).
We say that $u = (u^i)_{i=1,\dots,d} \in (C^2((0,+\infty)\times\mathbb{R}))^d$ is a classical solution of (1.3) if and only if

for all $\varphi \in C^2((0,+\infty)\times\mathbb{R})$*, if* (t_0,x_0) *is a maximum of* $u^i-\varphi$ *for every* $i=1,\dots,d$*, we have*

$$\partial_t u^i(t_0,x_0)+H^i(t_0,x_0,u(t_0,x_0),\partial_x\varphi(t_0,x_0),\partial^2_{xx}\varphi(t_0,x_0))\leq 0, \tag{1.5}$$

and for all $\varphi \in C^2((0,+\infty)\times\mathbb{R})$*, if* (t_0,x_0) *is a minimum of* $u^i-\varphi$ *for every* $i=1,\dots,d$*, we have*

$$\partial_t u^i(t_0,x_0)+H^i(t_0,x_0,u(t_0,x_0),\partial_x\varphi(t_0,x_0),\partial^2_{xx}\varphi(t_0,x_0))\geq 0. \tag{1.6}$$

We remark that (1.5) and (1.6) have sense even if $\partial_x u^i$ and $\partial^2_{xx}u^i$ are not defined at (t_0,x_0). In other words, (1.5) and (1.6) do not require the existence neither of $\partial_x u^i(t_0,x_0)$ nor of $\partial^2_{xx}u^i(t_0,x_0)$, but only that of $u^i(t_0,x_0)$. Thus, a natural condition that allows us to give sense to (1.5) and (1.6) is the continuity, or even less, the semi-continuity. From this point, the first definition of viscosity solutions was introduced, and it is quoted in the following definition (see Barles [12]).

Definition 1.1 (**Viscosity solutions**).
We say that $u=(u^i)_{i=1,\dots,d}\in(C((0,+\infty)\times\mathbb{R}))^d$ is a viscosity solution of (1.3) if and only if for all $\varphi\in C^2((0,+\infty)\times\mathbb{R})$, if $(t_0,x_0)\in(0,+\infty)\times\mathbb{R}$ is a local maximum of $u^i-\varphi$ for $i=1,\dots,d$, we have (1.5), and for all $\varphi\in C^2((0,+\infty)\times\mathbb{R})$, if $(t_0,x_0)\in(0,+\infty)\times\mathbb{R}$ is a local minimum of $u^i-\varphi$ for $i=1,\dots,d$, we have (1.6). If u verifies only (1.5) (resp. (1.6)), we say that u is viscosity sub-solution (resp. super-solution).

As we can see now, the ellipticity condition (1.4) plays an essential role in the previous definition.

We now come to realize that in the case where equation (1.3) is of first order, i.e. the Hamiltonians do not depend on $\partial^2_{xx}u^i$, inequalities (1.5) and (1.6) make no sense. However, they were the bedrock of the *vanishing viscosity solutions* in the sense given in the following subsection.

1.4.2 Vanishing viscosity solutions

This technique was developed in order to solve first order Hamilton-Jacobi equations. The idea was based on adding a certain *viscosity* to the equation, making it a second order one, in order to make use of the main definition of viscosity solutions (Definition 1.1). This technique is presented formally in what follows.

If we have, for $i=1,\dots,d$, the following problem

$$\begin{cases}\partial_t u^i(t,x)+H^i(t,x,u(t,x),\partial_x u^i(t,x))=0 & \text{in } (0,+\infty)\times\mathbb{R},\\ u^i(0,x)=u^i_0(x) & \text{in } \mathbb{R},\end{cases} \tag{1.7}$$

where the initial data and the Hamiltonians are given. For $\varepsilon > 0$, we consider

$$\begin{cases} \partial_t u^i_\varepsilon(t,x) + H^i(t,x,u_\varepsilon(t,x),\partial_x u^i_\varepsilon(t,x)) = \varepsilon \partial^2_{xx} u^i_\varepsilon(t,x) & \text{in } (0,+\infty)\times\mathbb{R}, \\ u^i_\varepsilon(0,x) = u^i_0(x) & \text{in } \mathbb{R}. \end{cases} \tag{1.8}$$

Under suitable conditions on u^i_0 and H^i for $i = 1,\dots,d$, the previous problem admits a smooth solution. Next, we hope that when we pass to the limit as $\varepsilon \to 0$, the smooth solution of (1.8) converges to some sort of weak solution of (1.7). First we must study the behavior of the sequence $(u^i_\varepsilon)_\varepsilon$. It turns out that in practice, we can always prove that u^i_ε, $\partial_x u^i_\varepsilon$, and $\partial_t u^i_\varepsilon$ are uniformly bounded in $L^\infty((0,+\infty)\times\mathbb{R})$. This shows that the sequence $(u^i_\varepsilon)_\varepsilon$ is compact in $C(K)$ for every compact $K \subset \mathbb{R}$. Thus, by the Ascoli-Arzela compactness theorem, we can extract a subsequence, denoted $(u^i_{\varepsilon_j})_{\varepsilon_j}$ that converges locally uniformly in $[0,+\infty)\times\mathbb{R}$ to some limit u^i for $i = 1,\dots,d$. The question now is whether $u = (u^i)_{i=1,\dots,d}$ solves (1.7) in some sense.

If we go back to the beginning of this section, we have said that the notion of vanishing viscosity solutions was inspired by the Maximum Principle and the idea of transferring the derivatives onto a smooth test function. We will illustrate the method in the case of continuous solutions:

Assume that the Hamiltonians and the initial data are continuous. We fix a smooth test function $\varphi \in C^\infty((0,+\infty)\times\mathbb{R})$ and we suppose that for every $i = 1,\dots,d$

$$u^i - \varphi \text{ has a strict local maximum at some point } (t_0,x_0) \in (0,+\infty)\times\mathbb{R}. \tag{1.9}$$

This means that

$$(u^i - \varphi)(t_0,x_0) > (u^i - \varphi)(t,x),$$

for all points (t,x) in the neighborhood of (t_0,x_0), such that $(t,x) \neq (t_0,x_0)$.

Now, as $u^i_{\varepsilon_j}$ converges to u^i locally uniformally, we claim that for every $\varepsilon_j > 0$ sufficiently small, there exists a point $(t_{\varepsilon_j}, x_{\varepsilon_j})$ such that

$$(t_{\varepsilon_j}, x_{\varepsilon_j}) \to (t_0,x_0) \quad \text{as } i \to \infty, \tag{1.10}$$

$$u^i_{\varepsilon_j} - \varphi \text{ has a local maximum at } (t_{\varepsilon_j}, x_{\varepsilon_j}). \tag{1.11}$$

Thus, we can see that

$$\begin{aligned} \partial_x u^i_{\varepsilon_j}(t_{\varepsilon_j}, x_{\varepsilon_j}) &= \partial_x \varphi(t_{\varepsilon_j}, x_{\varepsilon_j}), \\ \partial_t u^i_{\varepsilon_j}(t_{\varepsilon_j}, x_{\varepsilon_j}) &= \partial_t \varphi(t_{\varepsilon_j}, x_{\varepsilon_j}), \\ -\partial^2_{xx} u^i_{\varepsilon_j}(t_{\varepsilon_j}, x_{\varepsilon_j}) &\geq -\partial^2_{xx} \varphi(t_{\varepsilon_j}, x_{\varepsilon_j}). \end{aligned} \tag{1.12}$$

Consequently, by (1.10), (1.11) and (1.12), we can obtain

$$\begin{aligned}
\partial_t\varphi(t_{\varepsilon_j},x_{\varepsilon_j}) &+ H^i(t_{\varepsilon_j},x_{\varepsilon_j},u_{\varepsilon_j}(t_{\varepsilon_j},x_{\varepsilon_j}),\partial_x\varphi(t_{\varepsilon_j},x_{\varepsilon_j})) \\
&= \partial_t u^i_{\varepsilon_j}(t_{\varepsilon_j},x_{\varepsilon_j}) + H^i(t_{\varepsilon_j},x_{\varepsilon_j},u_{\varepsilon_j}(t_{\varepsilon_j},x_{\varepsilon_j}),\partial_x u^i_{\varepsilon_j}(t_{\varepsilon_j},x_{\varepsilon_j})) \\
&= \varepsilon_j\partial^2_{xx}u^i_{\varepsilon_j}(t_{\varepsilon_j},x_{\varepsilon_j}) \\
&\le \varepsilon_j\partial^2_{xx}\varphi(t_{\varepsilon_j},x_{\varepsilon_j}).
\end{aligned}$$

Finally, since φ is smooth and H^i, u^i_0 are continuous for every $i=1,\dots,d$, we pass to the limit as $\varepsilon_j\to 0$ to deduce

$$\partial_t\varphi(t_0,x_0) + H^i(t_0,x_0,u(t_0,x_0),\partial_x\varphi(t_0,x_0)) \le 0. \tag{1.13}$$

We say in this case that u is a continuous viscosity sub-solution of (1.7). Similarly, if we have assumed in (1.9) that $u^i-\varphi$ admits a strict local minimum at (t_0,x_0), then we would have reached

$$\partial_t\varphi(t_0,x_0) + H^i(t_0,x_0,u(t_0,x_0),\partial_x\varphi(t_0,x_0)) \ge 0, \tag{1.14}$$

and in this case we say that u is a continuous viscosity super-solution of (1.7). Finally, we say that u is a continuous viscosity solution of (1.7) if it is both a viscosity sub- and super- solution of (1.7).

Remark 1.1. The maximum in (1.9) does not necessarily have to be strict. If we assume that $u^i-\varphi$ has a local maximum at (t_0,x_0), then we can simply modify the test function by taking for example $\widetilde{\varphi}(t,x)=\varphi(t,x)+|t-t_0|^2+|x-x_0|^2$, so that we assure the new text function touches the graph of u^i from above.

Therefore, we have constructed the existence and the meaning of continuous vanishing viscosity solution in the case of first order Hamilton-Jacobi equations. There are other definitions to viscosity solutions, which are all based on inequalities (1.13) and (1.14), and can be found in Evans [49], Barles [8], and Tran [83].

It is important to mention that the definition of a continuous viscosity solutions can be generalized to the discontinuity case, which is in fact the framework of our results. Using the technique presented in this subsection, we will be able to formulate another meaning of a vanishing viscosity solution. This new formulation includes the fact that we regularize the velocities and initial data before adding a viscosity term. Thus, we will show the existence of discontinuous viscosity solutions using a careful analysis of the vanishing viscosity solutions technique presented in this subsection.

After defining the type of solutions we will be working with, we are now ready to present our contribution in the fields of dislocation dynamics and viscosity solutions in the following three sections.

2 Existence result of a discontinuous viscosity solution

In this section, we quote the existence result that we have proven to a 1-dimensional diagonal hyperbolic system, not necessarily strictly hyperbolic, of transport equations. This work is motivated by the following model

$$\partial_t u^i = \left(\sum_{j=1,\dots,d} A_{ij} u^j \right) \partial_x u^i \quad \text{for} \quad i = 1, \dots, d,$$

where $(A_{ij})_{i,j=1,\dots,d}$ is a non-positive symmetric matrix, which describes the dynamics of dislocation densities in the case of several sliding directions. We refer to Chapter 2 for the detailed modeling of the system considered.

More precisely, our main concern was to find solutions of the form $u(t,x) = (u^i(t,x))_{i=1,\dots,d}$ to the following system

$$\begin{cases} \partial_t u^i(t,x) = \lambda^i(t,x,u(t,x)) \partial_x u^i(t,x) & \text{in } \ (0,T) \times \mathbb{R}, \\ u^i(0,x) = u_0^i(x) & \text{in } \ \mathbb{R}, \end{cases} \tag{1.15}$$

where d is an integer, $T > 0$, and $i = 1, \dots, d$. We were able to prove the global in time existence of a discontinuous viscosity solution under the following condition on the velocities λ^i, for every $i = 1, \dots, d$

$$\lambda^i \in L^\infty((0,T) \times \mathbb{R} \times \mathcal{K}) \quad \text{for} \quad T > 0 \quad \text{and for all compact } \ \mathcal{K} \subset \mathbb{R}^d, \tag{1.16}$$

and assuming the initial data u_0^i satisfy for every $i = 1, \dots, d$

$$u_0^i \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}), \tag{1.17}$$

where $BV(\mathbb{R})$ is the space of functions of bounded variations given by

$$BV(\mathbb{R}) = \Big\{ f \in L^1_{loc}(\mathbb{R}); \ TV(f) < +\infty \Big\},$$

with $TV(f)$ being the total variation of f defined as

$$TV(f) = \sup \left\{ \int_{\mathbb{R}} f(x) \phi'(x) dx; \ \phi \in C^1_c(\mathbb{R}) \text{ and } \ \|\phi\|_{L^\infty(\mathbb{R})} \le 1 \right\}.$$

We mention earlier that we have taken the space $BV(\mathbb{R})$ endowed with the semi-norm $|f|_{BV(\mathbb{R})} = TV(f)$.

We recall now some results done in the framework of diagonal hyperbolic systems. In the case of strictly hyperbolic systems, Lax [63] proved the existence of a Lipschitz solution in the case (2×2) of system (1.15). This result was then extended to the case of $(d \times d)$ rich strictly hyperbolic systems by Serre [79, Vol. II]. Also, for general $(d \times d)$ strictly hyperbolic systems, Bianchini and Bressan [17] proved a global existence and uniqueness result assuming the initial data had small total variation. Their approach was mainly based on a careful analysis of the vanishing viscosity method. Furthermore, an existence result has first been proved by Glimm [52] in the special case of conservative equations. We can also mention that an existence result has been also obtained by LeFloch, Liu [65] and LeFloch [64, 66], in the non-conservative case. Moreover, El Hajj and Monneau [46] have shown the existence and uniqueness of a continuous solution for strictly hyperbolic systems assuming the initial data were non-decreasing functions.
For $(d \times d)$ systems that are not necessarily strictly hyperbolic, we mention a global existence and uniqueness result of a continuous solution for non-decreasing initial data by El Hajj and Monneau [45]. There result was based on the same entropy estimate that was proven in their previous work [46]. Moreover, El Hajj, Ibrahim and Rizik [44] have recently proven the existence of a discontinuous viscosity solution under certain monotony conditions on the velocities and the initial data.

We have established for system (1.15) the global in time existence of a discontinuous viscosity solution in some weak sense, which we define later, without any monotony conditions neither on the velocities nor on the initial data. In the case of non-decreasing initial data, we can obtain as a consequence of this result that the constructed solution is a classical discontinuous viscosity solution.

In order to prove so, we have first regularized the initial data and the velocities by classical convolution, then we have added the viscosity term $\eta \partial^2_{xx} u^i_\varepsilon$, where η denotes the diffusion coefficient of the parabolic regularization, and ε corresponds to the mollifiers. This brought us to consider the following system

$$\begin{cases} \partial_t u^i_{\varepsilon,\eta}(t,x) = \eta \partial^2_{xx} u^i_{\varepsilon,\eta}(t,x) + \lambda^i_\varepsilon(t,x,u_{\varepsilon,\eta}(t,x)) \partial_x u^i_{\varepsilon,\eta}(t,x) & \text{in } (0,T) \times \mathbb{R}, \\ u^i_{\varepsilon,\eta}(0,x) = u^i_{0,\varepsilon}(x) & \text{in } \mathbb{R}, \end{cases} \tag{1.18}$$

where for every $i = 1, \dots, d$

$$u^i_{0,\varepsilon} \in L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R}), \quad \text{and} \;\; \partial_x u^i_{0,\varepsilon} \in L^p(\mathbb{R}) \;\; \forall\, 1 \le p \le \infty, \tag{1.19}$$

and

$$\lambda^i_\varepsilon \in W^{1,\infty}((0,T) \times \mathbb{R} \times \mathcal{K}) \cap C^\infty((0,T) \times \mathbb{R} \times \mathbb{R}^d), \;\; \text{for all compact} \;\; \mathcal{K} \subset \mathbb{R}. \tag{1.20}$$

We can now quote in the following theorem the existence result we have proven on system (1.15), which is discussed elaborately in Chapter 3.

Theorem 1.2. ***(Global existence result in a weak sense)***
Suppose that assumptions (1.16) and (1.17) are satisfied. Then, we have

i) Global existence and uniqueness of a smooth solution
There exists a unique classical solution $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\dots,d}$ of (1.18) belonging to the space $(C^\infty((0,T) \times \mathbb{R}))^d \cap (W^{1,\infty}((0,T) \times \mathbb{R}))^d$, and satisfying for all $T > 0$ and $i = 1, \dots, d$, the following uniform a priori estimates

$$\left\| u^i_{\varepsilon,\eta} \right\|_{L^\infty((0,T)\times\mathbb{R})} \le \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})}, \tag{1.21}$$

$$\left\| \partial_x u^i_{\varepsilon,\eta} \right\|_{L^\infty((0,T);L^1(\mathbb{R}))} \le \left| u^i_0 \right|_{BV(\mathbb{R})}, \tag{1.22}$$

$$\left\| \partial_t u^i_{\varepsilon,\eta} \right\|_{L^\infty((0,T);W^{-1,1}(\mathbb{R}))} \le \left(1 + \left\| \lambda^i \right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)} \right) \left| u^i_0 \right|_{BV(\mathbb{R})}, \tag{1.23}$$

where $\mathcal{K}_0 = \prod_{i=1}^{d} \left[- \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})}, \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})} \right]$.

ii) Sub- and super-solutions of (1.15)

Let $u_{\varepsilon,\eta}$ be the solution of (1.18), given in (i). Then the upper and lower relaxed semi-limits $\overline{u} = \left(\overline{u}^i\right)_{i=1,\dots,d}$ and $\underline{u} = \left(\underline{u}^i\right)_{i=1,\dots,d}$, which are defined, respectively, as

$$\overline{u}^i(t,x) = \limsup{}^{\star} u^i_{\varepsilon,\eta}(t,x) = \limsup_{\substack{(\varepsilon,\eta)\longrightarrow(0,0)\\(s,y)\longrightarrow(t,x)}} u^i_{\varepsilon,\eta}(s,y), \tag{1.24}$$

and

$$\underline{u}^i(t,x) = \liminf{}_{\star} u^i_{\varepsilon,\eta}(t,x) = \liminf_{\substack{(\varepsilon,\eta)\longrightarrow(0,0)\\(s,y)\longrightarrow(t,x)}} u^i_{\varepsilon,\eta}(s,y), \tag{1.25}$$

are a couple of discontinuous viscosity sub- and super- solutions of system (1.18) (in the sense of Definition 3.2).

iii) Convergence and existence of a weak solution

Assume that $u^i_{\varepsilon,\eta}$ satisfies (1.21), (1.22) and (1.23) for $i = 1, \ldots, d$. Then, up to the extract of a subsequence, the function $u^i_{\varepsilon,\eta}$ converges, as ε and η tend to zero, to a function

$$u^i \in L^\infty\big((0,T)\times\mathbb{R}\big) \cap L^\infty\big((0,T);\, BV(\mathbb{R})\big) \cap C\left([0,T);\, L^1_{loc}(\mathbb{R})\right), \tag{1.26}$$

strongly in $C\left([0,T);\, L^1_{loc}(\mathbb{R})\right)$.
Moreover, u^i satisfies, for all $T > 0$ and for $i = 1, \ldots, d$, the following inequalities

$$\left\|u^i\right\|_{L^\infty((0,T)\times\mathbb{R})} \leqslant \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}, \tag{1.27}$$

$$\left\|u^i\right\|_{L^\infty((0,T);BV(\mathbb{R}))} \leqslant \left|u^i_0\right|_{BV(\mathbb{R})}, \tag{1.28}$$

and the following equality

$$u^i(t,\cdot) = \overline{u}^i(t,\cdot) = \underline{u}^i(t,\cdot), \text{ except at most on a countable set in } \mathbb{R},\ \forall\, t \in [0,T). \tag{1.29}$$

We will outline briefly the strategy of the proof to the previous theorem. The existence of a unique solution to (1.18) is based on a Fixed Point argument applied to the integral form of this equation. Then, by a Bootstrap argument, we can show that this solution is smooth. After that, we show that this smooth solution satisfies the L^∞ and the BV bounds (1.21) and (1.22) respectively. These estimates will allow us to pass to the limit when the regularization vanishes. Then, using the stability results of viscosity solutions, along with the finite speed propagation property (given in Lemma 3.4) that is proven on the smooth solution of (1.18), we will be able to pass to the limit as $\varepsilon, \eta \to 0$, and show that the upper and lower relaxed semi-limits, which are defined in (1.24) and (1.25), are discontinuous viscosity sub- and super- solutions of (1.15) respectively, in the sense of discontinuous viscosity solutions introduced by Ishii [58] for Hamilton-Jacobi systems (recalled in Definition 3.2). Finally, reaching some (ε, η)- independent *a priori* estimates, we will be able to show that $\overline{u}^i(t,\cdot)$ is equal to $\underline{u}^i(t,\cdot)$ in an almost everywhere sense.
Hence, we have established the existence of a function $u = (u^i)_{i=1,\ldots,d}$, such that its upper semi-continuous envelope (resp. lower semi-continuous envelope) coincides almost everywhere with its upper relaxed semi-limit (resp. lower relaxed semi-limit). This is what we mean by a discontinuous viscosity solution *in a weak sense* in this work.

In the case of non-decreasing initial data, we can actually prove the existence of a discontinuous viscosity solution, i.e., we will obtain a function u that is both a viscosity sub- and super- solution of (1.15). In this case, we have an absolute equality between the upper semi-continuous envelope of u (resp. the lower semi-continuous envelope) and the upper relaxed semi-limit (resp. lower relaxed semi-limit). This is announced in the following theorem.

Theorem 1.3. ***(Global existence of non-decreasing discontinuous viscosity solution)***
Assume that (1.16) and (1.17) are satisfied. Suppose that $u_0^i \in L^\infty(\mathbb{R})$ and the function u_0^i is non-decreasing for $i = 1, \ldots, d$, then system (1.15) admits a discontinuous non-decreasing viscosity solution $u = \left(u^i\right)_{i=1,\ldots,d}$ (in the sense of Definition 3.2), such that for $i = 1, \ldots, d$, u^i satisfies (1.21), (1.22), and (1.23).

In the proof of Theorem 1.3, we will employ the finite speed propagation property in order to prove that the upper semi-continuous envelope of $\overline{u}^i$ (resp. lower semi-continuous envelope) is in fact equal to the upper relaxed semi-limit (resp. lower relaxed semi-limit) of u^i, in the case where the initial data are non-decreasing functions. This will lead us to the existence of a classical viscosity solution. The proof is demonstrated in Section 6 of Chapter 3.

3 Existence and uniqueness result of a continuous viscosity solution

In this section, we show that under certain extra conditions on the velocities λ^i, we can obtain the existence and uniqueness of a continuous viscosity solution to (1.15). However, we will be studying a more general case of system (1.15). More precisely, we consider

$$\begin{cases} \partial_t u^i(t,x) = \lambda^i(t,x,u(t,x))|\partial_x u^i(t,x)| & \text{in } \ (0,T)\times\mathbb{R}, \\ u^i(0,x) = u_0^i(x) & \text{in } \ \mathbb{R}. \end{cases} \tag{1.30}$$

We have two main results for this system. First, we have proven the global in time existence of a discontinuous viscosity solution assuming the velocities λ^i verify (1.16) and the initial data u_0^i satisfy (1.17), for every $i = 1, \ldots, d$, as in the previous section.
As a second result, we show that this discontinuous viscosity solution would be continuous and unique if we assume that the initial data are also continuous, and the velocities satisfy,

for every $i = 1, \dots, d$, the following assumptions

$$\left|\begin{array}{l} \lambda^i \in C((0,T)\times\mathbb{R}\times\mathcal{K}) \quad \text{for} \quad T>0 \quad \text{and for all compact } \ \mathcal{K}\subset\mathbb{R}^d, \\ \text{there exists } M_1>0 \text{ such that, for all } x,y\in\mathbb{R} \text{ and } t\in(0,T), \\ |\lambda^i(t,x,u)-\lambda^i(t,y,u)| \le M_1|x-y|. \end{array}\right. \tag{1.31}$$

It is important to note here that the regularity assumptions imposed on the velocities and initial data in order to prove the existence and uniqueness of a continuous viscosity solution are optimal for such systems.

Let us now mention some results proven on system (1.30). El Hajj, Ibrahim, and Rizik have proved in [44] an existence result of a discontinuous viscosity solution in the same sense we have introduced in Section 2, but under a monotony condition on the velocities. In other words, they have shown the existence of a function $u = (u^i)_{i=1,\dots,d}$ such that its upper relaxed semi-limit $\overline{u} = (\overline{u}^i)_{i=1,\dots,d}$ is a viscosity sub-solution of (1.30), and its lower relaxed semi-limit $\underline{u} = (\underline{u}^i)_{i=1,\dots,d}$ is a viscosity super-solution of (1.30), with $\overline{u}^i(t,\cdot) = \underline{u}^i(t,\cdot)$ almost everywhere in $\mathbb{R}$. We will prove the same result without any monotony assumptions on the velocities. We also mention that, El Hajj and Oussaily [47] have proven the existence and uniqueness of a continuous viscosity solution to (1.30) under a certain monotony on the velocities using an entropy and a BV estimate. It is proven by considering a parabolic regularization of the main system and then passing to the limit when the regularization vanishes. This result is a generalization of their previous work [48] for a (2×2) eikonal system.

The proof of our first result, which is the global existence of a discontinuous viscosity solution, is quite similar to the proof of Theorem 1.2. We consider a parabolic regularization of the system understudy and we pass to the limit when the regularization vanishes using some uniform *a priori* estimates. To that end, we give the following theorem.

Theorem 1.4 (Existence of a discontinuous viscosity solution to (1.30)). *Assume that (1.16) and (1.17) are satisfied. Then the following points hold*

i) Existence and uniqueness to the regularized problem
There exists a unique Lipschitz solution $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\dots,d}$ of

$$\begin{cases} \partial_t u^i_{\varepsilon,\eta}(t,x) = \eta\partial^2_{xx}u^i_{\varepsilon,\eta}(t,x) + \lambda^i_\varepsilon(t,x,u_{\varepsilon,\eta}(t,x))|\partial_x u^i_{\varepsilon,\eta}(t,x)| & \text{in } (0,T)\times\mathbb{R}, \\ u^i_{\varepsilon,\eta}(0,x) = u^i_{0,\varepsilon}(x) & \text{in } \mathbb{R}. \end{cases} \tag{1.32}$$

where λ^i_ε and $u^i_{0,\varepsilon}$ are the regularization of λ^i and u^i_0 by classical convolution respectively, belonging to the space $(C([0,T);W^{1,\infty}(\mathbb{R})))^d$, and satisfying for all $T>0$ and $i=1,\dots,d$, the following uniform estimates

$$\left\|u^i_{\varepsilon,\eta}\right\|_{L^\infty((0,T)\times\mathbb{R})} \le \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}, \tag{1.33}$$

$$\left\|\partial_x u^i_{\varepsilon,\eta}\right\|_{L^\infty((0,T);L^1(\mathbb{R}))} \le \left\|\partial_x u^i_0\right\|_{L^1(\mathbb{R})}, \tag{1.34}$$

$$\left\|\partial_t u^i_{\varepsilon,\eta}\right\|_{L^\infty((0,T);W^{-1,1}(\mathbb{R}))} \le \left(1+\left\|\lambda^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)}\right)\left|u^i_0\right|_{BV(\mathbb{R})}, \tag{1.35}$$

where $W^{-1,1}(\mathbb{R})$ is the dual of $W^{1,\infty}(\mathbb{R})$, and

$$\mathcal{K}_0=\prod_{i=1}^d\left[-\left\|u^i_0\right\|_{L^\infty(\mathbb{R})},\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}\right].$$

ii) Sub- and super- solutions of (1.30)

Let $u_{\varepsilon,\eta}$ be the unique solution of (1.32) constructed in (i). Then the upper and lower relaxed semi-limits $\overline{u}=\left(\overline{u}^i\right)_{i=1,\dots,d}$ and $\underline{u}=\left(\underline{u}^i\right)_{i=1,\dots,d}$, are a couple of discontinuous discontinuous viscosity sub- and super- solutions of system (1.30) (in the sense of Definition 4.1).

iii) Convergence

Assume that the solution $u^i_{\varepsilon,\eta}$ of (1.32) satisfies (1.33), (1.34) and (1.35) for $i=1,\dots,d$. Then, up to the extraction of a subsequence, the function $u^i_{\varepsilon,\eta}$ converges, as ε and η tend to zero, to a function

$$u^i\in L^\infty\big((0,T)\times\mathbb{R}\big)\cap L^\infty\big((0,T);\,BV(\mathbb{R})\big)\cap C\left([0,T);\,L^1_{loc}(\mathbb{R})\right), \tag{1.36}$$

strongly in $C\left([0,T);\,L^1_{loc}(\mathbb{R})\right)$.
Moreover, u^i satisfies, for all $T>0$ and for $i=1,\dots,d$, the following inequalities

$$\left\|u^i\right\|_{L^\infty((0,T)\times\mathbb{R})} \leqslant \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}, \tag{1.37}$$

$$\left\|u^i\right\|_{L^\infty((0,T);BV(\mathbb{R}))} \leqslant \left|u^i_0\right|_{BV(\mathbb{R})}, \tag{1.38}$$

and the following equality

$$u^i(t,\cdot)=\overline{u}^i(t,\cdot)=\underline{u}^i(t,\cdot),\ \textit{except at most on a countable set in } \mathbb{R},\ \textit{for all } t\in[0,T).$$

The elaborate proof of this theorem can be found in Section 2 of Chapter 4.

Our second result in this work was based on the quasi-monotony assumption we imposed on (1.30), in the sense of Ishii, Koike [58, 59]. In other words, we assumed that the velocities λ^i verify, for every $i = 1, \ldots, d$

$$\left|\begin{array}{l} \lambda^j(t,x,s) - \lambda^j(t,x,r) \geq 0 \text{ for all vectors } r = (r^i)_{i=1,\ldots,d}, s = (s^i)_{i=1,\ldots,d} \\ \\ \text{such that } r^j - s^j = \max\limits_{i\in\{1,\ldots,d\}} (r^i - s^i) \geq 0. \end{array}\right. \tag{1.39}$$

Theorem 1.5 (**Existence and uniqueness of a continuous solution to** (**1.30**)). *Suppose that (1.16), (1.17), (1.31), (1.39) hold, and that the initial data u_0^i are also continuous functions on $\mathbb{R}$ for every $i = 1, \ldots, d$. Then, there exists a unique continuous viscosity solution of (1.30) satisfying (1.37) and (1.38).*

The proof of this theorem is based on the comparison principle. First, in order to prove the existence of a continuous viscosity solution, we will show that $\overline{u}^i$ and $\underline{u}^i$, the sub- and super- solutions constructed in Theorem 1.4, are equal for every $x \in \mathbb{R}$ and $t \in [0, T)$. As we already have from the definition of relaxed semi-limits that $\underline{u}^i \leq \overline{u}^i$, for every $i = 1, \ldots, d$, then by exploiting the comparison principle, we will be able to prove that $\underline{u}^i \geq \overline{u}^i$, for every $i = 1, \ldots, d$. This is announced in the following proposition.

Proposition 1.1 (**Comparison Principle**).
Assume (1.16), (1.17), (1.31), (1.39) hold, and that the initial data u_0^i are also continuous functions on $\mathbb{R}$ for every $i = 1, \ldots, d$. Let $\overline{u} = (\overline{u}^i)_{i=1,\ldots,d}$ and $\underline{u} = (\underline{u}^i)_{i=1,\ldots,d}$ be respectively discontinuous viscosity sub- and super solutions of (1.30), in the sense of Definition 4.1. Then, if $\overline{u}^i(\cdot, 0) \leq \underline{u}^i(\cdot, 0)$ in $\mathbb{R}$ we get $\overline{u}^i \leq \underline{u}^i$ in $\mathbb{R} \times [0, T)$ for every $i = 1, \ldots, d$.

As a consequence of this proposition, we can also obtain the uniqueness of the continuous solution.

We also apply the proven results to a 1-dimensional system describing the dynamics of dislocation densities that was initially proposed in 2 dimensions by Groma and Balogh [53, 54], where the dislocations are considered as points moving in the plane (x_1, x_2), propagating to the left and to the right, following two Burgers vectors $\pm(1, 0)$. In the 1-dimensional model, we assume that the dislocations depend on 1 variable $x = x_1 + x_2$ only, which reduces the $2D$ model into a $1D$ one. More precisely, this $1D$ system is of the form

$$\begin{cases} \partial_t v^1(t,x) = -\left((v^1-v^2)(t,x) + \beta \int_0^1 (v^1-v^2)(t,y)dy + a(t)\right)\left|\partial_x v^1(t,x)\right| & \text{in } (0,T)\times\mathbb{R}, \\ \partial_t v^2(t,x) = \left((v^1-v^2)(t,x) + \beta \int_0^1 (v^1-v^2)(t,y)dy + a(t)\right)\left|\partial_x v^2(t,x)\right| & \text{in } (0,T)\times\mathbb{R}, \end{cases} \tag{1.40}$$

where v^1, v^2 represent the left and right propagating dislocations, and $\partial_x v^1, \partial_x v^2$ are the dislocation densities corresponding to each type. The constant β depends on the elastic coefficients and the material size, while the function $a(t)$ represents the exterior shear stress. We refer to [41] for more details about the modeling.

We recall some results for system (1.40). El Hajj has shown in [39] the existence and uniqueness of a non-decreasing solution in the space $W^{1,2}_{loc}((0,T)\times\mathbb{R})$, based on an L^2 energy estimate. Also, El Hajj and Forcadel [41] have shown the existence and uniqueness of a Lipschitz continuous viscosity solution for non-decreasing initial data. In the framework of discontinuous solutions, El Hajj, Ibrahim, and Rizik [42] have proved a global existence result of a BV solution to system (1.40). In the case of (2×2) systems in 2-dimension, Cannone, El Hajj and Monneau [29] have proved a global existence and uniqueness result using an entropy estimate. In the same context, we mention a local existence result by El Hajj [40] in Hölder spaces.

We apply the result of Theorem 1.5 to the local case of system (1.40), i.e, when $\beta = 0$, in order to deduce the existence and uniqueness of a continuous viscosity solution to this system.

4 Convergence result of a finite difference scheme

Finally, in this section, we present a numerical study to the problem presented in the previous section. We propose a semi-explicit scheme, which preserves the L^∞ and BV estimates proven in the continuous case, that permits us to simulate the solutions we have constructed in Theorems 1.4, 1.5.

We first consider the discretization

$$\Xi = \{i\Delta x, i \in \mathbb{Z}\}, \quad \Xi_N = \{0, \ldots, (\Delta t)N\},$$

where N is a positive integer, and we take a time step $\Delta t > 0$ such that $\Delta t = T/N$, and a space step $\Delta x > 0$. We denote by u^α the continuous solution and by $u_i^{\alpha,n}$ the associated discrete solution defined as an approximation of $u^\alpha(n\Delta t, i\Delta x)$. For $u_i^n = (u_i^{\alpha,n})_{\alpha=1,\dots,d}$, we introduce the following semi-explicit scheme

$$\begin{cases} \dfrac{u_i^{\alpha,n+1} - \frac{1}{2}\left(u_{i+1}^{\alpha,n} + u_i^{\alpha,n}\right)}{\Delta t} - \lambda^\alpha(t_{n+1}, x_i, u_i^{n+1}) \dfrac{\left|u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\right|}{\Delta x} = 0, & \forall \alpha \in \{1, \dots, d\}, \\ u_i^{\alpha,0} = u_{|\varepsilon|}^{\alpha,0}(x_i), \end{cases} \tag{1.41}$$

where $u_{|\varepsilon|}^{\alpha,0}$ is a regularization by convolution of the initial data u_0^α to (1.30), where $\varepsilon = (\Delta t, \Delta x)$, for $\alpha = 1, \dots, d$.

We will mention some numerical results known in the framework of Hamilton-Jacobi equations. Using the notion of monotone numerical Hamiltonians introduced by Osher and Sethian [76], Alvarez, Carlini, Monneau, and Rouy [5, 4] have proved the convergence of explicit schemes approximating a non-local eikonal equation. In addition, Souganidis [81] have shown the convergence of general finite difference schemes approximating first order Hamilton-Jacobi equations. He also provides explicit error estimates.
In framework of non-decreasing solutions, Leveque [67] have considered approximations to conservative hyperbolic systems. For non-conservative systems, Monasse and Monneau [73] have presented a convergent semi-explicit scheme assuming the system is strictly hyperbolic.

Our purpose is first to recover the properties of the solution of (1.30) at the discrete level, then to prove the convergence of the discrete solution. In order to do so, we will first consider a continuous linear interpolation of the discrete points $(u_i^{\alpha,n})_{n,i}$, denoted by $u^{\alpha,\varepsilon}$ for $\varepsilon = (\Delta t, \Delta x)$. Then, we show that this interpolation function preserves the L^∞ and the BV estimates (1.37) and (1.38) respectively. These estimates, along with the discrete finite speed propagation property, and the stability, consistency, and monotony of the scheme, allow us to show that the upper and lower relaxed semi-limits $\overline{u}^\alpha$ and $\underline{u}^\alpha$ of $u^{\alpha,\varepsilon}$ are, respectively, discontinuous viscosity sub- and super- solutions of system (1.30) in the sense of discontinuous viscosity solutions introduced by Ishii in [58, Definition 2.1] for Hamilton Jacobi systems. Moreover, we will be able to prove that $\overline{u}^\alpha(t, \cdot)$ and $\underline{u}^\alpha(t, \cdot)$ coincide almost everywhere in space and uniformly in time for all $t \in [0, T)$, as in Theorem 1.4. Finally, in the case where system (1.30) verifies the comparison principle, i.e. under the conditions of Theorem 1.5, we will be able to prove that $u^{\alpha,\varepsilon}$ converges to the unique continuous solution of (1.30).

We set $u_i^n = (u_i^{\alpha,n})_{\alpha=1,\ldots,d}$, $u^n = (u_i^n)_{i\in\mathbb{Z}}$, and we introduce the box

$$\mathcal{U} = \prod_{\alpha=1}^{d} \left[-\|u_0^\alpha\|_{L^\infty(\mathbb{R})}, \|u_0^\alpha\|_{L^\infty(\mathbb{R})}\right].$$

We say that $u^n \in \mathcal{U}^{\mathbb{Z}}$ if $u_i^n \in \mathcal{U}$ for all $i \in \mathbb{Z}$. We also assume that the velocities satisfy the following assumption

$$\begin{cases} \text{there exists } M > 0 \;\text{ such that} \\ \displaystyle\sum_{\alpha=1}^{d} |\lambda^\alpha(t,x,u) - \lambda^\alpha(t,x,v)| \le M|u-v|, \quad \text{for all } u, v \in \mathbb{R}^d, \end{cases} \tag{1.42}$$

where $|w| = \displaystyle\sum_{\alpha=1}^{d} |w^\alpha|$, for $w = (w_1, \ldots, w_d)$.

Next, we assume that

$$\frac{\Delta t}{\Delta x} = \min\left(\frac{1}{2\Lambda}, \frac{1}{2M\|u_0\|_{(L^\infty(\mathbb{R}))^d}}\right) = \gamma, \tag{1.43}$$

where $\|u_0\|_{(L^\infty(\mathbb{R}))^d} = \displaystyle\sum_{\alpha=1}^{d} \|u_0^\alpha\|_{L^\infty(\mathbb{R})}$, and

$$\Lambda = \sup_{\alpha\in\{1,\ldots,d\}} \|\lambda^\alpha\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{U})}.$$

We can now present our results in this work

Theorem 1.6 (***Existence of*** BV ***discrete solution***).
Assume (1.16), (1.17), (1.42), and (1.43) hold. Then we have

i) (Existence)
Let $u^n \in \mathcal{U}^{\mathbb{Z}}$. Then there exists a unique solution $u^{n+1} \in \mathcal{U}^{\mathbb{Z}}$ to the semi-explicit scheme (1.41).

(ii) (Discrete BV ***estimate)***
The discrete gradient, which is defined as

$$\theta_{i+\frac{1}{2}}^{\alpha,n} = \frac{u_{i+1}^{\alpha,n} - u_i^{\alpha,n}}{\Delta x}, \tag{1.44}$$

verifies the following estimate

$$\sum_{i\in\mathbb{Z}} \left|\theta_{i+\frac{1}{2}}^{\alpha,n+1}\right| \le \sum_{i\in\mathbb{Z}} \left|\theta_{i+\frac{1}{2}}^{\alpha,n}\right|, \qquad \text{for } n = 0, \ldots, N-1. \tag{1.45}$$

Theorem 1.7 (***Convergence of the solution of the numerical scheme***)**.**
Assume (1.16), (1.17), (1.42), and (1.43) are satisfied. Consider the solution $(u^n)_{n=0,\dots,N}$ *of the scheme (1.41) for the time step* Δt *and the space step* Δx*. Let us denote by* $\varepsilon = (\Delta t, \Delta x)$ *and* u^ε *a continuous linear interpolation function defined as*

$$u^\varepsilon(n\Delta t, i\Delta x) = u_i^n, \quad \text{for} \;\; n = 0, \dots, N, \quad i \in \mathbb{Z}.$$

Then the following points hold

i) Estimates on u^ε
The function $u^\varepsilon = (u^{\alpha,\varepsilon})_{\alpha=1,\dots,d}$ *verifies*

$$\|u^{\alpha,\varepsilon}\|_{L^\infty((0,T)\times\mathbb{R})} \le \|u_0^\alpha\|_{L^\infty(\mathbb{R})}, \tag{1.46}$$

$$\|u^{\alpha,\varepsilon}\|_{L^\infty((0,T);BV(\mathbb{R}))} \le |u_0^\alpha|_{BV(\mathbb{R})}, \tag{1.47}$$

$$\|\partial_t u^{\alpha,\varepsilon}\|_{L^\infty((0,T);L^1(\mathbb{R}))} \le \left(1 + \frac{2}{\gamma} + \Lambda\right) |u_0^\alpha|_{BV(\mathbb{R})}. \tag{1.48}$$

ii) Convergence
The upper and lower relaxed semi-limits of $u^{\alpha,\varepsilon}$*, which are defined as*

$$\overline{u}^\alpha(t,x) = \lim sup^\star u^{\alpha,\varepsilon}(t,x) = \limsup_{\substack{\varepsilon \longrightarrow 0 \\ (s,y)\longrightarrow(t,x)}} u^{\alpha,\varepsilon}(s,y),$$

and

$$\underline{u}^\alpha(t,x) = \lim inf_\star u^{\alpha,\varepsilon}(t,x) = \liminf_{\substack{\varepsilon \longrightarrow 0 \\ (s,y)\longrightarrow(t,x)}} u^{\alpha,\varepsilon}(s,y),$$

are a couple of discontinuous viscosity sub- and super-solutions of (1.30) (in the sense of Definition 5.2).

iii) Equality between $\overline{u}^\alpha$ ***and*** $\underline{u}^\alpha$
Assume $u^{\alpha,\varepsilon}$ *satisfies (1.46), (1.47) and (1.48) for every* $\alpha = 1, \dots, d$*. Then, up to the extract of a subsequence, the function* $u^{\alpha,\varepsilon}$ *converges, as* $\varepsilon \to 0$*, to a function*

$$u^\alpha \in L^\infty\big((0,T)\times\mathbb{R}\big) \cap L^\infty\big((0,T);\, BV(\mathbb{R})\big) \cap C\left([0,T);\, L^1_{loc}(\mathbb{R})\right),$$

strongly in $C\left([0,T);\, L^1_{loc}(\mathbb{R})\right)$*.*
Moreover, u^α *satisfies, for all* $T > 0$ *and for* $\alpha = 1, \dots, d$*, estimates (1.37), (1.38) and the following equality*

$$u^\alpha(t,\cdot) = \overline{u}^\alpha(t,\cdot) = \underline{u}^\alpha(t,\cdot), \;\; \text{except at most on a countable set in } \mathbb{R}, \;\; \text{for all } t \in [0,T).$$

iv) Unique solution
If (1.31), (1.39) are also satisfied, and v_0^α are continuous for every $\alpha = 1, \dots, d$, then $u^{\alpha,\varepsilon}$ converges to the unique solution of (1.30).

We end this section with some numerical simulations illustrated in the local case of system (1.40), i.e for $\beta = 0$, based on scheme (1.41). We equip this system with non-decreasing initial data of the form

$$u_0^1(x) = u_0^2(x) = u_0(x) = u^{per}(x) + L_0 x,$$

where u^{per} are 1-periodic functions. We have illustrated numerical simulations under suitable time and space steps satisfying (1.43), for $a(t) = 3t$, and $L_0 = 0.5$.
As it is represented in Figure 1.8(a), we assumed that the dislocation densities $\partial_x u^1, \partial_x u^2$ are not uniformly distributed in space at $t = 0$. In other words, there exists regions with concentrated dislocation densities, and other regions with no dislocations at all. This assumption is quite natural as dislocations in materials are not uniformly distributed in reality. When we exert an exterior stress, we notice that they begin to diffuse inside the material (Figure 1.8(b), 1.8(c)), to reach a constant density that is equal to $L_0 = 0.5$, and fill the entire material at $t = 1$, as we can see in Figure 1.8(d).

5 New contraction result to the evolutionary p-Laplacian equation

Our purpose in this part of the thesis is to present a new contraction family for the evolutionary p-Laplacian equation

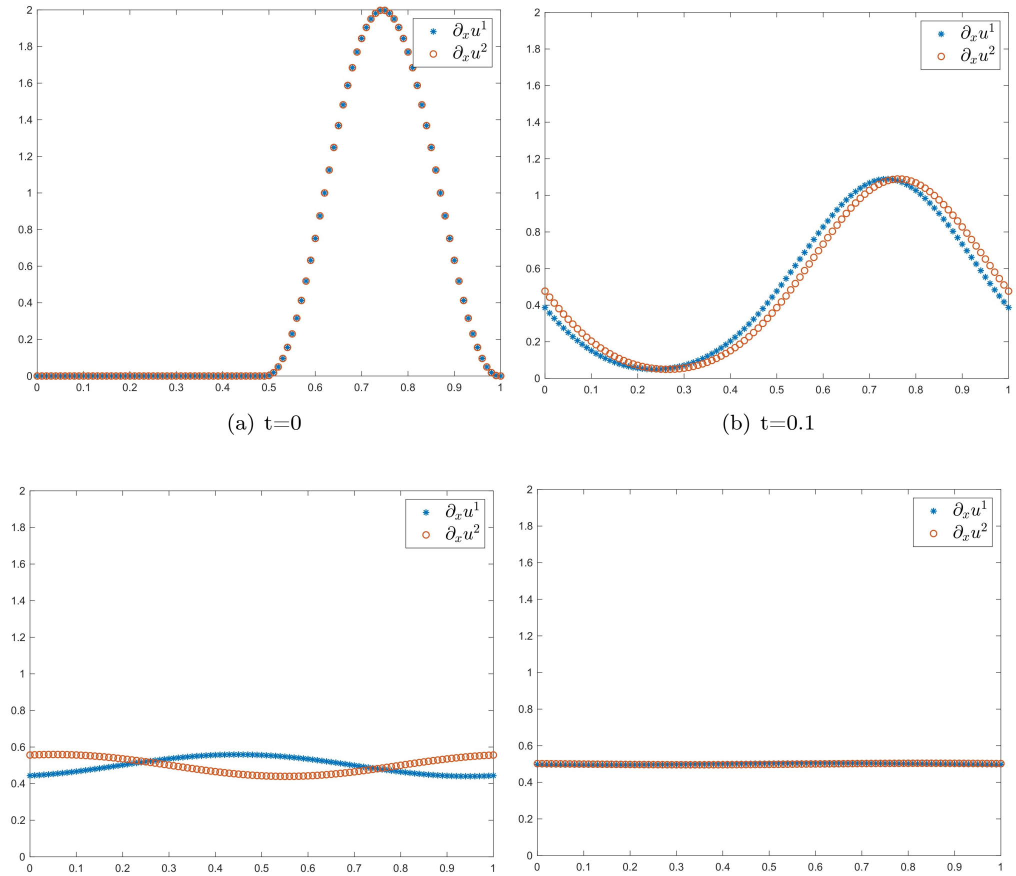
$$\partial_t u(t,x) = \Delta_p u(t,x) \quad \text{in} \quad Q_T = (0,T) \times \Omega, \tag{1.49}$$

where $T > 0$, and $\Omega \subset \mathbb{R}^n$, such that $n \geq 1$, is an open connected subset with smooth boundary. The symbol Δ_p represents the nonlinear p-Laplacian operator, which is defined as

$$\Delta_p u = \nabla \cdot (\nabla u |\nabla u|^{p-2}) = \operatorname{div}(\nabla u |\nabla u|^{p-2}) \ \text{ for } \ 1 \leq p \leq \infty. \tag{1.50}$$

It is a quasi-linear, elliptic partial differential operator of second order.

In simple terms, when we study the "contraction" of the solutions to a certain evolutionary equation, what we seek to know is whether these solutions are approaching each other with time or not. The use of the word "approach" implies the action of a distance, and



(a) t=0 (b) t=0.1

(c) t=0.5 (d) t=1

Figure 1.8: Dislocations densities distribution $\partial_x u^1(\cdot,t), \partial_x u^2(\cdot,t)$ at several instants.

this in fact is what is genuine about our work, the distance being used itself. Normally, contraction results are found in Sobolev spaces, such as $L^2(\Omega)$, $H^1(\Omega)$, or even $H^{-1}(\Omega)$, but in our studies, we will be introducing a new distance constructed between the solutions of the equation understudy, and thus developing a new meaning of "contracting solutions".

5.1 Physical interpretation

In what follows, $\Omega \subset \mathbb{R}^n$ is a bounded, open set with smooth boundary $\partial\Omega$. The p-Laplacian operator is the nonlinear generalization of the linear Laplacian operator

$$\Delta u = \Delta_x u = \sum_{i=1}^{n} \partial^2_{x_i x_i} u \quad \text{where} \quad x = (x_1, \ldots, x_n). \tag{1.51}$$

Notice that for $p = 2$, equation (1.49) reduces to the well-known *Heat equation*

$$\partial_t u = \Delta u. \tag{1.52}$$

These two equations are of diffusion type, and they are derived from the *Continuity equation*, which physically describes the evolution in time of the density of a certain quantity ϕ. In the book of Evans [49, chapter 2], it is mentioned that, for $V \subset \Omega$ being any smooth subset of Ω, the rate of change of the total quantity within V is equal to the negative of the net flux through ∂V. In other words, no material is created or destroyed. This phenomenon is translated mathematically as

$$\frac{d}{dt}\int_V \phi \, dx = -\int_{\partial V} J \cdot \nu \, dS,$$

J being the flux density. Thus, using *Stokes theorem*, we have

$$\partial_t \phi = -\nabla \cdot J, \tag{1.53}$$

since V was arbitrary. The flux J of a diffusing material moves, according to *Fick's first law*, from regions of high density into regions of low density with a magnitude that is proportional to the local density gradient, but points in the opposite direction. Thus, we have

$$J = -\mu(\phi, x)\nabla\phi(t, x), \tag{1.54}$$

where $\mu(\phi, x)$ is the diffusive coefficient of the density ϕ. Then, by replacing (1.54) in (1.53), we get the *Diffusion equation*

$$\partial_t \phi(t, x) = \nabla \cdot (\mu(\phi, x)\nabla\phi(t, x)). \tag{1.55}$$

This equation can intervene in the interpretation of a wide variety of physical phenomena, depending on the value of μ.

A work by Benedikt et al. [15], shows that a nonlinear form of *Darcy's Law* along with the *Continuity Equation* leads to the evolutionary p-Laplacian equation (1.49).

5.2 Some previous results

It is known that for $\Omega \subset \mathbb{R}^n$ bounded domain with Lipschitz continuous boundary, the energy functional

$$\varphi(u) = \int_\Omega |\nabla u|^p dx, \tag{1.56}$$

associated with the p-Laplacian operator $\Delta_p u = \nabla \cdot (\nabla u |\nabla u|^{p-2})$, gives rise to strongly continuous nonlinear (linear if $p = 2$) semigroups of contractions on $L^2(\Omega)$ (see for example [32, Theorem 4.8]). A classical result by Minty [71] (see also Evans [49, Chapter 9], or the monograph by Brézis [22]) shows that every convex, lower semi-continuous functional φ on a Hilbert space H generates a strongly continuous semigroup (nonlinear in general) of contractions on $\overline{D(\varphi)}$.

Moreover on the contraction of the solutions to the p-Laplacian equation (1.49), it is easy to show that $\|u(t,\cdot)\|_{W^{1,p}(\Omega)} \leq \|u_0\|_{W^{1,p}(\Omega)}$ for all t, since

$$\frac{d}{dt}\left(\frac{1}{p}\int_\Omega |\nabla u|^p dx\right) = \int_\Omega \nabla(\partial_t u)\nabla u|\nabla u|^{p-2}dx = -\int_\Omega (div(\nabla u|\nabla u|^{p-2}))^2 dx \leq 0.$$

Similarly,

$$\frac{d}{dt}\left(\frac{1}{p}\int_\Omega |u|^p dx\right) = -(p-1)\int_\Omega |\nabla u|^2|\nabla u|^{p-2}|u|^{p-2}dx \leq 0.$$

5.3 New contraction result

Here, we present our contribution in the theory of differential contraction to the evolutionary p-Laplacian equation. We consider, for $\Omega \subset \mathbb{R}^n$ an open connected subset with smooth boundary and $p > 1$, equation (1.49) equipped with the following initial and boundary conditions

$$\begin{cases} u(t,x) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.57}$$

The initial condition u_0 satisfies

$$u_0 \geq 0 \quad \text{in} \quad \Omega. \tag{1.58}$$

We announce our main result in the following theorem.

Theorem 1.8.
Let u and v be two solutions of (1.57), belonging to the space $C^2(Q_T)$, with initial data u_0 and v_0 respectively, both satisfying (1.58). For $p > 1$, $q > 1$, and $\alpha \in \mathbb{R}$ such that $0 < \alpha_- < \alpha < 1$, where

$$\alpha_- := \frac{4(q-1)(p-1)}{4q(q-1)(p-1) - p^2q^2} + 1,$$

we have

$$\int_\Omega (v^\alpha - u^\alpha)_+^q dx \leq \int_\Omega ((v_0)^\alpha - (u_0)^\alpha)_+^q \, dx. \tag{1.59}$$

As a consequence of this theorem, we obtain that the solutions of (1.57) have non-decreasing $L^r(\Omega)$ norm for $r = \alpha q \geq 1$, which is a quite known result for the p-Laplacian operator (See [16]). Thus, we have the following corollary.

Corollary 1.1.
Under the conditions of Theorem 1.8, a solution u satisfies, for all $r \geq 1$, the following estimate

$$\|u(t)\|_{L^r(\Omega)} \leq \|u_0\|_{L^r(\Omega)}, \quad \text{for all } t > 0. \tag{1.60}$$

Before giving the idea of the proof of Theorem 1.8, we first construct a distance between its positive solutions. We introduce the function

$$\begin{aligned} w_0 : [0,1] &\to C^2(\Omega) \\ s &\mapsto w_0(s) =: w_0^s, \end{aligned}$$

where $w_0 \in C^2\big([0,1]; C^2(\Omega)\big)$ is a path joining u_0 to v_0. In other words, we have $w_0^0 = u_0$ and $w_0^1 = v_0$.
Next, we construct the function

$$\begin{aligned} w : [0,1] \times Q_T &\to \mathbb{R} \\ (s,t,x) &\mapsto w(s,t,x), \end{aligned}$$

where for all $s \in [0,1]$, we have $w(s,\cdot,\cdot) =: w^s(\cdot,\cdot)$ is a solution of (1.57) with initial data w_0^s. It is clear that $w^0 = u$ and $w^1 = v$.
Now we define the set

$$\mathcal{E} = \Big\{ w \in C^2\big([0,1] \times Q_T\big) : w^0 = u \text{ and } w^1 = v \Big\}, \tag{1.61}$$

which is a set of paths connecting u to v.

Definition 1.2 (A pseudo-distance)**.**
Given two $C^2(Q_T)$-solutions u and v of (1.57), we define the following pseudo-distance

$$d(u,v) := \inf_{w \in \mathcal{E}} \mathcal{A}(w \quad \text{with} \quad \mathcal{A}(w) := \int_0^1 ds \int_\Omega (w_+)^\gamma \frac{(w'_+)^q}{q} dx, \tag{1.62}$$

where the set $\mathcal{E}$ was defined in (1.61), $\gamma \in \mathbb{R}$, and $q \in \mathbb{R}_+^*$.

We note here that we will working with positive paths only. Thus, the term $\mathcal{A}(w)$ becomes

$$\mathcal{A}(w) = \int_0^1 ds \int_\Omega w^\gamma \frac{(w'_+)^q}{q} dx. \tag{1.63}$$

In order to prove Theorem 1.8, first we show that the distance defined in the previous definition can be explicitly expressed in terms of the solutions as follows

$$d(u,v) = \inf_{w \in \mathcal{E}} \int_0^1 ds \int_\Omega w^\gamma \frac{(w')_+^q}{q} dx = \frac{1}{q\alpha^q} \int_\Omega (v^\alpha - u^\alpha)_+^q dx, \tag{1.64}$$

where $\alpha = 1 + \frac{\gamma}{q}$. Then, we differentiate the quantity $\mathcal{A}(w)$ defined in Definition 1.2 with respect to time, and we try to see where this derivative is negative. It does not necessarily admit a domain where it is negative, but in the case of system (1.57), we were able to construct such a domain. Then, using inequality (1.64), we can obtain the main result (1.59). The proof of Corollary 1.1 derives directly from Theorem 1.8 by assuming that u is identically zero.

2 Modeling

In this chapter, we present the physical derivation of a model describing the dynamics of dislocation densities. This chapter is based on the previous work of El Hajj, Monneau [45, Section 5].

In reality, models describing the dynamics of dislocations are three dimensional. However, we will assume that the geometry of the problem is invariant by translation in the x_3-direction. This reduces our problem to the study of planar dislocation densities propagating in the plane (x_1, x_2), following a Burgers vector b, which also belongs to the plane (x_1, x_2).

This chapter is divided into three sections. First in Section 1, we present the two dimensional model with multi-slip directions. Then in Section 2, we show how the two dimensional model reduces into a one dimensional one in a particular geometry, where we assume the dislocation densities depend on the variable $x = x_1 + x_2$. Finally in Section 3, we explain how the dynamics of dislocation densities can be described by the following system

$$\begin{cases} \partial_t u^i(t,x) + \left(\displaystyle\sum_{j=1}^{d} A_{ij} u^j\right) \partial_x u^i(t,x) = 0, & \text{on } (0,+\infty) \times \mathbb{R}, \text{ for } \ i = 1, \dots, d, \\ u^i(0,x) = u_0^i(x), & \text{on } \mathbb{R}, \text{ for } \ i = 1, \dots, d, \end{cases} \tag{2.1}$$

where $A = (A_{ij})_{i,j=1,\dots,d}$ is a non-negative symmetric matrix, which is a particular case of the systems studied in Chapters 3, 4, 5.

1 The bidimensional model

We denote by X the vector $X = (x_1, x_2) \in \mathbb{R}^2$. We consider a crystal filling the entire space $\mathbb{R}^2$, with $v = (v_1, v_2) : \mathbb{R}^2 \to \mathbb{R}^2$ as its displacement, where we have not yet introduced any time dependence.

Here, we are studying edge dislocations. We assume that we have d directions of sliding such that each direction corresponds to the Burgers vector $b^k = (b_1^k, b_2^k) \in \mathbb{R}^2$, for $k = 1, \ldots, d$. This leads to d different types of dislocations that propagate in the plane (x_1, x_2) following the direction of b^k, for $k = 1, \ldots, d$.

We introduce the total strain

$$\varepsilon(v) = \frac{1}{2}(\nabla v + {}^t\nabla v), \quad \text{where} \quad (\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j},$$

which is a symmetric matrix defined by

$$\varepsilon(v) = (\varepsilon_{ij}(v))_{i,j=1,2}\,, \quad \text{with} \quad \varepsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

The total strain can be divided into two parts

$$\varepsilon_{ij}(v) = \varepsilon_{ij}^e + \varepsilon_{ij}^p, \quad \text{with} \quad \varepsilon^p = \sum_{k=1}^{d} \varepsilon^{0,k} u^k,$$

where ε_{ij}^e is the elastic strain, and ε_{ij}^p is the plastic one. The scalar function u^k is the plastic displacement associated to the k-th slip system whose shear tensor $\varepsilon^{0,k}$ is defined by

$$\varepsilon_{ij}^{0,k} = \frac{1}{2}\left(b_i^k n_j^k + n_i^k b_j^k\right), \tag{2.2}$$

where $n^k = (n_1^k, n_2^k)$ is a unit vector orthogonal to the Burgers vector b^k.
In order to simplify the presentation, we assume the simplest possible periodicity of the unknowns.
<u>**Assumption** (H)</u>:

i) The function v is $\mathbb{Z}^2$-periodic with $\displaystyle\int_{(0,1)^2} v \; dX = 0.$

ii) For each $k = 1, \ldots, d$, there exists $L^k \in \mathbb{R}^2$ such that $u^k(X) - L^k \cdot X$ is $\mathbb{Z}^2$-periodic.

iii) The integer d is even with $d = 2N$ and we have for $k = 1, \ldots, N$, the following

$$L^{k+N} = L^k, \;\; n^{k+N} = n^k, \;\; b^{k+N} = -b^k \;, \varepsilon^{0,k+N} = -\varepsilon^{0,k}.$$

iv) We denote by $\tau^k \in \mathbb{R}^2$ a unit vector parallel to b^k such that $\tau^{k+N} = \tau^k$.
We choose L^k such that $\tau^k \cdot L^k \geq 0$.

We note that, as a consequence of Assumption (H), the plastic strain ε^p_{ij} is $\mathbb{Z}^2$-periodic. The stress matrix is then given by

$$\sigma = \Lambda \colon \varepsilon^e(v), \quad i.e, \quad \sigma_{ij} = \sum_{k,l=1,2} \Lambda_{ijkl}\ \varepsilon^e_{kl} \quad \text{for} \quad i,j = 1,2,$$

where $\Lambda = (\Lambda_{ijkl})_{j,j,k,l=1,2}$ are the constant elastic coefficients of the material, satisfying for some constant $m > 0$

$$\sum_{i,j,k,l=1,2} \Lambda_{ijkl}\ \varepsilon_{ij}\ \varepsilon_{kl} \geq m \sum_{i,j=1,2} \varepsilon^2_{ij}, \tag{2.3}$$

for all symmetric matrices $\varepsilon = (\varepsilon_{ij})_{ij}$, that is, $\varepsilon_{ij} = \varepsilon_{ji}$.
Then, the stress is assumed to satisfy the elasticity equation

$$\operatorname{div} \sigma = 0, \quad i.e, \quad \sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \text{for} \quad i = 1,2. \tag{2.4}$$

On the other hand, the plastic displacement u^k is assumed to satisfy the following transport equation

$$\partial_t u^k = c^k \tau^k \cdot \nabla u^k, \quad \text{with} \quad c^k = \sum_{i,j=1,2} \sigma_{ij}\ \varepsilon^{0,k}_{ij}.$$

This equation can be interpreted by observing that

$$\theta^k = \tau^k \cdot \nabla u^k \geq 0, \tag{2.5}$$

is the density of edge dislocations associated to the Burgers vector b^k moving in the direction τ^k with the velocity c^k. Here, c^k is also called the resolved Peach-Koehler force in the physical literature. In particular, we see that the dislocation density θ^k satisfies the following conservation law

$$\partial_t \theta^k = \operatorname{div}\left(c^k \tau^k \theta^k\right).$$

Finally, for $k = 1, \ldots, d$, the functions u^k and v are then assumed to depend on $(t, X) \in (0, +\infty) \times \mathbb{R}^2$ and to be solutions of the coupled system (see Yefimov [85, Ch. 5] and Yefimov, Van der Giessen [86])

$$\begin{cases} \operatorname{div}\ \sigma & = 0 & \text{on } (0,T) \times \mathbb{R}^2, \\ \sigma & = \Lambda : (\varepsilon(v) - \varepsilon^p) & \text{on } (0,T) \times \mathbb{R}^2, \\ \varepsilon(v) & = \dfrac{1}{2}\left(\nabla v + {}^t\nabla v\right) & \text{on } (0,T) \times \mathbb{R}^2, \\ \varepsilon^p & = \displaystyle\sum_{k=1,\ldots,d} \varepsilon^{0,k} u^k & \text{on } (0,T) \times \mathbb{R}^2, \\ \partial_t u^k & = (\sigma : \varepsilon^{0,k}) \tau^k . \nabla u^k & \text{on } (0,T) \times \mathbb{R}^2, \quad \text{for } k = 1, \ldots, d, \end{cases} \tag{2.6}$$

which is, in coordinates, equivalent to

$$
\left\{
\begin{array}{lll}
\displaystyle\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 & \text{on } (0,T)\times\mathbb{R}^2, & \text{for } i=1,2,\\
\displaystyle\sigma_{ij} = \sum_{k,l=1,2} \Lambda_{ijkl}\left(\varepsilon_{kl}(v) - \varepsilon^p_{kl}\right) & \text{on } (0,T)\times\mathbb{R}^2, & \\
\displaystyle\varepsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) & \text{on } (0,T)\times\mathbb{R}^2, & \text{for } i,j=1,2\\
\displaystyle\varepsilon^p_{ij} = \sum_{k=1,\ldots,d} \varepsilon^{0,k}_{ij} u^k & \text{on } (0,T)\times\mathbb{R}^2, & \\
\displaystyle\partial_t u^k = \left(\sum_{i,j\in\{1,2\}} \sigma_{ij}\varepsilon^{0,k}_{ij}\right)\tau^k.\nabla u^k & \text{on } (0,T)\times\mathbb{R}^2, & \text{for } k=1,\ldots,d,
\end{array}
\right.
\tag{2.7}
$$

where the unknowns of this system are $u = (u^k)_{k=1,\ldots,d}$ and the displacement $v = (v_1, v_2)$, and $\varepsilon^{0,k}_{ij}$ is defined in (2.2). We also note that these equations are compatible with the periodicity conditions (i)-(ii) of Assumption (H).

For a detailed physical presentation of a model with multi-slip directions, we refer to Yefimov, Van der Giessen [86] and Yefimov [85, Ch. 5], and to Groma, Balogh [54] for the case of a model with a single slip direction. See also Cannone et al. [29] for a mathematical analysis of the Groma, Balogh model.

2 The unidimensional model

In this section, we are interested in a particular geometry where the dislocation densities depend only on one variable $x = x_1 + x_2$, which transforms the 2-dimensional model presented in the previous section into a 1-dimensional one. More precisely, we assume the following

Assumption (H'):

i) The functions $v(t,X)$ and $u^k(t,X) - L^k \cdot X$ depend only on the variable $x = x_1 + x_2$.

ii) For $k = 1,\ldots,d$, the vector $\tau^k = (\tau^k_1, \tau^k_2)$ satisfies $\tau^k_1 + \tau^k_2 = 1$.

iii) For $k = 1, \dots, d$, the vector $L^k = (L_1^k, L_2^k)$ satisfies $L_1^k = L_2^k$.

In this particular 1-dimensional geometry, we denote by the function $v = v(t,x)$, which is 1-periodic in x. If we set $l^k = \dfrac{L_1^k + L_2^k}{2}$, we get

$$L^k \cdot X = l^k \cdot x + \left(\frac{L_1^k + L_2^k}{2}\right)(x_1 - x_2).$$

By (iii) of Assumption (H'), we see that $u = \left(u^k(t,x)\right)_{k=1,\dots,d}$ is defined such that $u^k(t,x) - l^k \cdot x$ is 1-periodic in x for every $k = 1, \dots, d$.

Now, we can integrate the equation of elasticity, i.e. the first equation of (2.6). Using the $\mathbb{Z}^2$-periodicity of the unknowns (see (i)-(ii) of assumption (H)), and the fact that $\varepsilon^{0,k+N} = -\varepsilon^{0,k}$ (see (iii) of Assumption (H)), we can easily conclude that the strain

$$\varepsilon^p \text{ is a linear function of } \left(u^j - u^{j+N}\right)_{j=1,\dots,N}, \text{ and } \left(\int_0^1 \left(u^j - u^{j+N}\right) dx\right)_{j=1,\dots,N} \tag{2.8}$$

This leads to the following lemma.

Lemma 2.1 (Stress for the 1D model).
Under (i)-(iii) of Assumption (H), and (i)-(iii) of Assumptions (H'), we have

$$-\sigma : \varepsilon^{0,i} = \sum_{j=1}^{d} A_{ij} u^j + \sum_{j=1}^{d} Q_{ij} \int_0^1 u^j dx, \quad \textit{for } i = 1, \dots, N, \tag{2.9}$$

where for $i, j = 1, \dots, N$

$$\begin{cases} A_{ij} = Aji & \textit{and} \quad A_{i+N\ j} = -A_{ij} = A_{i\ j+N}, \\ Q_{ij} = Q_{ji} & \textit{and} \quad Q_{i+N\ j} = -Q_{ij} = Q_{i\ j+N}. \end{cases} \tag{2.10}$$

Moreover, the matrix $A = (A_{ij})_{i,j=1,\dots,d}$ is non-negative.

The proof of this lemma will be given at the end of this section.

Finally, using Lemma 2.1, we can eliminate the stress and reduce the problem to a 1-dimensional system of d transport equations depending only on the function u^i, for $i = 1, \dots, d$. Naturally, from (2.9) and (ii) of (H'), this 1D model has the following form

$$\partial_t u^i + \left(\sum_{i=1}^{d} A_{ij} u^j + \sum_{j=1}^{d} Q_{ij} \int_0^1 u^j dx\right) \partial_x u^i = 0, \quad \text{on } (0,T) \times \mathbb{R}, \text{ for } i = 1, \dots, d, \tag{2.11}$$

with

$$\partial_x u^i \geq 0, \text{ for } i = 1, \ldots, d, \tag{2.12}$$

using (2.5).

Now, we give the proof of Lemma 2.1.

Proof of Lemma 2.1.
For the 2-dimensional model, using the fact that ε^e is $\mathbb{Z}^2$-periodic, we consider the following elastic energy on the periodic cell

$$E^{el} = \frac{1}{2} \int_{(0,1)^2} \Lambda \colon \varepsilon^e \colon \varepsilon^e dX.$$

By definition of σ and ε^e, we have for $i = 1, \ldots, d$

$$\sigma \colon \varepsilon^{0,i} = -\nabla_{u^i} E^{el}. \tag{2.13}$$

On the other hand, using (i)-(iii) of (H'), with $x = x_1 + x_2$, we can verify that the elastic energy can be rewritten as

$$E^{el} = \frac{1}{2} \int_0^1 \Lambda \colon \varepsilon^e \colon \varepsilon^e dx.$$

Replacing ε^e by its expression (2.8), we get

$$\begin{aligned} E^{el} =& \frac{1}{2} \int_0^1 \sum_{j=1}^N A_{ij} (u^j - u^{j+N})(u^i - u^{i+N}) dx \\ &+ \frac{1}{2} \sum_{j=1}^N Q_{ij} \left(\int_0^1 (u^j - u^{j+N}) dx \right) \left(\int_0^1 (u^i - u^{i+N}) dx \right), \end{aligned}$$

for some symmetric matrices $A_{ij} = A_{ji}$, $Q_{ij} = Q_{ji}$. In particular, joint to (2.13) gives exactly (2.1) and (2.10).

Let us now consider the functions $w^i = u^i - u^{i+N}$ such that

$$\int_0^1 w^i dx = 0, \quad \text{for } i = 1, \ldots, N. \tag{2.14}$$

From (2.3), we deduce that

$$0 \leq E^{el} = \frac{1}{2} \int_0^1 \sum_{i,j=1}^N A_{ij} w^i w^j dx.$$

More precisely, for all $i = 1, \dots, N$ and for all $\bar{w}^i \in \mathbb{R}$, we set

$$w^i = \begin{cases} \bar{w}^i & \text{on } \left[0, \frac{1}{2}\right], \\ -\bar{w}^i & \text{on } \left[\frac{1}{2}, 1\right], \end{cases}$$

which satisfies (2.14). Finally, we obtain that

$$0 \le E^{el} = \frac{1}{2} \int_0^1 \sum_{i,j=1}^{N} A_{ij} \bar{w}^i \bar{w}^j dx.$$

As this is valid for every $\bar{w}^i$, we deduce that A is a non-negative matrix.

□

3 Derivation of the non-periodic model

Starting from the model (2.11)-(2.12) where the function $u^i(t,x) - l^i \cdot x$ is 1-periodic in x for every $i = 1, \dots, d$, we now wish to get rid of the periodicity. More precisely, we have the following lemma.

Lemma 2.2 (**Non-periodic model**).
Let u be a solution of (2.11)-(2.12). Suppose that Lemma 2.1 is verified and $u^i(t,x) - l^k \cdot x$ is 1-periodic in x. Let

$$u_\delta^j(t,x) = u^j(\delta t, \delta x), \quad \textit{for a small } \delta > 0, \ \textit{and for } j = 1, \dots, d,$$

such that, for every $j = 1, \dots, d$

$$u_\delta^j(0,\cdot) \to \bar{u}^j(0,\cdot), \quad \textit{as } \delta \to 0, \ \textit{and } \ \bar{u}^j(0, \pm\infty) = \bar{u}^{j+N}(0, \pm\infty). \tag{2.15}$$

Then, $\bar{u} = (\bar{u}^j)_{j=1,\dots,d}$ formally is a solution of

$$\partial_t \bar{u}^i + \left(\sum_{j=1}^{d} A_{ij} \bar{u}^j \right) \partial_x \bar{u}^i = 0, \quad \textit{on } (0,T) \times \mathbb{R}, \tag{2.16}$$

where the matrix A is non-negative and $\partial_x \bar{u}^i \ge 0$ for every $i = 1, \dots, d$.

We remark that the limit problem (2.16) is of type of (2.1).

Now we give the proof of Lemma 2.2 formally.

Formal proof of Lemma 2.2.
Here, we know that $u^i_\delta - \delta l^i \cdot x$ is $(1/\delta)$-periodic in x, and satisfies for $i = 1, \dots, d$

$$\partial_t u^i_\delta + \left(\sum_{j=1}^{d} A_{ij} u^j_\delta + \delta \sum_{j=1}^{d} Q_{ij} \int_0^{\frac{1}{\delta}} u^j_\delta \, dx \right) \partial_x u^i_\delta = 0, \quad \text{on } (0,T) \times \mathbb{R}. \tag{2.17}$$

For simplification, we assume that the initial data $u_\delta(0,\cdot)$ converges to a function $\bar{u}(0,\cdot)$ such that $\partial_x u_\delta(0,\cdot)$ has a support in $(-R,R) \subset \left(-\frac{1}{2\delta}, \frac{1}{2\delta}\right)$ uniformly in δ, where R is a positive constant. Using (2.15) and the fact that the matrix Q is an antisymmentric matrix given by (2.10), we expect that the velocity in (2.17) remains uniformly bounded as $\delta \to 0$.

Thus, using the finite speed propagation property, we can see that there exists a constant C independent of δ, such that $\partial_x u_\delta(t,\cdot)$ has a support in $(-R-Ct, R+Ct) \subset \left(-\frac{1}{2\delta}, \frac{1}{2\delta}\right)$ uniformly in δ. Moreover, from (2.15) and the fact that

$$\sum_{j=1}^{d} Q_{ij} \int_0^{\frac{1}{\delta}} u^j_\delta \, dx = \sum_{j=1}^{N} Q_{ij} \int_0^{\frac{1}{\delta}} \left(u^j - u^{j+N}\right) dx,$$

we deduce that

$$\sum_{j=1}^{d} Q_{ij} \int_0^{\frac{1}{\delta}} u^j_\delta \, dx,$$

remains bounded uniformly in δ. Then, formally, the non-local term vanishes and we get for $i = 1, \dots, d$

$$\sum_{j=1}^{d} A_{ij} \, u^j_\delta + \delta \sum_{j=1}^{d} Q_{ij} \int_0^{\frac{1}{\delta}} u^j_\delta \, dx \longrightarrow \sum_{j=1}^{d} A_{ij} \bar{u}^j, \text{ as } \delta \to 0,$$

which proves that $\bar{u}$ is a solution of (2.16), with A being a non-negative matrix.

$\square$

3 Global existence to a diagonal hyperbolic system

This chapter is an article, written in collaboration with Ahmad El Hajj and Mustapha Jazar, that was accepted in *Nonlinearity* journal.

In this work, we study a non-linear diagonal hyperbolic system in one space dimension. Using a BV estimate, we will be able to show the existence of a discontinuous viscosity solution to system understudy. In order to achieve this, we consider first a parabolic regularization of the main system, and we prove that the regularized system admits a unique solution. Then, using some uniform *a priori*, along with stability results of viscosity solutions, we will be able to pass to the limit and prove the existence to the main problem.

Global existence to a diagonal hyperbolic system for any BV initial data

Maryam Al Zohbi, Ahmad El Hajj, Mustapha Jazar

Abstract

In this paper, we study the existence of solutions for a diagonal hyperbolic system, that is not necessarily strictly hyperbolic, in one space dimension, considering discontinuous BV initial data without any restrictions on the size of its norm. This system appears naturally in various physical domains, particularly in isentropic gas dynamics and dislocation dynamics in materials. In the case of strictly hyperbolic systems, an existence and uniqueness of a discontinuous solution result is available for BV initial data with small norm, whereas several existence and uniqueness results have been presented for non-decreasing continuous solutions. In the present paper, we show the global in time existence of discontinuous viscosity solutions to a diagonal hyperbolic system for every initial data of bounded total variation, without the assumption that the system is strictly hyperbolic. Up to our knowledge, this is the first global existence result of large discontinuous solutions to this system.

AMS Classification: 35L45, 35F55, 35A23, 35D40.

Key words: Non-linear hyperbolic system, non-linear transport system, BV estimate, discontinuous viscosity solution.

1 Introduction and main result

1.1 Setting of the problem

In this paper, we are interested in existence results for solutions of the form $u(t,x) = (u^i(t,x))_{i=1,\dots,d}$, to the following one dimensional hyperbolic system

$$\begin{cases} \partial_t u^i(t,x) = \lambda^i(t,x,u(t,x))\partial_x u^i(t,x) & \text{in } (0,T)\times\mathbb{R}, \\ u^i(0,x) = u_0^i(x) & \text{in } \mathbb{R}, \end{cases} \tag{3.1}$$

for $T > 0$ and $i = 1,\dots,d$, where $d \in \mathbb{N}^*$. The functions u^i are real valued, $\partial_t u^i$ and $\partial_x u^i$ represent the time and spatial derivatives of u^i respectively. Here, the velocity λ^i is assumed to satisfy, for all $i = 1,\dots,d$, the following assumption

$$\lambda^i \in L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}) \quad \text{for} \quad T>0 \quad \text{and for all compact} \;\; \mathcal{K}\subset\mathbb{R}^d. \tag{3.2}$$

Our purpose in this work is to establish the global existence of discontinuous viscosity solutions to system (3.1) assuming (3.2) and the following regularity on the initial data

$$u_0^i \in L^\infty(\mathbb{R})\cap BV(\mathbb{R}), \tag{3.3}$$

where $BV(\mathbb{R})$ is the space of functions of bounded variations given by

$$BV(\mathbb{R}) = \Big\{ f\in L^1_{loc}(\mathbb{R});\; TV(f) < +\infty \Big\},$$

with $TV(f)$ being the total variation of f defined as

$$TV(f) = \sup\left\{ \int_{\mathbb{R}} f(x)\phi'(x)dx;\;\; \phi\in C^1_c(\mathbb{R}) \text{ and } \|\phi\|_{L^\infty(\mathbb{R})}\le 1 \right\}.$$

In the following, we take the space $BV(\mathbb{R})$ endowed with the semi-norm $|f|_{BV(\mathbb{R})} = TV(f)$. Note that BV functions are integrable functions whose distributional derivative is a finite Radon measure.

Our study of system (3.1) is initially motivated by the consideration of a model describing the dynamics of dislocation densities (see [45, Section 5] for more details about the model), which is given by

$$\partial_t u^i = \left(\sum_{j=1,\dots,d} A_{ij}u^j + a(t) \right)\partial_x u^i \quad \text{for} \quad i=1,\dots,d, \tag{3.4}$$

where $(A_{ij})_{i,j=1,\dots,d}$ is a non-positive symmetric matrix. This model can be seen as a special case of system (3.1).

Let us mention that El Hajj and Forcadel proved in [41] the existence and uniqueness of a non-decreasing Lipschitz continuous viscosity solution of (3.4), in the particular case where $d=2$. Moreover, the existence and uniqueness of a non-decreasing solution was shown by El Hajj [39], in $[W^{1,2}_{loc}([0,+\infty)\times\mathbb{R})]^2$.

For system (3.1), most of the results are done in the case where the system is strictly hyperbolic. We will recall some of the most significant ones.

First, in the case of (2×2) strictly hyperbolic systems, Lax proved in [63], the existence of a Lipschitz solution to system (3.1). This result was extended by Serre [79, Vol II] to the case of $(d\times d)$ rich strictly hyperbolic systems.

For general $(d \times d)$ strictly hyperbolic systems, Bianchini and Bressan proved in [17] a striking global existence and uniqueness result assuming that the initial data has small total variation. This approach is mainly based on a careful analysis of the vanishing viscosity approximation. An existence result has first been proved by Glimm [52] in the special case of conservative equations. We can also mention that an existence result has been also obtained by LeFloch, Liu [65] and LeFloch [64, 66], in the non-conservative case. Moreover, based a new entropy estimate, El Hajj and Monneau were able to prove in [46] the existence and uniqueness of a continuous solution for strictly hyperbolic systems with non-decreasing initial data.

On the other hand, in the general case of $(d \times d)$ hyperbolic systems (not necessarily strictly hyperbolic), it is worth mentioning that the global existence of a continuous solution has been shown in El Hajj, Monneau [45] for non-decreasing initial data, basing on the same entropy estimate used in [46].

Let us also mention that, using the framework of discontinuous viscosity solutions, a similar result to that presented in this paper for a particular equation of one-dimensional scalar eikonal type was initially shown in [19], which corresponds to system (3.1) in the case $d = 1$, assuming that the velocity is independent of the solution and considering additionally non-decreasing initial data. After that, this work was generalized in the thesis of Vivian Rizik, first in [43] for a particular quasi-monotone (2×2) system (similar to (3.4)) and then in [44], for a more general $(d \times d)$ system of non-linear eikonal type. This last result gives the existence of a global discontinuous viscosity solution for system (3.1) only in the case of non-decreasing initial data and under some monotonicity conditions on the velocities λ^i. We show in this paper the global in time existence of a discontinuous solution for the $(d \times d)$ hyperbolic system (3.1) without any monotonicity condition on the velocity and considering any initial data with bounded total variation (not necessarily non-decreasing). To achieve this result, we consider the parabolic regularization of system (3.1) and we show that the smooth solution satisfies some new fundamental *a priori* estimates, in particular a BV bound and a finite speed propagation property, which remains valid even for data that is not necessarily non-decreasing (contrary to what was done in [44]). Moreover, thanks to these key estimates, we were able to pass here to the limit when the regularization vanishes, without any monotonicity restrictions on the velocity. This is what makes the presented work interesting and novel, and up to our knowledge this is the first work established in this direction, with very weak regularity on the data.

In the framework of viscosity solutions, Ishii, Koike [59] and Ishii [58], have shown existence and uniqueness of continuous viscosity solutions for Hamilton-Jacobi systems of the form

$$\begin{cases} \partial_t u^i + H_i(t,x,u,Du^i) = 0 & \text{with } \ u=(u^1,\dots,u^d)\in\mathbb{R}^d, x\in\mathbb{R}^N, \ \text{and } t\in(0,+\infty), \\ u^i(0,x)=u_0^i(x) & \text{for } \ x\in\mathbb{R}^N, \end{cases}$$

where the Hamiltonian H_i is quasi-monotone in u (see the definition in Ishii, Koike [59, Th. 4.7]).

In this paper, we present a global in time existence of a discontinuous viscosity solution to the hyperbolic system (3.1), without any monotony conditions neither on the velocities λ^i nor on the solution, considering any BV initial data.

Let us return to the key steps followed to prove our existence results. First, we regularize by classical convolution the velocities and the initial data which were announced in (3.1) as follows

$$u_{0,\varepsilon}^i(x) = u_0^i \star \rho_\varepsilon^1(x) \ \text{ and } \ \lambda_\varepsilon^i(t,x,w) = \hat{\lambda}^i \star \rho_\varepsilon^{d+2}(t,x,w) \quad \forall\ (t,x,w)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d, \tag{3.5}$$

where $\hat{\lambda}^i$ is an extension of λ^i by 0 for all $i=1,\dots,d$. Moreover, ρ_ε^n for $n=1$ and $n=d+2$ are the standard mollifiers defined as

$$\rho_\varepsilon^n(\cdot) = \frac{1}{\varepsilon^n}\rho^n\left(\frac{\cdot}{\varepsilon}\right), \ \text{such that } \rho^n\in C_c^\infty(\mathbb{R}^n), \ \text{supp}\{\rho^n\}\subseteq B(0,1), \ \rho^n\geq 0, \ \text{and } \int_{\mathbb{R}^n}\rho^n = 1.$$

This approximation brings us to consider, for every $0<\varepsilon\leq 1$ and $i=1,\dots,d$, the following system

$$\begin{cases} \partial_t u_\varepsilon^i(t,x) = \lambda_\varepsilon^i(t,x,u_\varepsilon(t,x))\partial_x u_\varepsilon^i(t,x) & \text{in } \ (0,T)\times\mathbb{R}, \\ u_\varepsilon^i(0,x) = u_{0,\varepsilon}^i(x) & \text{in } \ \mathbb{R}. \end{cases} \tag{3.6}$$

In order to obtain a better regularity on the solution, we will add the viscosity term $\eta\partial_{xx}^2 u_\varepsilon^i$ to the first equation of the previous system, for $0<\eta\leq 1$. As a first step, we prove the existence and uniqueness of a smooth solution to the following parabolic regularized system

$$\begin{cases} \partial_t u^i_{\varepsilon,\eta}(t,x) = \eta \partial^2_{xx} u^i_{\varepsilon,\eta}(t,x) + \lambda^i_\varepsilon(t,x,u_{\varepsilon,\eta}(t,x))\partial_x u^i_{\varepsilon,\eta}(t,x) & \text{in } (0,T)\times\mathbb{R}, \\ u^i_{\varepsilon,\eta}(0,x) = u^i_{0,\varepsilon}(x) & \text{in } \mathbb{R}, \end{cases} \tag{3.7}$$

through a fixed point argument. After that, using the stability result of viscosity solutions, we pass to the limit as $(\varepsilon,\eta) \to (0,0)$ to go back to the solution of (3.1). In other words, we will show that the upper and lower relaxed semi-limits, of Barles and Perthame [10, 11], which are defined as follows

$$\overline{u}^i(t,x) = \limsup{}^\star u^i_{\varepsilon,\eta}(t,x) = \limsup_{\substack{(\varepsilon,\eta)\longrightarrow(0,0)\\(s,y)\longrightarrow(t,x)}} u^i_{\varepsilon,\eta}(s,y), \tag{3.8}$$

and

$$\underline{u}^i(t,x) = \liminf{}_\star u^i_{\varepsilon,\eta}(t,x) = \liminf_{\substack{(\varepsilon,\eta)\longrightarrow(0,0)\\(s,y)\longrightarrow(t,x)}} u^i_{\varepsilon,\eta}(s,y), \tag{3.9}$$

are a couple of discontinuous viscosity sub- and super-solutions of system (3.1) in the sense of discontinuous viscosity solutions introduced by Ishii in [58, Definition 2.1] for the Hamilton-Jacobi system and recalled below in Definition 3.2. Finally, reaching some (ε,η)-independent *a priori* estimates, we will be able to prove an almost everywhere equality between $\overline{u}^i$ and $\underline{u}^i$ in $\mathbb{R}$, for all $t > 0$. This proves the existence of a function $u = (u^i)_{i=1,\dots,d}$, where u^i is defined as a strong limit of $u^i_{\varepsilon,\eta}$ in $C([0,T); L^1_{loc}(\mathbb{R}))$, such that the upper semi-continuous envelope (resp. lower semi-continuous envelope) of u coincides with the upper relaxed semi-limit (resp. lower relaxed semi-limit). This leads us later on into using a weaker notion of the standard one that is usually used for discontinuous viscosity solutions. All of this is possible thanks to the uniform BV bound obtained on $u^i_{\varepsilon,\eta}$, and the finite speed propagation property of the equation.

1.2 Main results

In this subsection, we first present, in Theorem 3.1, a global existence result of a discontinuous viscosity solution of (3.1) in a certain weak sense. As a consequence, we show in Theorem 3.2 that this solution is a classical discontinuous viscosity solution of (3.1) in the case of non-decreasing initial data.

Remark 3.1. Before stating our main results, we will first clarify what we mean by a discontinuous viscosity solution "in a certain weak sense". We will show that there exists a sub-solution $\overline{u}^i$ and a super-solutions $\underline{u}^i$ for (3.1), that are equal in space except on a countable set of points in $\mathbb{R}$, that is, we will have $\overline{u}^i(t,\cdot) = \underline{u}^i(t,\cdot) = u^i(t,\cdot)$ almost everywhere in $\mathbb{R}$, for every $i = 1,\dots,d$ and $t \in [0,T)$. In other words, we will show that the upper semi-continuous envelope (resp. the lower semi-continuous envelope) of the solution u^i coincides with the upper relaxed semi-limit of u^i (resp. lower relaxed semi-limit) almost everywhere only, in contrary to the classical definition where they coincide everywhere.

Theorem 3.1. ***(Global existence result in a weak sense)***
Suppose that assumptions (3.2) and (3.3) are satisfied. Then, we have

i) Global existence and uniqueness of a smooth solution
There exists a unique classical solution $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\dots,d}$ of (3.7) belonging to the space $(C^\infty((0,T)\times\mathbb{R}))^d \cap (W^{1,\infty}((0,T)\times\mathbb{R}))^d$, and satisfying for all $T>0$ and $i=1,\dots,d$, the following uniform a priori estimates

$$\left\|u^i_{\varepsilon,\eta}\right\|_{L^\infty((0,T)\times\mathbb{R})} \le \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}, \tag{3.10}$$

$$\left\|\partial_x u^i_{\varepsilon,\eta}\right\|_{L^\infty((0,T);L^1(\mathbb{R}))} \le \left|u^i_0\right|_{BV(\mathbb{R})}, \tag{3.11}$$

$$\left\|\partial_t u^i_{\varepsilon,\eta}\right\|_{L^\infty((0,T);W^{-1,1}(\mathbb{R}))} \le \left(1 + \left\|\lambda^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)}\right) \left|u^i_0\right|_{BV(\mathbb{R})}, \tag{3.12}$$

where $\mathcal{K}_0 = \prod_{i=1}^{d} \left[-\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}, \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}\right]$.

ii) Sub- and super-solutions of (3.1)
Let $u_{\varepsilon,\eta}$ be the solution of (3.7), given in (i). Then the relaxed semi-limits $\overline{u} = \left(\overline{u}^i\right)_{i=1,\dots,d}$ and $\underline{u} = (\underline{u}^i)_{i=1,\dots,d}$, are a couple of discontinuous viscosity sub- and super- solutions of system (3.1) (in the sense of Definition 3.2), where $\overline{u}^i$ and $\underline{u}^i$ are, respectively, the upper relaxed semi-limit and the lower relaxed semi-limit defined in (3.8) and (3.9).

iii) Convergence and existence of a weak solution
Assume that $u^i_{\varepsilon,\eta}$ satisfies (3.10), (3.11) and (3.12) for $i=1,\dots,d$. Then, up to the extraction of a subsequence, the function $u^i_{\varepsilon,\eta}$ converges, as ε and η tend to zero, to a function

$$u^i \in L^\infty\big((0,T)\times\mathbb{R}\big) \cap L^\infty\big((0,T);\, BV(\mathbb{R})\big) \cap C\left([0,T);\, L^1_{loc}(\mathbb{R})\right), \tag{3.13}$$

strongly in $C\left([0,T);\, L^1_{loc}(\mathbb{R})\right)$.
Moreover, u^i satisfies, for all $T>0$ and for $i=1,\dots,d$, the following inequalities

$$\left\|u^i\right\|_{L^\infty((0,T)\times\mathbb{R})} \leqslant \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}, \tag{3.14}$$

$$\left\|u^i\right\|_{L^\infty((0,T);BV(\mathbb{R}))} \leqslant \left|u_0^i\right|_{BV(\mathbb{R})}, \tag{3.15}$$

and the following equality

$$u^i(t,\cdot) = \overline{u}^i(t,\cdot) = \underline{u}^i(t,\cdot), \text{ except at most on a countable set in } \mathbb{R}, \text{ for all } t \in [0,T). \tag{3.16}$$

The key point to establish this theorem is the uniform BV estimate (3.11) on $u^i_{\varepsilon,\eta}$. We first show that the smooth solution of the parabolic regularized system (3.7) admits the L^∞ bound (3.10) and the fundamental BV estimate (3.11). These estimates will allow us to pass to the limit when the regularization vanishes. Then we will show, from the classical stability properties of viscosity solutions, that the relaxed semi-limits $\overline{u}$ and $\underline{u}$ are, respectively, sub- and super-solutions of (3.1) . These estimates also imply that the set of the discontinuous points, with respect to x, of the solution u is at most countable. Taking into account the finite speed propagation property of (3.1) and the time continuous estimate (3.12), it is then possible to show this property uniformly in time, which proves in particular (3.16).

We note that, the solution $u = (u^i)_{i=1,\dots,d}$ constructed in Theorem 3.1-(iii) as the limit of $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\dots,d}$, is in the sense on Remark 3.1. In other words, due to the discontinuity of the solutions, we are not capable to affirm that the upper semi-continuous envelope (resp. the lower semi-continuous envelope) of u is equal to the sub-solution $\overline{u}$ (resp. the super-solution $\underline{u}$) over the entire space, as it's usually considered in the classical study of discontinuous viscosity solutions. Nevertheless, thanks to the discontinuity property of BV functions, we can prove that this is valid almost everywhere, which is reflected by equality (3.16).

However, in the framework of non-decreasing solutions, it is possible to prove the existence of a standard discontinuous viscosity solution, namely proving the equality between the sub-solution $\overline{u}$ (resp. the super-solution $\underline{u}$) and the upper semi-continuous envelope (resp. the lower semi-continuous envelope) over $(0,T) \times \mathbb{R}$. Thus we announce the following theorem.

Theorem 3.2. ***(Global existence of non-decreasing discontinuous viscosity solution)***
Assume that (3.2) and (3.3) are satisfied. Suppose that $u_0^i \in L^\infty(\mathbb{R})$ and the function u_0^i is non-decreasing for $i = 1,\dots,d$, then system (3.1) admits a discontinuous non-decreasing

viscosity solution $u = \left(u^i\right)_{i=1,\dots,d}$ *(in the sense of Definition 3.2), such that for* $i = 1, \dots, d$, u^i *satisfies (3.13), (3.14), and (3.15).*

1.3 Organization of the paper

This paper is organized as follows: in Section 2, we show the local existence and uniqueness of a Lipschitz solution to the parabolic regularized system (3.7). Then in Section 3, we prove the (ε, η)-uniform BV estimate, and another uniform a priori estimate. Section 4 is devoted to the proof of the global in time existence result of the parabolic regularized system (3.7), announced in Theorem 3.1 (i). Then, in Section 5, using the finite speed propagation property of (3.7), we prove Theorem 3.1 (ii). After that, by passing to the limit as ε and η tend to zero and using a compactness argument, we display the proof of Theorem 3.2. Finally, in Section 7, we present the proof of Theorem 3.1 (iii), using again the finite speed propagation property.

2 Local solution for parabolic regularized equation

In this section, we prove the existence and uniqueness of a solution to a parabolic regularized equation obtained by the regularization of problem (3.1). More precisely, we consider, for $0 < \eta \leq 1$ and $i = 1, \dots, d$, the following system

$$\begin{cases} \partial_t v_\eta^i(t,x) = \eta \partial_{xx}^2 v_\eta^i(t,x) + \widetilde{\lambda}^i(t,x,v_\eta(t,x))\partial_x v_\eta^i(t,x) & \text{in } (0,T)\times\mathbb{R}, \\ v_\eta^i(0,x) = v_0^i(x) & \text{in } \mathbb{R}, \end{cases} \tag{3.17}$$

where for all $i = 1, \dots, d$, we have

$$v_0^i \in L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R}) \quad \text{and} \quad \partial_x v_0^i \in L^p(\mathbb{R}) \ \text{ for all } \ 1 \leq p \leq +\infty, \tag{3.18}$$

and

$$\widetilde{\lambda}^i \in W^{1,\infty}((0,T)\times\mathbb{R}\times\mathcal{K}) \cap C^\infty((0,T)\times\mathbb{R}\times\mathbb{R}^d), \quad \text{for all compact } \mathcal{K} \subset \mathbb{R}. \tag{3.19}$$

Theorem 3.3. ***(Existence and uniqueness of a Lipschitz solution to (3.17))*** *Assume that (3.18) and (3.19) hold. Then, there exists* $T^* > 0$*, such that system (3.17) admits a unique solution* $v_\eta = (v_\eta^i)_{i=1,\dots,d}$ *belonging to the space* $(C^\infty((0,T^*)\times\mathbb{R}))^d \cap (W^{1,\infty}((0,T^*)\times\mathbb{R}))^d$ *and satisfying, for all* $i = 1, \dots, d$*, the following estimate*

$$\left\| v_\eta^i \right\|_{L^\infty((0,T^*)\times\mathbb{R})} \leq \left\| v_0^i \right\|_{L^\infty(\mathbb{R})}. \tag{3.20}$$

Moreover, there exists a constant ξ_p *depending on* $\|v_0^i\|_{L^\infty(\mathbb{R})}$, $\|\partial_x v_0^i\|_{L^p(\mathbb{R})}$ *and* T^* *such that*

$$\left\|\partial_x v_\eta^i\right\|_{L^\infty((0,T^*);L^p(\mathbb{R}))} \leq \xi_p, \qquad \text{for all } \ 1 \leq p \leq +\infty. \tag{3.21}$$

To prove this theorem, we need the following Lemma.

Lemma 3.1.
Let $G_\eta(t,x) = \dfrac{1}{\sqrt{4\pi t\eta}} e^{-\frac{x^2}{4t\eta}}$ *be the standard heat kernel. If we note* $G_\eta(t) = G_\eta(t,\cdot)$, *then for all* $t > 0$, *we have*

$$\text{(i) } \|G_\eta(t)\|_{L^p(\mathbb{R})} = \begin{cases} K_p t^{\frac{1}{2p}-\frac{1}{2}} \text{ where } K_p = \left(\dfrac{1}{(4\pi\eta)^{\frac{p-1}{2}}\sqrt{p}}\right)^{\frac{1}{p}}, & \text{for } 1 \leq p < +\infty \\ \dfrac{1}{\sqrt{4\pi\eta t}}, & \text{for } p = +\infty. \end{cases}$$

$$\text{(ii) } \|\partial_x G_\eta(t)\|_{L^p(\mathbb{R})} = \zeta_p t^{\frac{1}{2p}-1} \text{ where } \zeta_p = \left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{2^{p-1} p^{\frac{p+1}{2}} \pi^{\frac{p}{2}} \eta^{p-\frac{1}{2}}}\right)^{\frac{1}{p}} \text{ for all } 1 \leq p < +\infty.$$

For the proof of this Lemma, we refer to Pazy [77, Theorem 5.2].

Proof of Theorem 3.3.
The proof is outlined in four steps.

Step 1. (Rewriting the equation in its integral form):
Problem (3.17) can be written in its integral form, for every $i = 1, \ldots, d$, as follows

$$v_\eta^i(t,x) = G_\eta(t) \star v_0^i(x) + \int_0^t \left(G_\eta(t-s) \star \widetilde{\lambda}^i(s,\cdot,v_\eta(s,\cdot))\partial_x v_\eta^i(s,\cdot)\right)(x)ds.$$

In other words, we consider the following problem

$$\begin{cases} v_\eta(t,x) = (v_\eta^i(t,x))_{i=1,\ldots,d}, \quad v_0(x) = (v_0^i(x))_{i=1,\ldots,d}, \\ v_\eta(t,x) = G_\eta(t) \star v_0(x) + B(v_\eta)(t,x), \end{cases} \tag{3.22}$$

where for $r(t,x) = (r^i(t,x))_{i=1,\ldots,d}$, and $\partial_x r(t,x) = (\partial_x r^i(t,x))_{i=1,\ldots,d}$, we have

$$B(r)(t,x) = \int_0^t \Big(G_\eta(t-s) \star A\big(s,\cdot,r(s,\cdot)\big) \cdot \partial_x r(s,\cdot)\Big)(x)ds,$$

with

$$A\big(s,\cdot,r(s,\cdot)\big)=\begin{pmatrix}\widetilde{\lambda}^1(s,\cdot,r(s,\cdot)) & 0 & 0 & \cdots & 0\\ 0 & \widetilde{\lambda}^2(s,\cdot,r(s,\cdot)) & 0 & \cdots & 0\\ \vdots & & \ddots & & \vdots\\ \\ 0 & \cdots & & 0 & \widetilde{\lambda}^d(s,\cdot,r(s,\cdot))\end{pmatrix}.$$

In what follows, we show some estimates to prove in particular the contraction of the operator B, and therefore the existence and uniqueness of a solution to (3.17) by applying a fixed point argument.

Step 2. (A priori estimates on B and $G_\eta(\cdot)\star v_0$):
Given any Banach space $(Y,\|\cdot\|_Y)$, in the rest of the paper we consider the norm on Y^d as

$$\|w\|_{Y^d}=\sum_{i=1,\dots,d}\left\|w^i\right\|_Y,\quad\text{for}\quad w=(w^1,\dots,w^d)\in Y^d.$$

Consider, for $T>0$, the spaces

$$F=\Big\{r=(r^i)_{i=1,\dots,d}:r\in(L^\infty(\mathbb{R}))^d\ \text{ and }\ \partial_x r\in(L^1(\mathbb{R}))^d\Big\},$$

equipped with the norm $\|r\|_F=\|r\|_{(L^\infty(\mathbb{R}))^d}+\|\partial_x r\|_{(L^1(\mathbb{R}))^d}$, and

$$F_T=\Big\{r=(r^i)_{i=1,\dots,d}:r\in(L^\infty((0,T)\times\mathbb{R}))^d\text{ and }\partial_x r\in(L^\infty((0,T);L^1(\mathbb{R})))^d\Big\},$$

equipped with the norm $\|r\|_{F_T}=\|r\|_{(L^\infty((0,T)\times\mathbb{R}))^d}+\|\partial_x r\|_{(L^\infty((0,T);L^1(\mathbb{R})))^d}$.

We introduce the subspace X_T of F_T defined as follows

$$X_T=\Big\{r\in F_T:\|r\|_{F_T}\le\|v_0\|_F+1\Big\}.\tag{3.23}$$

We define, for $T>0$, the two constants Λ_1 and Λ_2 as follows

$$\begin{cases}\Lambda_1=\max\limits_{i\in\{1,\dots,d\}}\left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_1)},\quad\text{where}\quad\mathcal{K}_1=\Big[-\|v_0\|_F-1,\|v_0\|_F+1\Big]^d,\\[2em] \left|\widetilde{\lambda}^i(t,x,u)-\widetilde{\lambda}^i(t,x,v)\right|\le\Lambda_2\|u-v\|_{(L^\infty((0,T)\times\mathbb{R}))^d}\quad\text{for all}\quad i=1,\dots,d.\end{cases}\tag{3.24}$$

We remark that, for all $r \in X_T$, we have

$$\begin{aligned}
\|B(r)(t,\cdot)\|_F &= \left\| \int_0^t G_\eta(t-s) \star A\big(s,\cdot,r(s,\cdot)\big) \cdot \partial_x r(s,\cdot) ds \right\|_{(L^\infty(\mathbb{R}))^d} \\
&+ \left\| \int_0^t \partial_x G_\eta(t-s) \star A\big(s,\cdot,r(s,\cdot)\big) \cdot \partial_x r(s,\cdot) ds \right\|_{(L^1(\mathbb{R}))^d} \\
&\leq \sum_{i=1}^d \int_0^t \left\| G_\eta(t-s) \star \widetilde{\lambda}^i(s,\cdot,r(s,\cdot)) \partial_x r^i(s,\cdot) \right\|_{L^\infty(\mathbb{R})} ds \\
&+ \sum_{i=1}^d \int_0^t \left\| \partial_x G_\eta(t-s) \star \widetilde{\lambda}^i(s,\cdot,r(s,\cdot)) \partial_x r^i(s,\cdot) \right\|_{L^1(\mathbb{R})} ds \\
&\leq \sum_{i=1}^d \int_0^t \left\| G_\eta(t-s) \right\|_{L^\infty(\mathbb{R})} \left\| \widetilde{\lambda}^i \right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_1)} \left\| \partial_x r^i(s,\cdot) \right\|_{L^1(\mathbb{R})} \\
&+ \sum_{i=1}^d \int_0^t \left\| \partial_x G_\eta(t-s) \right\|_{L^1(\mathbb{R})} \left\| \widetilde{\lambda}^i \right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_1)} \left\| \partial_x r^i(s,\cdot) \right\|_{L^1(\mathbb{R})}.
\end{aligned}$$

We employ Lemma 3.1 (i) for $p = +\infty$ in the first line of the previous inequality, and Lemma 3.1 (ii) for $p = 1$ in the second line. Thus we get, for $0 < t \leq T$, the following estimate

$$\|B(r)\|_{F_T} \leq \frac{3\Lambda_1}{\sqrt{\eta\pi}} \sqrt{T} \, \|r\|_{F_T}. \tag{3.25}$$

In addition, we have

$$\begin{aligned}
\|G_\eta(t) \star v_0\|_F &= \|G_\eta(t) \star v_0\|_{(L^\infty(\mathbb{R}))^d} + \|G_\eta(t) \star \partial_x v_0\|_{(L^1(\mathbb{R}))^d} \\
&= \sum_{i=1}^d \left\| G_\eta(t) \star v_0^i \right\|_{L^\infty(\mathbb{R})} + \sum_{i=1}^d \left\| G_\eta(t) \star \partial_x v_0^i \right\|_{L^1(\mathbb{R})} \\
&\leq \sum_{i=1}^d \left\| v_0^i \right\|_{L^\infty(\mathbb{R})} + \sum_{i=1}^d \left\| \partial_x v_0^i \right\|_{L^1(\mathbb{R})} = \|v_0\|_F,
\end{aligned}$$

where we have used Lemma 3.1 (i) for $p = 1$ in the last line. Thus we have

$$\|G_\eta(\cdot) \star v_0\|_{F_T} \leq \|v_0\|_F. \tag{3.26}$$

Step 3. (Fixed point argument):
We choose to work in the Banach space X_T, defined in (3.23). Now we introduce the

mapping

$$J : X_T \to X_T$$

$$r \mapsto J(r) = G_\eta(\cdot) \star v_0 + B(r).$$

We will show that J is well-defined and contracting for T small enough. First, we will prove that J is well-defined.
Let $r \in X_T$, using (3.25) and (3.26), we get

$$\begin{aligned}\|J(r)\|_{F_T} &\leq \|v_0\|_F + \frac{3\Lambda_1}{\sqrt{\eta\pi}}\sqrt{T}\,\|r\|_{F_T}\\ &\leq \|v_0\|_F + \frac{3\Lambda_1}{\sqrt{\eta\pi}}\sqrt{T}\big(\,\|v_0\|_F + 1\big),\end{aligned}$$

Then, taking

$$T \leq \frac{\eta\pi}{\Big(3\Lambda_1\big(\,\|v_0\|_F + 1\big)\Big)^2} := T_1^*,$$

implies that J is well-defined in $X_{T_1^*}$.

It remains to show that J is a contraction on X_T for a certain T. Indeed, let $r_1, r_2 \in X_T$,

$$\begin{aligned}&\|J(r_1(s,\cdot)) - J(r_2(s,\cdot))\|_F = \|B(r_1(s,\cdot)) - B(r_2(s,\cdot))\|_F\\ &\leq \sum_{i=1}^d \int_0^t \left\| G_\eta(t-s) \star \Big(\widetilde{\lambda}^i(s,\cdot,r_1(s,\cdot))\partial_x r_1^i(s,\cdot) - \widetilde{\lambda}^i(s,\cdot,r_2(s,\cdot))\partial_x r_2^i(s,\cdot)\Big)\right\|_{L^\infty(\mathbb{R})} ds\\ &+ \sum_{i=1}^d \int_0^t \left\| \partial_x G_\eta(t-s) \star \Big(\widetilde{\lambda}^i(s,\cdot,r_1(s,\cdot))\partial_x r_1^i(s,\cdot) - \widetilde{\lambda}^i(s,\cdot,r_2(s,\cdot))\partial_x r_2^i(s,\cdot)\Big)\right\|_{L^1(\mathbb{R})} ds.\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}&\|J(r_1(s,\cdot)) - J(r_2(s,\cdot))\|_F\\ &\leq \sum_{i=1}^d \int_0^t \Big\| G_\eta(t-s)\Big\|_{L^\infty(\mathbb{R})} \Big\|\widetilde{\lambda}^i(s,\cdot,r_1(s,\cdot))\partial_x r_1^i(s,\cdot) - \widetilde{\lambda}^i(s,\cdot,r_2(s,\cdot))\partial_x r_2^i(s,\cdot)\Big\|_{L^1(\mathbb{R})} ds\\ &+ \sum_{i=1}^d \int_0^t \Big\| \partial_x G_\eta(t-s)\Big\|_{L^1(\mathbb{R})} \Big\|\widetilde{\lambda}^i(s,\cdot,r_1(s,\cdot))\partial_x r_1^i(s,\cdot) - \widetilde{\lambda}^i(s,\cdot,r_2(s,\cdot))\partial_x r_2^i(s,\cdot)\Big\|_{L^1(\mathbb{R})} ds.\end{aligned}$$

Using again Lemma 3.1 (i) for $p = +\infty$ and Lemma 3.1 (ii) for $p = 1$, we obtain

$$\|J(r_1) - J(r_2)\|_{F_T} \leq \frac{3}{\sqrt{\eta\pi}}\sqrt{T}\big(\Lambda_1 + \Lambda_2(\|v_0\|_F + 1)\big)\,\|r_1 - r_2\|_{F_T}.$$

Then, taking

$$T < \frac{\eta\pi}{\Big(3\big(\Lambda_1 + \Lambda_2(\|v_0\|_F + 1)\big)\Big)^2} := T_2^*.$$

This implies that J is a contraction on the space $X_{T_2^*}$.
Hence, by Fixed Point Theorem, there exists a unique solution v_η to system (3.22) in X_{T^*}, where $T^* = \min(T_1^*, \frac{T_2^*}{2})$, that verifies in particular (3.21). This shows that v_η is a solution of system (3.17) over $(0, T^*) \times \mathbb{R}$.

Step 4. (L^p-Regularity):
In this step, we prove estimate (3.21) for some $T > 0$.
We will start with the case where $1 \le p < +\infty$. Differentiating the integral equation in (3.22) with respect to x, then considering the norm in $L^p(\mathbb{R})$, we get

$$\begin{aligned}
\left\|\partial_x v_\eta^i(t,\cdot)\right\|_{L^p(\mathbb{R})} &\le \left\|G_\eta(t) \star \partial_x v_0^i\right\|_{L^p(\mathbb{R})} + \int_0^t \left\|\partial_x G_\eta(t-s) \star \widetilde{\lambda}^i(s,\cdot,v_\eta(s,\cdot))\partial_x v_\eta^i(s,\cdot)\right\|_{L^p(\mathbb{R})} ds \\
&\le \left\|G_\eta(t)\right\|_{L^1(\mathbb{R})} \left\|\partial_x v_0^i\right\|_{L^p(\mathbb{R})} \\
&\quad + \int_0^t \left\|\partial_x G_\eta(t-s)\right\|_{L^p(\mathbb{R})} \left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_1)} \left\|\partial_x v_\eta^i(s,\cdot)\right\|_{L^1(\mathbb{R})} ds.
\end{aligned}$$

We employ Lemma 3.1 (i) for $p = 1$ in the second line, and Lemma 3.1 (ii) in the third line, to get

$$\begin{aligned}
\left\|\partial_x v_\eta^i(t,\cdot)\right\|_{L^p(\mathbb{R})} &\le \left\|\partial_x v_0^i\right\|_{L^p(\mathbb{R})} + \zeta_p \Lambda_1 \left\|v_\eta\right\|_{F_T} \int_0^t (t-s)^{\frac{1}{2p}-1} ds \\
&\le \left\|\partial_x v_0^i\right\|_{L^p(\mathbb{R})} + 2p\zeta_p\Lambda_1\big(\left\|v_0\right\|_F + 1\big)T^{\frac{1}{2p}}.
\end{aligned}$$

Then, taking the supremum over t, leads to the following estimate for all $1 \le p < +\infty$

$$\left|\begin{aligned}
&\left\|\partial_x v_\eta^i\right\|_{L^\infty((0,T);L^p(\mathbb{R}))} \le M_p, \\
&M_p := \left\|\partial_x v_0\right\|_{(L^p(\mathbb{R}))^d} + 2p\zeta_p\Lambda_1\Big(\left\|v_0\right\|_{(L^\infty(\mathbb{R}))^d} + \left\|\partial_x v_0\right\|_{(L^1(\mathbb{R}))^d} + 1\Big)T^{\frac{1}{2p}}.
\end{aligned}\right. \tag{3.27}$$

Next, we consider $1 < p < +\infty$ in order to obtain an estimate on $\left\|\partial_x v_\eta^i\right\|_{(L^\infty((0,T)\times\mathbb{R}))^d}$. In this case, we have from estimate (3.27) that $v_\eta^i \in L^\infty((0,T); L^p(\mathbb{R}))$ for all $i = 1, \dots, d$. Again, differentiating equation (3.22) with respect to x, then taking the norm in $L^\infty(\mathbb{R})$, we obtain

$$\begin{aligned}\left\|\partial_x v_\eta^i(t,\cdot)\right\|_{L^\infty(\mathbb{R})} &\le \left\|G_\eta(t)\star\partial_x v_0^i\right\|_{L^\infty(\mathbb{R})} + \int_0^t \left\|\partial_x G_\eta(t-s)\star\widetilde{\lambda}^i(s,\cdot,v_\eta(s,\cdot))\partial_x v_\eta^i(s,\cdot)\right\|_{L^\infty(\mathbb{R})} ds \\ &\le \left\|G_\eta(t)\right\|_{L^1(\mathbb{R})}\left\|\partial_x v_0^i\right\|_{L^\infty(\mathbb{R})} \\ &+ \int_0^t \left\|\partial_x G_\eta(t-s)\right\|_{L^q(\mathbb{R})}\left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_1)}\left\|\partial_x v_\eta^i(s,\cdot)\right\|_{L^p(\mathbb{R})} ds,\end{aligned}$$

where q is the conjugate of p (i.e., $\frac{1}{p}+\frac{1}{q}=1$). We employ Lemma 3.1 (i) for $p=1$ in the second line, and Lemma 3.1 (ii) in the third line. Thus we get

$$\begin{aligned}\left\|\partial_x v_\eta^i(t,\cdot)\right\|_{L^\infty(\mathbb{R})} &\le \left\|\partial_x v_0^i\right\|_{L^\infty(\mathbb{R})} + \zeta_q\Lambda_1\left\|\partial_x v_\eta^i\right\|_{L^\infty((0,T);L^p(\mathbb{R}))}\int_0^t (t-s)^{\frac{1}{2q}-1}ds \\ &\le \left\|\partial_x v_0^i\right\|_{L^\infty(\mathbb{R})} + 2q\zeta_q\Lambda_1\left\|\partial_x v_\eta^i\right\|_{L^\infty((0,T);L^p(\mathbb{R}))}T^{\frac{1}{2q}},\end{aligned}$$

which leads to the following estimate

$$\left\|\partial_x v_\eta^i\right\|_{(L^\infty(0,T)\times\mathbb{R})} \le \left\|\partial_x v_0\right\|_{(L^\infty(\mathbb{R}))^d} + 2q\zeta_q\Lambda_1 M_p T^{\frac{1}{2q}} := N_p. \tag{3.28}$$

Thus, estimates (3.27) and (3.28) prove estimate (3.21), in particular for $T=T^*$.

Step 5. (Existence of a bounded smooth solution):
Using L^p-regularity for parabolic equations, we can show by classical Bootstrap argument that the solution v_η of (3.17) belongs to $(C^\infty((0,T^*)\times\mathbb{R}))^d \cap (W^{1,\infty}((0,T^*)\times\mathbb{R}))^d \cap X_{T^*}$. Also, by applying the Maximum principle Theorem for parabolic equations (see Lieberman [68, Th. 2.10]) on system (3.17), we can obtain estimate (3.20). We remark that Fixed Point theorem gives us existence and uniqueness in X_{T^*}. However, by an independent argument valid for regular and Lipschitz solutions, which is based on applying a Maximum Principle to the equation satisfied by the difference of two distinct solutions, we can have the uniqueness in the space $(W^{1,\infty}((0,T^*)\times\mathbb{R}))^d$.

□

3 *A priori* uniform estimates on the smooth solution

In this section, we show two η-uniform estimates on the local solution of equation (3.17) obtained in Theorem 3.3.

The first one concerns the BV estimate of the equation and is a key result.

Lemma 3.2 (*BV* ***estimate***).
Assume that (3.18) and (3.19) hold. Let $v_\eta = (v_\eta^i)_{i=1,\dots,d}$ *be a solution of (3.17) in* $(C^\infty((0,T)\times\mathbb{R}))^d \cap (W^{1,\infty}((0,T)\times\mathbb{R}))^d \cap F_T$*, that satisfies, for all* $i = 1,\dots,d$*, estimates (3.20) and (3.21) for some* $T > 0$*. Then, for all* $0 < \eta \leq 1$*, the solutions* v_η^i *satisfy the following estimate*

$$\int_{\mathbb{R}} |\partial_x v_\eta^i(t,x)|dx \leq \int_{\mathbb{R}} |\partial_x v_0^i(x)|dx, \quad \textit{for all } i = 1,\dots,d, \textit{ and } 0 < t \leq T. \tag{3.29}$$

Proof of Lemma 3.2.
First, we introduce the smooth function

$$\beta_\delta(x) = \sqrt{x^2+\delta^2} \quad \text{for all} \quad 0 < \delta \leq 1.$$

Differentiating the first equation in (3.17) with respect to x and then multiplying by $\beta'_\delta(\partial_x v_\eta^i)$, we get

$$\partial_t\left(\beta_\delta(\partial_x v_\eta^i)\right) = \eta(\partial^3_{xxx} v_\eta^i)\beta'_\delta(\partial_x v_\eta^i) + \frac{d}{dx}\left(\widetilde{\lambda}^i(t,x,v_\eta)\partial_x v_\eta^i\right)\beta'_\delta(\partial_x v_\eta^i). \tag{3.30}$$

Let $\phi \in C^\infty(\mathbb{R})$ be a cut-off function taking values in $[0,1]$, supported by the interval $[-2,2]$ and $\phi(x) \equiv 1$ on $[-1,1]$. Multiplying (3.30) by $\phi_R(\cdot) = \phi(\frac{\cdot}{R})$, for $R > 0$, and integrating over the spatial variable, we get

$$\begin{aligned}\partial_t\left[\int_{-2R}^{2R} \beta_\delta(\partial_x v_\eta^i(t,x))\phi_R(x)dx\right] &= \eta \underbrace{\int_{-2R}^{2R} \partial_x(\partial^2_{xx} v_\eta^i(t,x))\beta'_\delta(\partial_x v_\eta^i(t,x))\phi_R(x)dx}_{I_1} \\ &+ \underbrace{\int_{-2R}^{2R} \frac{d}{dx}\left(\widetilde{\lambda}^i(t,x,v_\eta(t,x))\partial_x v_\eta^i(t,x)\right)\beta'_\delta(\partial_x v_\eta^i(t,x))\phi_R(x)dx}_{I_2}.\end{aligned} \tag{3.31}$$

We note that these computations are justified since $v_\eta^i \in C^\infty((0,T)\times\mathbb{R})$ for all $i = 1,\dots,d$. In what follows, we will show that the terms I_1 and I_2 are bounded uniformly in η and δ.

Step 1. (Estimate of I_1):

From integration by parts, we have

$$I_1 = -\eta \left[\int_{-2R}^{2R} \left(\partial^2_{xx} v^i_\eta(t,x)\right)^2 \beta''_\delta(\partial_x v^i_\eta(t,x))\phi_R(x)dx \right.$$
$$\left. + \frac{1}{R}\int_{-2R}^{2R} \partial^2_{xx} v^i_\eta(t,x)\beta'_\delta(\partial_x v^i_\eta(t,x))\phi'\left(\frac{x}{R}\right)dx\right].$$

Thanks to the convexity of β_δ and the positivity of ϕ_R, we know that

$$\left(\partial^2_{xx} v^i_\eta(t,x)\right)^2 \beta''_\delta(\partial_x v^i_\eta(t,x))\phi_R(x) \geq 0.$$

This implies that I_1 satisfies the following inequality

$$I_1 \leq -\frac{\eta}{R}\int_{-2R}^{2R} \partial^2_{xx} v^i_\eta(t,x)\beta'_\delta(\partial_x v^i_\eta(t,x))\phi'\left(\frac{x}{R}\right)dx$$
$$= -\frac{\eta}{R}\int_{-2R}^{2R} \partial_x\left(\beta_\delta(\partial_x v^i_\eta(t,x))\right)\phi'\left(\frac{x}{R}\right)dx.$$

Integrating again by parts the right hand side of the above inequality, we get

$$I_1 \leq \frac{\eta}{R^2}\int_{-2R}^{2R} \beta_\delta(\partial_x v^i_\eta(t,x))\phi''\left(\frac{x}{R}\right)dx.$$

Using the fact that $\beta_\delta(x) \leq \delta + |x|$ and estimate (3.21) for $p = +\infty$, we obtain

$$I_1 \leq \frac{\eta}{R}\left(4\delta + \frac{\xi_\infty}{R}\right)\|\phi''\|_{L^\infty(\mathbb{R})}. \tag{3.32}$$

Step 2. (Estimate of I_2):
We start by splitting the integral as follows

$$I_2 = \int_{-2R}^{2R} \frac{d}{dx}\left(\widetilde{\lambda}^i(t,x,v_\eta(t,x))\right)\partial_x v^i_\eta(t,x)\beta'_\delta(\partial_x v^i_\eta(t,x))\phi\left(\frac{x}{R}\right)dx$$
$$+ \int_{-2R}^{2R} \widetilde{\lambda}^i(t,x,v_\eta(t,x))\partial^2_{xx} v^i_\eta(t,x)\beta'_\delta(\partial_x v^i_\eta(t,x))\phi\left(\frac{x}{R}\right)dx.$$

Using the property $\beta'_\delta(x) = \dfrac{x}{\beta_\delta(x)}$ in the first integral of I_2, we get

$$
\begin{aligned}
I_2 &= \int_{-2R}^{2R} \frac{\left(\partial_x v_\eta^i(t,x)\right)^2}{\beta_\delta(\partial_x v_\eta^i(t,x))} \left(\frac{d}{dx}\left(\widetilde{\lambda}^i(t,x,v_\eta)\right)\right)\phi\left(\frac{x}{R}\right)dx \\
&+ \int_{-2R}^{2R} \widetilde{\lambda}^i(t,x,v_\eta)\partial_x\left(\beta_\delta(\partial_x v_\eta^i(t,x))\right)\phi\left(\frac{x}{R}\right)dx \\
&= \int_{-2R}^{2R} \frac{\delta^2 + \left(\partial_x v_\eta^i(t,x)\right)^2}{\beta_\delta(\partial_x v_\eta^i(t,x))} \left(\frac{d}{dx}\left(\widetilde{\lambda}^i(t,x,v_\eta)\right)\right)\phi\left(\frac{x}{R}\right)dx \\
&+ \int_{-2R}^{2R} \widetilde{\lambda}^i(t,x,v_\eta)\partial_x\left(\beta_\delta(\partial_x v_\eta^i(t,x))\right)\phi\left(\frac{x}{R}\right)dx \\
&- \int_{-2R}^{2R} \frac{\delta^2}{\beta_\delta(\partial_x v_\eta^i(t,x))} \left(\frac{d}{dx}\left(\widetilde{\lambda}^i(t,x,v_\eta)\right)\right)\phi\left(\frac{x}{R}\right)dx \\
&= \int_{-2R}^{2R} \left(\beta_\delta(\partial_x v_\eta^i(t,x))\left(\frac{d}{dx}\left(\widetilde{\lambda}^i(t,x,v_\eta)\right)\right)\right)\phi\left(\frac{x}{R}\right)dx \\
&+ \int_{-2R}^{2R} \widetilde{\lambda}^i(t,x,v_\eta)\partial_x\left(\beta_\delta(\partial_x v_\eta^i(t,x))\right)\phi\left(\frac{x}{R}\right)dx \\
&- \int_{-2R}^{2R} \frac{\delta^2}{\beta_\delta(\partial_x v_\eta^i(t,x))} \left(\frac{d}{dx}\left(\widetilde{\lambda}^i(t,x,v_\eta)\right)\right)\phi\left(\frac{x}{R}\right)dx \\
&= \underbrace{\int_{-2R}^{2R} \partial_x\left(\beta_\delta(\partial_x v_\eta^i(t,x))\widetilde{\lambda}^i(t,x,v_\eta)\right)\phi\left(\frac{x}{R}\right)dx}_{J_1} \\
&\underbrace{- \int_{-2R}^{2R} \frac{\delta^2}{\beta_\delta(\partial_x v_\eta^i(t,x))} \left(\frac{d}{dx}\left(\widetilde{\lambda}^i(t,x,v_\eta)\right)\right)\phi\left(\frac{x}{R}\right)dx}_{J_2}.
\end{aligned}
$$

Integrating by parts in J_1, and using again the property $\beta_\delta(x) \le \delta + |x|$, we get

$$
\begin{aligned}
|J_1| &\le \left|\frac{1}{R}\int_{-2R}^{2R}\left(\delta + \left|\partial_x v_\eta^i(t,x)\right|\right)\widetilde{\lambda}^i(t,x,v_\eta)\phi'\left(\frac{x}{R}\right)dx\right| \\
&\le \frac{\delta}{R}\int_{-2R}^{2R}\left|\widetilde{\lambda}^i(t,x,v_\eta)\phi'\left(\frac{x}{R}\right)\right|dx + \frac{1}{R}\int_{-2R}^{2R}\left|\partial_x v_\eta^i(t,x)\widetilde{\lambda}^i(t,x,v_\eta)\phi'\left(\frac{x}{R}\right)\right|dx,
\end{aligned}
$$

which implies, using (3.20) and (3.21) for $p = 1$, that

$$\begin{aligned} J_1 &\leq \left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)} \left(4\delta + \frac{1}{R}\|\partial_x v^i\|_{L^\infty((0,T);L^1(\mathbb{R}))}\right) \|\phi'\|_{L^\infty(\mathbb{R})} \\ &\leq \left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)} \left(4\delta + \frac{\xi_1}{R}\right) \|\phi'\|_{L^\infty(\mathbb{R})}, \end{aligned} \tag{3.33}$$

where $\widetilde{\mathcal{K}}_0 = \prod_{i=1}^{d} \Big[- \|v_0^i\|, \|v_0^i\| \Big]$.

Using the property $\beta_\delta(x) \geq \delta$ in J_2 and again Hölder's inequality, we get

$$|J_2| \leq \delta R \left\|\frac{d}{dx}\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)} \Big\|\phi\Big\|_{L^1(\mathbb{R})}.$$

We have

$$\frac{d}{dx}\widetilde{\lambda}^i(t,x,v_\eta(t,x)) = \partial_x\widetilde{\lambda}^i(t,x,v_\eta(t,x)) + \sum_{j=1}^{d} \partial_x v_\eta^i(t,x)\frac{\partial\widetilde{\lambda}^i}{\partial v^j}(t,x,v_\eta(t,x)).$$

Thus we can show that

$$\begin{aligned} \left\|\frac{d}{dx}\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)} &\leq \left\|\partial_x\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)} + \max_{1\leq j\leq d}\left\|\frac{\partial\widetilde{\lambda}^i}{\partial v^j}\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)} \Big\|\partial_x v_\eta\Big\|_{(L^\infty((0,T)\times\mathbb{R}))^d} \\ &\leq \|\widetilde{\lambda}^i\|_{W^{1,\infty}((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)}\big(1 + d\xi_\infty\big), \end{aligned} \tag{3.34}$$

where we have used inequality (3.21) for $p = +\infty$ in the second line. Using estimate (3.34) in J_2 we obtain

$$J_2 \leq \delta R\big(1 + d\xi_\infty\big)\|\phi\|_{L^1(\mathbb{R})}\|\widetilde{\lambda}^i\|_{W^{1,\infty}((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)}. \tag{3.35}$$

Combining (3.33) and (3.35), we get

$$I_2 \leq \widetilde{\Lambda}_1 \left(4\delta + \frac{\xi_1}{R}\right) \|\phi'\|_{L^\infty(\mathbb{R})} + \delta R\big(1 + d\xi_\infty\big)\|\phi\|_{L^1(\mathbb{R})}\|\widetilde{\lambda}^i\|_{W^{1,\infty}((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)}, \tag{3.36}$$

where $\widetilde{\Lambda}_1 = \max_{i\in\{1,\ldots,d\}} \left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)}$.

Step 3. (Passage to the limit):
Integrating (3.31) in time on $(0,t)$, for $0 < t < T$, we get

$$\int_{-2R}^{2R} \beta_\delta(\partial_x v_\eta^i(t,x))\phi_R(x)dx \leq \int_{-2R}^{2R} \beta_\delta(\partial_x v_0^i(x))\phi_R(x)dx + T(I_1 + I_2).$$

Passing to the limit in the previous estimate, as δ goes to 0, we obtain

$$\int_{-2R}^{2R} |\partial_x v_\eta^i(t,x)|\phi_R(x)dx \leq \int_{-2R}^{2R} |\partial_x v_0^i(x)|\phi_R(x)dx + \frac{\eta T \xi_\infty}{R^2}\|\phi''\|_{L^\infty(\mathbb{R})} + \frac{\widetilde{\Lambda}_1 T \xi_1}{R}\|\phi'\|_{L^\infty(\mathbb{R})},$$

where we have used estimates (3.32) and (3.36). According to the monotone convergence theorem, we get, by passing to the limit as $R \to +\infty$ in the previous inequality

$$\int_{\mathbb{R}} |\partial_x v_\eta^i(t,x)|dx \leq \int_{\mathbb{R}} |\partial_x v_0^i(x)|dx,$$

which is the desired result. □

The following estimate will provide the compactness in time of the solution $v_\eta = (v_\eta^i)_{i=1,\dots,d}$ given in Theorem 3.3, uniformly with respect to η.

Lemma 3.3 (***Estimate on the time derivative of the solution***).
Assume that (3.18) and (3.19) hold. Let $W^{-1,1}(\mathbb{R})$ be the dual space of $W^{1,\infty}(\mathbb{R})$, and $v_\eta = (v_\eta^i)_{i=1,\dots,d}$ be a solution of (3.17) in $(C^\infty((0,T)\times\mathbb{R}))^d \cap (W^{1,\infty}((0,T)\times\mathbb{R}))^d \cap F_T$, that satisfies, for all $i = 1,\dots,d$, estimates (3.20) and (3.21) for some $T > 0$. Then, for all $0 < \eta \leq 1$ and $i = 1,\dots,d$, the solutions v_η^i of (3.17) satisfy the following estimate

$$\|\partial_t v_\eta^i\|_{L^\infty((0,T);W^{-1,1}(\mathbb{R}))} \leq \left(1 + \|\widetilde{\lambda}^i\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)}\right)\|\partial_x v_0^i\|_{L^1(\mathbb{R})}, \tag{3.37}$$

where $\widetilde{\mathcal{K}}_0 = \prod_{i=1}^{d}\Big[-\|v_0^i\|, \|v_0^i\|\Big]$.

Proof of Lemma 3.3.
The idea is somehow to bound $\partial_t v_\eta^i$ using the available bounds on the right hand side of equation (3.17). The proof is given by duality. Multiplying equation (3.17) by $\phi \in L^1((0,T);W^{1,\infty}(\mathbb{R}))$ and integrating on $(0,T)\times\mathbb{R}$, we get

$$\int_{(0,T)\times\mathbb{R}} \phi\partial_t v_\eta^i = \underbrace{\eta\int_{(0,T)\times\mathbb{R}} \phi\partial_{xx}^2 v_\eta^i}_{I_1} + \underbrace{\int_{(0,T)\times\mathbb{R}} \phi\widetilde{\lambda}^i(t,x,v_\eta)\partial_x v_\eta^i}_{I_2}.$$

Integrating by parts in I_1, for all $0 < \eta \leq 1$, we obtain

$$\begin{aligned} |I_1| \leq \left|\int_{(0,T)\times\mathbb{R}} \partial_x\phi\cdot\partial_x v_\eta^i\right| &\leq \|\partial_x\phi\|_{L^1((0,T),L^\infty(\mathbb{R}))}\|\partial_x v_\eta^i\|_{L^\infty((0,T);L^1(\mathbb{R}))} \\ &\leq \|\phi\|_{L^1((0,T);W^{1,\infty}(\mathbb{R}))}\|\partial_x v_0^i\|_{L^1(\mathbb{R})}, \end{aligned} \tag{3.38}$$

where we have used the BV estimate (3.29) in the second line.

Similarly for the term I_2, from (3.20) and (3.29), we have

$$\begin{aligned} |I_2| &\leq \left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)} \|\phi\|_{L^1((0,T);L^\infty(\mathbb{R}))} \left\|\partial_x v^i_\eta\right\|_{L^\infty((0,T);L^1(\mathbb{R}))} \\ &\leq \left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)} \|\phi\|_{L^1((0,T);W^{1,\infty}(\mathbb{R}))} \left\|\partial_x v^i_0\right\|_{L^1(\mathbb{R})}. \end{aligned} \tag{3.39}$$

Collecting (3.38) and (3.39), we get

$$\int_{(0,T)\times\mathbb{R}} \phi\partial_t v^i_\eta \leq \left(1 + \left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)}\right) \left\|\partial_x v^i_0\right\|_{L^1(\mathbb{R})} \|\phi\|_{L^1((0,T);W^{1,\infty}(\mathbb{R}))}$$

Then, by duality, we deduce that

$$\left\|\partial_t v^i_\eta\right\|_{L^\infty((0,T);W^{-1,1}(\mathbb{R}))} \leq \left(1 + \left\|\widetilde{\lambda}^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)}\right) \left\|\partial_x v^i_0\right\|_{L^1(\mathbb{R})}.$$

□

4 Global existence of a solution to (3.7)

In this section, we give the proof of Theorem 3.1 (i). First of all, we are going to prove that the local in time solution obtained in Section 2 can be extended to a global one.

Theorem 3.4 (***Global existence of Lipschitz continuous solution of (3.17)***). *Assume that (3.18) and (3.19) hold. Then, for all $0 < \eta \leq 1$ and all $T > 0$, system (3.17) admits a unique solution $v_\eta = (v^i_\eta)_{i=1,\ldots,d}$ belonging to the space $(C^\infty((0,T) \times \mathbb{R}))^d \cap (W^{1,\infty}((0,T) \times \mathbb{R}))^d$ and satisfying, for all $i = 1, \ldots, d$, estimates (3.20), (3.29), and (3.37), for all $T > 0$.*

Proof of Theorem 3.4.
From Theorem 3.3, we have the local existence of a solution $v_\eta = (v^i_\eta)_{i=1,\ldots,d}$ to system (3.17). It remains to show that this local solution can be indeed extended into a global one.
We argue by contradiction. Assume there exists a maximum time T_{max} such that, we have existence of solutions to system (3.17) in the function space $F_{T_{max}}$. Similarly, as in Steps 4 and 5 of the proof of Theorem 3.3, we can obtain by a Bootstrap argument that $v^i_\eta \in W^{1,\infty}((0,T_{max}) \times \mathbb{R}) \cap C^\infty((0,T_{max}) \times \mathbb{R})$ for every $i = 1, \ldots, d$. Consequently, v^i_η verifies (3.20) for $T^* = T_{max}$, and (3.21) for a certain constant $\xi_p = \xi_p(\|v^i_0\|_{L^\infty(\mathbb{R})}, \|\partial_x v^i_0\|_{L^p(\mathbb{R})}, T_{\max})$. Thus, estimates (3.29) and (3.37) hold for $T = T_{max}$.
Now, for every $\mu > 0$, we consider system (3.17) with the initial conditions

$$v^i_{0,\mu}(x) = v^i_\eta(T_{max} - \mu, x) \ \text{ for all } i = 1, \ldots, d.$$

We apply, for the second time, the same techniques of the proof of Theorem 3.3 with $v^i_{0,\mu}$ to deduce that there exists a time T^*_μ such that system (3.17) admits a unique solution defined until the time

$$T_0 = (T_{max} - \mu) + T^*_\mu,$$

with

$$T^*_\mu = \min\left(\frac{\eta\pi}{\left(3\Lambda_\mu(\|v_{0,\mu}\|_F + 1)\right)^2}, \frac{\eta\pi}{\left(3\big(\Lambda_\mu + \Lambda_2(\|v_{0,\mu}\|_F + 1)\big)\right)^2} \right),$$

where $v_{0,\mu} = (v^i_{0,\mu})_{i=1,\dots,d}$ and Λ_μ is defined as Λ_1 with replacing v_0 by $v_{0,\mu}$ in (3.24). According to (3.20) and (3.29) we know that $v_{0,\mu}$ is μ-uniformly bounded in F and therefore there exists a constant

$$C\big(\eta, \Lambda_1, \Lambda_2, T_{max}, \|v_0\|_{(L^\infty(\mathbb{R}))^d}, \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d}\big) > 0,$$

independent of μ such that $T^*_\mu \geq C > 0$. Passing to the limit $\mu \to 0$, we can see that $\lim\limits_{\mu\to 0} T^*_\mu \geq C > 0$. This implies that $T_0 > T_{max}$ (for small μ). Thus we have a contradiction, and then we can construct a solution $v_\eta \in W^{1,\infty}((0,T)\times\mathbb{R}) \cap C^\infty((0,T)\times\mathbb{R})$ for all $T > 0$. □

4.1 Proof of Theorem 3.1 (i)

In this subsection, we prove the global existence and uniqueness result of a smooth solution to system (3.7), that was announced in Theorem 3.1 (i).

By assumptions (3.2) and (3.3) and by classical properties of the mollifiers $\rho^1_\varepsilon, \rho^{d+2}_\varepsilon$, we can see that, for all $\varepsilon, \eta > 0$, the functions $u^i_{0,\varepsilon}$ and λ^i_ε (defined in (3.5)) satisfy assumptions (3.18) and (3.19) for every $i = 1, \dots, d$. Thus, we can apply Theorem 3.4, with $\widetilde{\lambda}^i = \lambda^i_\varepsilon$ and $v^i_0 = u^i_{0,\varepsilon}$, in order to prove that system (3.7) admits a unique smooth solution $u_{\varepsilon,\eta} = \big(u^i_{\varepsilon,\eta}\big)_{i=1,\dots,d}$ satisfying, for all $i = 1, \dots, d$, the following estimates

$$\left\|u^i_{\varepsilon,\eta}\right\|_{L^\infty((0,T)\times\mathbb{R})} \leq \left\|u^i_{0,\varepsilon}\right\|_{L^\infty(\mathbb{R})}, \tag{3.40}$$

$$\left\|\partial_x u^i_{\varepsilon,\eta}(t,\cdot)\right\|_{L^1(\mathbb{R})} \leq \left\|\partial_x u^i_{0,\varepsilon}\right\|_{L^1(\mathbb{R})}, \quad \text{for all} \quad t \in [0,T), \tag{3.41}$$

$$\left\|\partial_t u^i_{\varepsilon,\eta}\right\|_{L^\infty((0,T);W^{-1,1}(\mathbb{R}))} \leq \left(1 + \left\|\lambda^i_\varepsilon\right\|_{L^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d)}\right) \left\|\partial_x u^i_{0,\varepsilon}\right\|_{L^1(\mathbb{R})}. \tag{3.42}$$

If we note by $\hat{\lambda}^i$ the extension of λ^i by zero outside $(0,T)\times\mathbb{R}\times\mathcal{K}_0$ where

$$\mathcal{K}_0 = \prod_{i=1}^d \left[-\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}, \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}\right],$$

then by classical properties of the mollifiers, we know that (see Ambrosio *et al.* [6, Theorem 2.2 (b)], for the third estimate)

$$\begin{aligned} \left\|u^i_{0,\varepsilon}\right\|_{L^\infty(\mathbb{R})} &\le \left\|u^i_0\right\|_{L^\infty(\mathbb{R})} \\ \left\|\lambda^i_\varepsilon\right\|_{L^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d)} &\le \left\|\hat{\lambda}^i\right\|_{L^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d)} \le \left\|\lambda^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)} \\ \left\|\partial_x u^i_{0,\varepsilon}\right\|_{L^1(\mathbb{R})} &\le TV(u^i_0) = \left|u^i_0\right|_{BV(\mathbb{R})}, \end{aligned} \tag{3.43}$$

which joint to (3.40), (3.41) and (3.42) imply (3.10), (3.11) and (3.12).

5 Discontinuous viscosity sub- and super- solutions

In this section, we prove the existence of discontinuous viscosity sub- and super- solutions to system (3.1), as it was announced in Theorem 3.1 (*ii*). This section is divided into two subsections. First, in Subsection 5.1, we introduce some useful results for viscosity solutions. Then, in Subsection 5.2, we give the proof of Theorem 3.1 (*ii*).

5.1 Some useful results

We begin by giving the following finite speed propagation property, valid on the continuous viscosity solutions of (3.7).

Lemma 3.4. ***(Finite speed propagation property)***
Under assumptions (3.2) and (3.3), if $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\dots,d}$ is the unique continuous viscosity solution of (3.7), given by Theorem 3.1 (i), then $u^i_{\varepsilon,\eta}$ satisfies, for all $h \ge 0$, the following estimate

$$\begin{aligned} \int_{\mathbb{R}} G_\eta(t,y) \min_{|z-(x-y)|\le\Lambda t} u^i_{\varepsilon,\eta}(h,z)dy &\le u^i_{\varepsilon,\eta}(t+h,x) \\ &\le \int_{\mathbb{R}} G_\eta(t,y) \max_{|z-(x-y)|\le\Lambda t} u^i_{\varepsilon,\eta}(h,z)dy, \end{aligned} \tag{3.44}$$

for all $(t,x) \in [0,T-h)\times\mathbb{R}$, where $G_\eta(t,x) = \dfrac{1}{\sqrt{4\pi\eta t}}e^{-\frac{x^2}{4\eta t}}$ is the standard heat kernel given in Lemma 3.1, and Λ is defined as

$$\Lambda = \max_{i\in\{1,\dots,d\}} \left\|\lambda^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)}. \tag{3.45}$$

Proof of Lemma 3.4.
Let us start by proving the right hand side of (3.44), in viscosity sense, namely

$$u^i_{\varepsilon,\eta}(t+h,x) \le \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\eta t}}e^{-\frac{y^2}{4\eta t}} \max_{|z-(x-y)|\le\Lambda t} u^i_{\varepsilon,\eta}(h,z)dy. \tag{3.46}$$

We proceed in two steps.

Step 1. First, we note that $u^i_{\varepsilon,\eta}$ is also a continuous viscosity solution of (3.7) since it is a smooth solution. Let $u^{i,h}_{\varepsilon,\eta}(t,x) = u^i_{\varepsilon,\eta}(t+h,x)$. Then, we can see that

$$\begin{aligned}\partial_t u^{i,h}_{\varepsilon,\eta}(t,x) = \partial_t u^i_{\varepsilon,\eta}(t+h,x) &= \lambda^i_\varepsilon\big(t+h,x,u_{\varepsilon,\eta}(t+h,x)\big)\partial_x u^i_{\varepsilon,\eta}(t+h,x) + \eta\partial^2_{xx}u^i_{\varepsilon,\eta}(t+h,x)\\ &= \lambda^i_\varepsilon\big(t+h,x,u^h_{\varepsilon,\eta}(t,x)\big)\partial_x u^{i,h}_{\varepsilon,\eta}(t,x) + \eta\partial^2_{xx}u^{i,h}_{\varepsilon,\eta}(t,x)\\ &\leq \Lambda\big|\partial_x u^{i,h}_{\varepsilon,\eta}(t,x)\big| + \eta\partial^2_{xx}u^{i,h}_{\varepsilon,\eta}(t,x),\end{aligned}$$

since we have by (3.43) that $\lambda^i_\varepsilon(t+h,x,u^h_{\varepsilon,\eta}(t,x)) \leq \|\lambda^i\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)} \leq \Lambda$. This implies that $u^{i,h}_{\varepsilon,\eta}$ is a viscosity sub-solution to the following system

$$\begin{cases}\partial_t w(t,x) - \eta\partial^2_{xx}w(t,x) - \Lambda|\partial_x w(t,x)| = 0,\\ w(0,x) = w_0(x) = u^i_{\varepsilon,\eta}(h,x)\end{cases} \tag{3.47}$$

Step 2. We will try to find a viscosity super-solution to (3.47). Consider the function

$$w(t,x) = \int_{\mathbb{R}} G_\eta(t,y)\psi(t,x-y)dy, \tag{3.48}$$

where ψ is the unique continuous viscosity solution of the system

$$\begin{cases}|\partial_t\psi(t,x)| = \Lambda|\partial_x\psi(t,x)|\\ \psi(0,x) = u^i_{\varepsilon,\eta}(h,x),\end{cases} \tag{3.49}$$

which is given by the Lax-Oleinik formula in [9, Lemma 2.1], as

$$\psi(t,x) = \sup_{|y-x|\leq\Lambda t} w_0(y).$$

Using the property $\partial_t G_\eta(t,y) = \eta\partial^2_{yy}G_\eta(t,y)$ for all $t>0$, we get

$$\begin{aligned}\partial_t w(t,x) &= \int_{\mathbb{R}}\partial_t G_\eta(t,y)\psi(t,x-y)dy + \int_{\mathbb{R}} G_\eta(t,y)\partial_t\psi(t,x-y)dy\\ &= \int_{\mathbb{R}}\eta\partial^2_{yy}G_\eta(y,t)\psi(t,x-y)dy + \int_{\mathbb{R}} G_\eta(t,y)\Lambda\big|\partial_x\psi(t,x-y)\big|dy\\ &= \eta\int_{\mathbb{R}} G_\eta(t,y)\partial^2_{xx}\psi(t,x-y)dy + \Lambda\int_{\mathbb{R}}\big|G_\eta(t,y)\partial_x\psi(t,x-y)\big|dy\\ &\geq \eta\partial^2_{xx}w(t,x) + \Lambda\big|\partial_x w(t,x)\big|,\end{aligned}$$

this implies that w is a viscosity super-solution to equation (3.47).
Therefore, using the comparison principle (see [13, Theorem 1.1]), we deduce that

$$u^i_{\varepsilon,\eta}(t+h,x) \leq w(t,x), \quad \text{on} \quad (0,T)\times\mathbb{R},$$

which implies (3.46).

The same proof is done for the inequality

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\eta t}} e^{-\frac{y^2}{4\eta t}} \min_{|z-(x-y)|\leq \Lambda t} u^i_{\varepsilon,\eta}(h,z)dy \leq u^i_{\varepsilon,\eta}(t+h,x),$$

by considering the equation $\partial_t w(t,x) - \eta\partial^2_{xx} w(t,x) + \Lambda|\partial_x w(t,x)| = 0$. In this case, we take $\psi(t,x) = \inf\limits_{|y-x|\leq \Lambda t} w_0(y)$.

□

Now we will give, in the following Lemma, an important relation between the envelopes of the velocities λ^i and their relaxed semi-limits, and another relation between the initial data u^i_0 and its regularization.

Lemma 3.5. ***(Envelopes)***
i) ***(Envelopes of the initial data)***
Assume that u^i_0 is a locally bounded function on $\mathbb{R}$. Let $u^i_{0,\varepsilon}$ be the standard regularization of the function u^i_0 defined in (3.5). Then, we have

$$\max_{|x-x_0|\leq c} u^i_{0,\varepsilon}(x) \leq \max_{|x-x_0|\leq c+\varepsilon} u^i_0(x), \qquad \textit{where} \quad c>0. \tag{3.50}$$

ii) ***(Envelopes of the velocity)***
Assume that λ^i is locally bounded on $(0,T)\times\mathbb{R}\times\mathbb{R}^d$ for all $T>0$. Let λ^i_ε be the standard regularization of the functions λ^i defined in (3.5). Noting

$$\overline{\lambda}^i(t,x,r) = \limsup_{\substack{\varepsilon\longrightarrow 0\\ (s,y,w)\longrightarrow(t,x,r)}} \lambda^i_\varepsilon(s,y,w), \quad \textit{and} \quad \underline{\lambda}^i(t,x,r) = \liminf_{\substack{\varepsilon\longrightarrow 0\\ (s,y,w)\longrightarrow(t,x,r)}} \lambda^i_\varepsilon(s,y,w).$$

Then, we have

$$\overline{\lambda}^i(t,x,r) \leq (\lambda^i)^\star(t,x,r) \quad \textit{and} \quad (\lambda^i)_\star(t,x,r) \leq \underline{\lambda}^i(t,x,r) \quad \textit{for all} \quad (t,x,r)\in[0,T)\times\mathbb{R}\times\mathbb{R}^d,$$

where $(\lambda^i)^\star$ and $(\lambda^i)_\star$ are respectively the upper and lower semi-continuous envelopes of λ^i.

Proof of Lemma 3.5

Proof of (i):

Using classical properties of mollifiers, we have

$$u^i_{0,\varepsilon}(x) = u^i_0 \star \rho^1_\varepsilon(x) = \int_{\mathbb{R}} u^i_0(y)\rho^1_\varepsilon(x-y)dy \leq \max_{|x-y|\leq\varepsilon} u^i_0(y) \int_{\mathbb{R}} \rho^1_\varepsilon(x-y)dy$$
$$\leq \max_{|x-y|\leq\varepsilon} u^i_0(y),$$

this implies that

$$\max_{|x-y|\leq c} u^i_{0,\varepsilon}(y) \leq \max_{|x-y|\leq c+\varepsilon} u^i_0(y).$$

Proof of (ii):

We only show the proof of the first inequality, the second is proved similarly. Indeed, we know that there exists a sequence $(\epsilon_n, t_{\epsilon_n}, x_{\epsilon_n}, r_{\epsilon_n}) \to (0, t, x, r)$, as n goes to $+\infty$, such that

$$\overline{\lambda}^i(t,x,r) = \lim_{n\longrightarrow+\infty} \lambda^i_{\varepsilon_n}(t_{\epsilon_n}, x_{\epsilon_n}, r_{\epsilon_n}).$$

From (3.5), we can see that

$$\lambda^i_{\varepsilon_n}(t_{\epsilon_n}, x_{\epsilon_n}, r_n) = \int_{(0,T)\times\mathbb{R}\times\mathcal{K}_0} \lambda^i(\tau,y,w)\rho^{d+2}_{\epsilon_n}(t_{\epsilon_n}-\tau, x_{\epsilon_n}-y, r_{\epsilon_n}-w)\, dyd\tau dw$$

$$\leq \max_{\substack{|y-x_{\epsilon_n}|\leq\epsilon_n, |\tau-t_{\epsilon_n}|\leq\epsilon_n \\ |w-r_{\epsilon_n}|\leq\epsilon_n}} \lambda^i(\tau,y,w),$$

where we have used the fact that $\rho^{d+2}_{\epsilon_n} \geq 0$ and $\int_{\mathbb{R}^{d+2}} \rho^{d+2}_{\epsilon_n} = 1$. Thanks to the convergence, as $n \to +\infty$, of $(\epsilon_n, t_{\epsilon_n}, x_{\epsilon_n}, r_{\epsilon_n})$ to $(0, t, x, r)$, we can deduce that for every $\alpha > 0$ there exists $n_\alpha > 0$, such that, for all $n \geq n_\alpha$, we have

$$\lambda^i_{\varepsilon_n}(t_{\epsilon_n}, x_{\epsilon_n}, r_n) \leq \max_{\substack{|y-x|\leq 2\alpha, |\tau-t|\leq 2\alpha \\ |w-r|\leq 2\alpha}} \lambda^i(\tau,y,w).$$

Now, we pass to the limit, $n \to +\infty$, firstly and then $\alpha \to 0$, we get $\overline{\lambda}^i(t,x,r) \leq (\lambda^i)^\star(t,x,r)$.

□

5.2 Existence of sub and super solutions of (3.1)

In this subsection, we prove Theorem 3.1 (ii). Before illustrating the proof, we introduce the definitions of continuous and discontinuous viscosity solutions for systems (3.17) and (3.1) respectively. For a complete overview of viscosity solutions, we refer the reader to [9, 35, 37].

5.2.1 Definitions of viscosity solutions

Definition 3.1. ***(Continuous viscosity sub-solution, super-solution, and solution)***
Assume that $\widetilde{\lambda}^i$ is a continuous function on $(0,T)\times\mathbb{R}\times\mathbb{R}^d$, and $v_0 = (v_0^i)_{i=1,\dots,d}$ is a continuous function on $\mathbb{R}$. Let $u = (u^i)_{i=1,\dots,d}$ be a continuous function defined on $(0,T)\times\mathbb{R}$.
(1) (Continuous viscosity sub-solution)
We call u a continuous viscosity sub-solution of (3.17) if it satisfies

(*i*) $u^i(0,x) \le v_0^i(x)$, for every $i = 1,\dots,d$, and $x \in \mathbb{R}$.

(*ii*) If whenever $\phi \in C^2((0,T)\times\mathbb{R})$, $i = 1,\dots,d$, and $u^i - \phi$ attains its local maximum at $(t_0,x_0) \in (0,T)\times\mathbb{R}$, then we have

$$\partial_t\phi(t_0,x_0) - \widetilde{\lambda}^i\big(t_0,x_0,u(t_0,x_0)\big)\partial_x\phi(t_0,x_0) - \eta\partial_{xx}^2\phi(t_0,x_0) \le 0. \tag{3.51}$$

(2) (Continuous viscosity super-solution)
We call u a continuous viscosity super-solution of (3.17) if it satisfies

(*i*) $u^i(0,x) \ge v_0^i(x)$, for every $i = 1,\dots,d$, and $x \in \mathbb{R}$.

(*ii*) If whenever $\phi \in C^2((0,T)\times\mathbb{R})$, $i = 1,\dots,d$, and $u^i - \phi$ attains its local minimum at $(t_0,x_0) \in (0,T)\times\mathbb{R}$, then we have

$$\partial_t\phi(t_0,x_0) - \widetilde{\lambda}^i\big(t_0,x_0,u(t_0,x_0)\big)\partial_x\phi(t_0,x_0) - \eta\partial_{xx}^2\phi(t_0,x_0) \ge 0. \tag{3.52}$$

(3) (Continuous viscosity solution)
A continuous function u is a viscosity solution of (3.17) if and only if it is a viscosity sub- and super-solution of (3.17).

Next, we are going to recall the definition of discontinuous viscosity solutions for system (3.1) introduced by Ishii in [58, Definition 2.1].
For a given function f with values into $\mathbb{R}$, we write $f = f^+ - f^-$ where f^+ and f^- are defined respectively as

$$f^+ = \frac{|f| + f}{2} \quad \text{and} \quad f^- = \frac{|f| - f}{2}.$$

It is clear that $f^+ \ge 0$ and $f^- \ge 0$.
Also, we denote by $f^\star$ and $f_\star$ the respective upper and lower semi-continuous envelopes of a locally bounded function f defined on an open domain in $\mathbb{R}^n$ and given by

$$f^\star(X) = \limsup_{Y\to X} f(Y) \quad \text{and} \quad f_\star(X) = \liminf_{Y\to X} f(Y) \quad \text{for } X \in \mathbb{R}^n. \tag{3.53}$$

For a vector $u = (u^1, \dots, u^d)$ locally bounded on $[0,T)\times\mathbb{R}$ for all $T>0$, we write $u^\star = ((u^1)^\star, \dots, (u^d)^\star)$ and $u_\star = ((u^1)_\star, \dots, (u^d)_\star)$.
Given two locally bounded functions $v = (v^i)_{i=1,\dots,d}$ and $u = (u^i)_{i=1,\dots,d}$ on $[0,T)\times\mathbb{R}$ such that $(v^i)_\star \le (u^i)^\star$ for every $i = 1,\dots,d$, we define the set

$$\mathcal{E}_v^u(t,x) = \prod_{i=1}^d \Big[(v^i)_\star(t,x), (u^i)^\star(t,x)\Big].$$

Definition 3.2. ***(Discontinuous viscosity sub-solution, super-solution and solution)***
Assume that $\lambda = (\lambda^i)_{i=1,\dots,d}$ is locally bounded on $(0,T)\times\mathbb{R}\times\mathbb{R}^d$ and $u_0 = (u_0^i)_{i=1,\dots,d}$ is locally bounded on $\mathbb{R}$. Let $v = (v^i)_{i=1,\dots,d}$, $u = (u^i)_{i=1,\dots,d}$ be two locally bounded functions on $[0,T)\times\mathbb{R}$ such that $(v^i)_\star \le (u^i)^\star$ for every $i = 1,\dots,d$. We say that u and v are a couple of discontinuous viscosity sub- and super- solutions of (3.1) if they satisfy the following two conditions

- (i) • $(u^i)^\star(0,x) \le (u_0^i)^\star(x)$, for all $i = 1,\dots,d$ and $x\in\mathbb{R}$.

  • $(v^i)_\star(0,x) \ge (u_0^i)_\star(x)$, for all $i = 1,\dots,d$ and $x\in\mathbb{R}$.

- (ii) • Whenever a test function $\phi \in C^2((0,T)\times\mathbb{R})$, $i = 1,\dots,d$ and $(u^i)^\star - \phi$ attains a local maximum at $(t_0,x_0)\in(0,T)\times\mathbb{R}$, then we have

$$\begin{aligned}\min\Big\{\partial_t\phi(t_0,x_0) - (\lambda^i)^\star(t_0,x_0,r)(\partial_x\phi)^+(t_0,x_0) + (\lambda^i)_\star(t_0,x_0,r)(\partial_x\phi)^-(t_0,x_0) :\\ r\in\mathcal{E}_v^u(t_0,x_0),\ r^i = (u^i)^\star(t_0,x_0)\Big\} \le 0.\end{aligned} \tag{3.54}$$

  • Whenever $\phi \in C^2((0,T)\times\mathbb{R})$, $i = 1,\dots,d$ and $(v^i)_\star - \phi$ attains a local minimum at $(t_0,x_0)\in(0,T)\times\mathbb{R}$, then we have

$$\begin{aligned}\max\Big\{\partial_t\phi(t_0,x_0) - (\lambda^i)_\star(t_0,x_0,r)(\partial_x\phi)^+(t_0,x_0) + (\lambda^i)^\star(t_0,x_0,r)(\partial_x\phi)^-(t_0,x_0) :\\ r\in\mathcal{E}_v^u(t_0,x_0),\ r^i = (v^i)_\star(t_0,x_0)\Big\} \ge 0.\end{aligned} \tag{3.55}$$

Finally, we call a function $w = (w^i)_{i=1,\dots,d}$ a discontinuous viscosity solution of (3.1) if $w^\star$ and $w_\star$ verify conditions (i) and (ii).

Remark 3.2. We can replace $\phi \in C^2((0,T)\times\mathbb{R})$ by $\phi\in C^1((0,T)\times\mathbb{R})$, in the previous definition.

Noting that the minimum and the maximum in (3.54) and (3.55) are attained, since the sets

$$\left\{r \in \mathbb{R}^d \ : \ r \in \mathcal{E}_v^u(t_0,x_0),\, r^i = (u^i)^\star(t_0,x_0)\right\} \text{ and } \left\{r \in \mathbb{R}^d \ : \ r \in \mathcal{E}_v^u(t_0,x_0),\, r^i = (u^i)_\star(t_0,x_0)\right\}$$

are non-empty and compact and moreover $(\lambda^i)^\star$ and $(\lambda^i)_\star$ are upper and lower semi-continuous, respectively.

5.2.2 Proof of Theorem 3.1 (ii)

We only prove the result for the sub-solution case, the super-solution case can be proved analogously. Let $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\dots,d}$ be the solution of (3.7), constructed in Theorem 3.1 (i). We have to prove that the relaxed semi-limit $(\overline{u}^i)^\star = \overline{u}^i$ is a discontinuous viscosity sub-solution of (3.1), in the sense of Definition 3.2. We do this in two steps.

Step 1. (Meaning of the initial data):
We will prove that $\overline{u} = (\overline{u}^1,\dots,\overline{u}^d) = (\overline{u})^*$, satisfies (1)-(i) in Definition 3.2. It is sufficient to prove the following inequality

$$\overline{u}^i(0,x) \le (u^i_0)^\star(x) \quad \text{for all} \quad x \in \mathbb{R},\ i = 1,\dots,d. \tag{3.56}$$

From the definition of $\overline{u}^i$, we know that there exists a sequence $(\varepsilon_n, \eta_n, t_{\varepsilon_n,\eta_n}, x_{\varepsilon_n,\eta_n}) \to (0,,0,0,x)$ as $n \to +\infty$, such that

$$\overline{u}^i(0,x) = \lim_{n\longrightarrow+\infty} u^i_{\varepsilon_n,\eta_n}(t_{\varepsilon_n,\eta_n}, x_{\varepsilon_n,\eta_n}).$$

For the sake of simplicity, we will use the notation $(\varepsilon_n,\eta_n) = d_n$.
Using Lemma 3.4 with $h = 0$, $t = t_{d_n}$ and $x = x_{d_n}$, we get

$$\begin{aligned}
u^i_{d_n}(t_{d_n},x_{d_n}) &\le \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\eta_n t_{d_n}}} e^{-\frac{y^2}{4\eta_n t_{d_n}}} \max_{|z-(x_{d_n}-y)|\le \Lambda t_{d_n}} u^i_{d_n}(0,z)dy \\
&\le \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \max_{\left|z-\left(x_{d_n}-y\sqrt{4\eta_n t_{d_n}}\right)\right|\le \Lambda t_{d_n}} u^i_{d_n}(0,z)dy \\
&\le \underbrace{\frac{1}{\sqrt{\pi}} \int_{|y|\le\beta} e^{-y^2} \max_{\left|z-\left(x_{d_n}-y\sqrt{4\eta_n t_{d_n}}\right)\right|\le \Lambda t_{d_n}} u^i_{d_n}(0,z)dy}_{K_1} \\
&\quad + \underbrace{\frac{1}{\sqrt{\pi}} \int_{|y|\ge\beta} e^{-y^2} \max_{\left|z-\left(x_{d_n}-y\sqrt{4\eta_n t_{d_n}}\right)\right|\le \Lambda t_{d_n}} u^i_{d_n}(0,z)dy,}_{K_2}
\end{aligned}$$

where $\beta \in \mathbb{R}$. The convergence of $(\varepsilon_n, \eta_n, t_{d_n}, x_{d_n})$ to $(0,0,0,x)$ as $n \longrightarrow +\infty$, implies that for all $\alpha > 0$, there exists $n_\alpha > 0$, such that, for all $n \geq n_\alpha$, we have

$$\varepsilon_n \leq \alpha, \quad \eta_n \leq \alpha, \quad |x_{d_n} - x| \leq \alpha \text{ and } t_{d_n} \leq \alpha.$$

Then by Lemma 3.5 (i), for $\overline{\Lambda} = \alpha(\Lambda + 2\beta + 1)$, we get

$$K_1 \leq \frac{1}{\sqrt{\pi}} \max_{|z-x|\leq\overline{\Lambda}} u^i_{0,\varepsilon_n}(z) \int_{-\beta}^{\beta} e^{-y^2} dy \leq \frac{1}{\sqrt{\pi}} \max_{|z-x|\leq\overline{\Lambda}+\alpha} u^i_0(z) \int_{-\beta}^{\beta} e^{-y^2} dy. \tag{3.57}$$

For K_2, as $y^2 e^{-y^2} \leq 1$, we get by using the classical properties of the mollifiers that

$$\begin{aligned} K_2 \leq \frac{1}{\sqrt{\pi}} \max_{z\in\mathbb{R}} \left|u^i_{0,\varepsilon_n}(z)\right| \int_{|y|\geq\beta} e^{-y^2} dy \leq \frac{1}{\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})} \\ \int_{|y|\geq\beta} \frac{1}{y^2} dy \leq \frac{2}{\beta\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}. \end{aligned} \tag{3.58}$$

Collecting (3.57) and (3.58), we obtain

$$u^i_{d_n}(t_{d_n}, x_{d_n}) \leq \frac{1}{\sqrt{\pi}} \max_{|z-x|\leq\overline{\Lambda}+\alpha} u^i_0(z) \int_{-\beta}^{\beta} e^{-y^2} dy + \frac{2}{\beta\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}.$$

Passing to the limit in the previous inequality as $n \longrightarrow +\infty$ first, we get

$$\overline{u}^i(0,x) \leq \frac{1}{\sqrt{\pi}} \max_{|z-x|\leq\overline{\Lambda}+\alpha} u^i_0(z) \int_{-\beta}^{\beta} e^{-y^2} dy + \frac{2}{\beta\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}.$$

Then, we pass to the limit, as $\alpha \longrightarrow 0$, to get

$$\overline{u}^i(0,x) \leq \frac{1}{\sqrt{\pi}} (u^i_0)^\star(x) \int_{-\beta}^{\beta} e^{-y^2} dy + \frac{2}{\beta\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}.$$

Lastly, passing to the limit as $\beta \longrightarrow +\infty$, we deduce inequality (3.56).

□

Step 2. (Meaning of the equation):
We will show that $\overline{u} = (\overline{u}^1, \dots, \overline{u}^d) = (\overline{u})^*$, satisfies (1)-(ii) in Definition 3.2. Indeed, let $\phi \in C^2((0,T) \times \mathbb{R})$, and suppose that, for $i = 1, \dots, d$, the function $\overline{u}^i - \phi$ attains its local maximum at $(t_0, x_0) \in (0,T) \times \mathbb{R}$. Then, (t_0, x_0) is strict local maximum of $\overline{u}^i - \tilde{\phi}$, where $\tilde{\phi}(t,x) = \phi(t,x) + |t - t_0|^2 + |x - x_0|^2$. By a usual technique used in the theory of viscosity solutions (see Barles [9, Lemma 4.2]), we can say that there exists a subsequence $(\epsilon^i_m, t_{\epsilon^i_m}, x_{\epsilon^i_m}) \to (0, t_0, x_0)$ when $m \to +\infty$, such that $(t_{\epsilon^i_m}, x_{\epsilon^i_m})$ is local maximum of $u^i_{\varepsilon^i_m, \eta^i_m} - \tilde{\phi}$ and

$$\overline{u}^i(t_0, x_0) = \lim_{m\longrightarrow+\infty} u^i_{\varepsilon^i_m, \eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}).$$

Moreover, from Theorem 3.1 (i), we know that $u_{\varepsilon^i_m,\eta^i_m} = (u^j_{\varepsilon^i_m,\eta^i_m})_{j=1,\dots,d}$ is a continuous viscosity solution of system (3.7) in the sense of Definition 3.1, thus

$$\partial_t\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m}) - \lambda^i_{\epsilon^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}, u^1_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}), \dots, u^i_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}), \dots, u^d_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}))\partial_x\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m})$$
$$- \eta^i_m\partial^2_{xx}\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m}) \leq 0.$$

If we write $\partial_x\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m}) = (\partial_x\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m}))^+ - (\partial_x\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m}))^-$, we get

$$\partial_t\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m}) - \lambda^i_{\epsilon^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}, u^1_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}), \dots, u^i_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}), \dots, u^d_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}))(\partial_x\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m}))^+$$
$$+ \lambda^i_{\epsilon^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}, u^1_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}), \dots, u^i_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}), \dots, u^d_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}))(\partial_x\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m}))^-$$
$$- \eta^i_m\partial^2_{xx}\tilde{\phi}(t_{\epsilon^i_m}, x_{\epsilon^i_m}) \leq 0.$$

Since $u^j_{\varepsilon^i_m,\eta^i_m}$ are uniformly bounded for $j = 1, \dots, d$, we can extract a subsequence (independent of j), still noted ϵ^i_m, such that

$$\left|\begin{array}{l} \lim\limits_{m\longrightarrow+\infty} u^j_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}) = r^j \quad \text{for} \quad j \neq i, \\ \\ \lim\limits_{m\longrightarrow+\infty} u^i_{\varepsilon^i_m,\eta^i_m}(t_{\epsilon^i_m}, x_{\epsilon^i_m}) = r^i = \overline{u}^i(t_0, x_0). \end{array}\right.$$

Now, passing to the inferior limit $m \longrightarrow +\infty$ in the previous inequality satisfied by $\tilde{\phi}$, we get

$$\partial_t\phi(t_0, x_0) - \overline{\lambda}^i(t_0, x_0, r^1, \dots, r^i, \dots, r^d)(\partial_x\phi(t_0, x_0))^+$$
$$+ \underline{\lambda}^i(t_0, x_0, r^1, \dots, r^i, \dots, r^d)(\partial_x\phi(t_0, x_0))^- \leq 0 \quad \text{with} \quad r^i = \overline{u}^i(t_0, x_0).$$

Which proves, using Lemma 3.5 (ii), that

$$\min\Big\{\partial_t\phi(t_0, x_0) - (\lambda^i)^\star(t_0, x_0, r)(\partial_x\phi)^+(t_0, x_0) + (\lambda^i)_\star(t_0, x_0, r)(\partial_x\phi)^-(t_0, x_0) :$$
$$r \in \mathcal{E}^{\overline{u}}_{\underline{u}}(t_0, x_0),\ r^i = \overline{u}^i(t_0, x_0)\Big\} \leq 0.$$

and therefore $\overline{u} = (\overline{u}^1, \cdots, \overline{u}^d)$ is viscosity sub-solution of (3.1). Similarly, we can verify that $\underline{u} = (\underline{u}^1, \cdots, \underline{u}^d)$ satisfies (3.55).

□

6 Existence of a viscosity solution for non-decreasing initial data

In this section, we will prove Theorem 3.2. First, we show some preliminary results in Subsection 6.1. Then in Subsection 6.2, we demonstrate the proof of Theorem 3.2.

6.1 Preliminary results

First we recall some properties valid on bounded $BV(\mathbb{R})$-functions.

Lemma 3.6. ***(Properties of BV-functions, [6])***
Let f be a bounded $BV(\mathbb{R})$-function. Then, the following hold
i) f is continuous except at most on a countable set,
ii) The right and left limits

$$f(x^+) = \lim_{\substack{y \to x \\ y > x}} f(y), \qquad f(x^-) = \lim_{\substack{y \to x \\ y < x}} f(y)$$

exists at every point $x \in \mathbb{R}$. Moreover, there exists a unique right-continuous function f_r (resp. left-continuous function f_l) coinciding with f except on a countable set.
iii) There exists a pair of non-decreasing functions f^1, $f^2 \in L^\infty(\mathbb{R})$ such that $f = f^1 - f^2$.

The following lemma shows a local estimate valid on sequences of non-decreasing functions converging locally and strongly in $L^1(\mathbb{R})$.

Lemma 3.7. ***(Sequences of non-decreasing functions)***
i) ***Sequence of non-decreasing functions strongly convergent in $L^1_{loc}(\mathbb{R})$***
Let $(\phi_{\epsilon,\eta})_{(\epsilon,\eta)}$ be a sequence of non-decreasing functions defined on $\mathbb{R}$ such that $\phi_{\epsilon,\eta} \to \phi$ strongly in $L^1_{loc}(\mathbb{R})$, as $(\epsilon,\eta) \to (0,0)$, with ϕ a non-decreasing function also defined on $\mathbb{R}$. Then, for all $a > 0$ and $0 < \delta \le \frac{a}{2}$, there exists $\epsilon^\delta_a, \eta^\delta_a > 0$, such that, for every $0 < \epsilon \le \epsilon^\delta_a$ and $0 < \eta \le \eta^\delta_a$, the following estimate holds

$$-\delta + \phi(x-\delta) \le \phi_{\epsilon,\eta}(x) \le \delta + \phi(x+\delta), \quad \forall x \in [-a,a]. \tag{3.59}$$

ii) ***Sequence of non-decreasing functions strongly convergent in $C([0,T];L^1_{loc}(\mathbb{R}))$***
Let $(\phi_{\epsilon,\eta})_{(\epsilon,\eta)}$ be a sequence of functions defined on $[0,T) \times \mathbb{R}$ such that, for all $t \in [0,T)$, the function $\phi_{\epsilon,\eta}(t,\cdot)$ is non-decreasing on $\mathbb{R}$. Assume, moreover, that $\phi_{\epsilon,\eta} \to \phi$ strongly in $C([0,T);L^1_{loc}(\mathbb{R}))$, as $(\epsilon,\eta) \to (0,0)$, with, for all $t \in [0,T)$, the function $\phi(t,\cdot)$ is defined and non-decreasing on $\mathbb{R}$. Then, for all $a > 0$ and $0 < \delta \le \frac{a}{2}$, there exists $\epsilon^\delta_{a,T}, \eta^\delta_{a,T} > 0$, such that, for every $0 < \epsilon \le \epsilon^\delta_{a,T}$ and $0 < \eta \le \eta^\delta_{a,T}$, the following estimate holds

$$-\delta + \phi(t,x-\delta) \le \phi_{\epsilon,\eta}(t,x) \le \delta + \phi(t,x+\delta), \quad \forall x \in [-a,a],\ \forall t \in [0,T).$$

Lemma 3.8. ***(Sequence of $BV(\mathbb{R})$ functions)***
Let $(\phi_{\epsilon,\eta})_{(\epsilon,\eta)}$ be a sequence of functions, defined on $\mathbb{R}$, uniformly bounded in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and strongly convergent to $\phi \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ in $L^1_{loc}(\mathbb{R})$, with ϕ a right-continuous function. Then, there exists a subsequence $(\phi_{\epsilon',\eta'})_{(\epsilon',\eta')}$ such that, for all $a > 0$

and for all $0 < \delta \leq \frac{a}{2}$, there exists ϵ_a^δ, $\eta_a^\delta > 0$ such that, for all $0 < \epsilon \leq \epsilon_a^\delta$ and $0 < \eta \leq \eta_a^\delta$, the following estimate holds

$$-2\delta + \phi^1(x-\delta) - \phi^2(x-\delta) \leq \phi_{\epsilon',\eta'} \leq 2\delta + \phi^1(x+\delta) - \phi^2(x-\delta), \quad \forall x \in [-a,a] \quad (3.60)$$

where ϕ^1 and ϕ^2 are two bounded, right-continuous and non-decreasing functions on $\mathbb{R}$ satisfying $\phi = \phi^1 - \phi^2$.

For the proof of these two previous lemmas see [19, section 6.1].
We end this subsection with the following compactness lemma.

Lemma 3.9. ***(Simon's Lemma [80, Corollary 4])***
Let X, B and Y be three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(\theta_n)_n$ is a sequence uniformly bounded in $L^\infty((0,T);X)$ and $(\partial_t\theta_n)_n$ is uniformly bounded in $L^r((0,T);Y)$ where $r > 1$, then, $(\theta_n)_n$ is relatively compact in $C((0,T);B)$.

6.2 Proof of Theorem 3.2

6.2.1 Passing to the limit as ε and η tend to zero

Let $u_{\varepsilon,\eta} = \left(u^i_{\varepsilon,\eta}\right)_{i=1,\dots,d}$ be the solution of (3.7), constructed in Theorem 3.1 (i). From estimates (3.10), (3.11) and (3.12), we can say that, for all compact $K_0 \subset \mathbb{R}$, $(u^i_{\varepsilon,\eta})_{\varepsilon,\eta}$ is uniformly bounded in $L^\infty((0,T);BV(K_0)) \cap L^\infty((0,T)\times K_0)$ and $(\partial_t u^i_{\varepsilon,\eta})_{\varepsilon,\eta}$ is uniformly bounded in $L^\infty((0,T);W^{-1,1}(K_0))$. Using Simon's lemma in the particular case $X = BV(K_0)$, $B = L^1(K_0)$, $Y = W^{-1,1}(K_0)$ and the following compact embedding $BV(K_0) \hookrightarrow L^1(K_0)$, we can extract a subsequence, denoted by $(u^i_{(\varepsilon_n,\eta_n)_{K_0}})_{(\varepsilon_n,\eta_n)_{K_0}}$, that converges strongly in $L^\infty((0,T);L^1(K_0))$ to some limit u^i, as $n \to +\infty$. By a standard diagonalization procedure, we can extract a subsequence $(u^i_{\varepsilon_n,\eta_n})_{\varepsilon_n,\eta_n}$ (independent of i and K) that converges to the limit u^i strongly in $C([0,T);L^1(K))$ for all compact $K \subset \mathbb{R}$. Now, thanks to estimates (3.10) and (3.11) we can extract a subsequence, still denoted by $(u^i_{\varepsilon_n,\eta_n})_{\varepsilon_n,\eta_n}$, satisfying the following convergences

$$\left|\begin{array}{llll} u^i_{\varepsilon_n,\eta_n} \longrightarrow u^i, & \text{strongly} & \text{in} & C([0,T);L^1(K)), \ \text{for all compact } K \subset \mathbb{R}, \\ u^i_{\varepsilon_n,\eta_n} \longrightarrow u^i, & \text{weakly-}\star & \text{in} & L^\infty((0,T)\times\mathbb{R}), \\ u^i_{\varepsilon_n,\eta_n} \longrightarrow u^i, & \text{weakly-}\star & \text{in} & L^\infty((0,T);BV(\mathbb{R})). \end{array}\right. \quad (3.61)$$

Taking the lim inf in estimates (3.10), (3.11) and using the lower semi-continuity of $\|\cdot\|_{L^\infty(\mathbb{R})}$ and $|\cdot|_{BV(\mathbb{R})}$, we can prove that u^i satisfies (3.13), (3.14) and (3.15). Since, for all $t \in [0,T)$, the function $u^i(t,\cdot) \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, then by property (ii) of Lemma 3.6, we know that this function coincides with a right-continuous function almost everywhere in $\mathbb{R}$ and consequently in $L^1_{loc}(\mathbb{R})$. This allows us to consider, in the following, a right-continuous limit with respect to the space variable.

6.2.2 Existence of a discontinuous viscosity solution

It remains to prove that the limit u is a discontinuous viscosity solution of (3.1). Since we have proved in Theorem 3.1 (ii) that $\overline{u}$ and $\underline{u}$ are respectively discontinuous viscosity sub- and super- solutions, then it is sufficient to show that

$$\overline{u}^i(t,x) = (u^i)^\star(t,x) \quad \text{and} \quad \underline{u}^i(t,x) = u^i_\star(t,x) \quad \text{for all } (t,x) \in [0,T)\times\mathbb{R}, \quad \text{and } i = 1,\dots,d.$$

We will only show the proof of the first equality, the second can be proved in a similar way. We proceed in two steps.

Step 1. We will prove the following inequality for $i = 1,\dots,d$,

$$\overline{u}^i(t,x) \leq (u^i)^\star(t,x). \tag{3.62}$$

Let $a > 0$, $x \in [-\frac{a}{2}, \frac{a}{2}]$ and $t \in [0,T)$. In fact, by the definition of $\overline{u}^i$, we know that there exists a sequence $(\varepsilon_m, \eta_m, t_{\varepsilon_m,\eta_m}, x_{\varepsilon_m,\eta_m}) \to (0,0,t,x)$, when $m \to +\infty$, such that

$$\overline{u}^i(t,x) = \lim_{m \longrightarrow +\infty} u^i_{\varepsilon_m,\eta_m}(t_{\varepsilon_m,\eta_m}, x_{\varepsilon_m,\eta_m}).$$

We will use the notation $(\varepsilon_m, \eta_m) = d_m$.

For all $\alpha > 0$, we can state that, there exists $m_\alpha > 0$, such that, for all $m \geq m_\alpha$, we have

$$\varepsilon_m \leq \alpha, \quad \eta_m \leq \alpha, \quad |x_{d_m} - x| \leq \alpha \quad \text{and} \quad |t_{d_m} - t| \leq \alpha.$$

Using Lemma 3.4, with

$$h_\alpha = \begin{cases} t - \alpha & \text{if} \quad t > 0, \\ \\ 0 & \text{if} \quad t = 0, \end{cases}$$

we get that, for all $m \geq m_\alpha$ and $\alpha > 0$ such that $h_\alpha \geq 0$,

$$\begin{aligned}
u^i_{d_m}(t_{d_m}, x_{d_m}) &\leq \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\eta_m(t_{d_m} - h_\alpha)}} e^{-\frac{y^2}{4\eta_m(t_{d_m}-h_\alpha)}} \max_{|z-(x_{d_m}-y)|\leq \Lambda(t_{d_m}-h_\alpha)} u^i_{d_m}(h_\alpha, z)dy \\
&\leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-y^2} \max_{\left|z-\left(x_{d_m}-y\sqrt{4\eta_m(t_{d_m}-h_\alpha)}\right)\right|\leq 2\alpha\Lambda} u^i_{d_m}(h_\alpha, z)dy \\
&\leq \underbrace{\frac{1}{\sqrt{\pi}} \int_{|y|\leq\beta} e^{-y^2} \max_{\left|z-\left(x_{d_m}-y\sqrt{4\eta_m(t_{d_m}-h_\alpha)}\right)\right|\leq 2\alpha\Lambda} u^i_{d_m}(h_\alpha, z)dy}_{L_1} \\
&\quad \underbrace{+ \frac{1}{\sqrt{\pi}} \int_{|y|\geq\beta} e^{-y^2} \max_{\left|z-\left(x_{d_m}-y\sqrt{4\eta_m(t_{d_m}-h_\alpha)}\right)\right|\leq 2\alpha\Lambda} u^i_{d_m}(h_\alpha, z)dy,}_{L_2}
\end{aligned}$$

where $\beta \in \mathbb{R}$. Then, for $\widetilde{\Lambda} = \alpha(2\Lambda + 2\sqrt{2}\beta + 1)$, we get

$$L_1 \leq \frac{1}{\sqrt{\pi}} \max_{|z-x|\leq\widetilde{\Lambda}} u^i_{d_m}(h_\alpha, z) \int_{-\beta}^{\beta} e^{-y^2} dy. \tag{3.63}$$

For L_2, as $y^2 e^{-y^2} \leq 1$, we get

$$\begin{aligned}
L_2 \leq \frac{1}{\sqrt{\pi}} \max_{z\in\mathbb{R}} \left|u^i_{d_m}(h_\alpha, z)\right| \int_{|y|\geq\beta} e^{-y^2} dy \leq \\
\frac{1}{\sqrt{\pi}} \left\|u^i_{d_m}\right\|_{L^\infty(\mathbb{R})} \int_{|y|\geq\beta} \frac{1}{y^2} dy \leq \frac{2}{\beta\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})},
\end{aligned} \tag{3.64}$$

where we have used estimate (3.10). Collecting (3.63) and (3.64), we obtain

$$u^i_{d_m}(t_{d_m}, x_{d_m}) \leq \frac{1}{\sqrt{\pi}} \max_{|z-x|\leq\widetilde{\Lambda}} u^i_{d_m}(h_\alpha, z) \int_{-\beta}^{\beta} e^{-y^2} dy + \frac{2}{\beta\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}. \tag{3.65}$$

Moreover, from the Maximum Principle of (3.7) and since the initial data is non-decreasing, we know that $u^i_{d_m}$ is non-decreasing (with respect to x) and therefore, for all $m \geq m_\alpha$, we have

$$u^i_{d_m}(t_{d_m}, x_{d_m}) \leq \frac{1}{\sqrt{\pi}} u^i_{d_m}\left(h_\alpha, x + \alpha(2\Lambda + 2\sqrt{2}\beta + 1)\right) \int_{-\beta}^{\beta} e^{-y^2} dy + \frac{2}{\beta\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}.$$

Now, as we have indicated in Subsubection 6.2.1, since $u^i_{d_m}$ satisfies estimates (3.10) and (3.11),we can extract a subsequence, still denoted by $(u^i_{d_m})_{d_m}$, that converges in the sense of (3.61) to a function u^i. Since $u^i_{d_m}$ is non-decreasing (with respect to x), then for all

$t \in [0,T)$ the limit $u^i(t,\cdot)$ can be considered non-decreasing and defined on $\mathbb{R}$. By the previous inequality and Lemma 3.7 (ii), we obtain that, for all $0 < \alpha \leq \alpha_a$, where

$$\alpha_a = \begin{cases} \min\left(f\frac{t}{2}, \dfrac{a}{2(2\Lambda + 2\sqrt{2}\beta + 1)}\right) & \text{if} \quad t > 0, \\ \\ \dfrac{a}{2(2\Lambda + 2\sqrt{2}\beta + 1)} & \text{if} \quad t = 0, \end{cases}$$

there exists $m^\alpha_{a,T} > 0$, such that, for every $m \geq m^\alpha_{a,T}$, we have

$$u^i_{d_m}(t_{d_m}, x_{d_m}) \leq \frac{1}{\sqrt{\pi}} u^i\left(h_\alpha, x + 2\alpha(\Lambda + \sqrt{2}\beta + 1)\right) \int_{-\beta}^{\beta} e^{-y^2} dy + \alpha + \frac{2}{\beta\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}.$$

Passing to the limit $m \to +\infty$, then $\alpha \to 0$, and lastly $\beta \to +\infty$, we obtain (3.62).

Step 2. It remains to show that

$$(u^i)^\star(t,x) \leq \overline{u}^i(t,x). \tag{3.66}$$

Consider $a > 0$, $x \in [-\frac{a}{2}, \frac{a}{2}]$ and $t \in [0,T)$. In fact, from the definition of $(u^i)^\star$ we know that there exists a sequence $(t_{\epsilon_m}, x_{\epsilon_m}) \to (t,x)$, when $m \to +\infty$, such that

$$(u^i)^\star(t,x) = \lim_{m \longrightarrow +\infty} u^i(t_{\epsilon_m}, x_{\epsilon_m}).$$

Similarly, as in Step 1, we can state that, for all $\alpha > 0$, there exists $m_\alpha > 0$, such that, for all $m \geq m_\alpha$, we have

$$|x_{\epsilon_m} - x| \leq \alpha \quad \text{and} \quad |t_{\epsilon_m} - t| \leq \alpha.$$

However, using Lemma 3.7 (ii), we know that, for all $0 < \alpha \leq \frac{a}{2}$, there exists $k^\alpha_{a,T}, \ell^\alpha_{a,T} > 0$ and two subsequences $0 < \varepsilon_{\alpha_k} \leq \alpha$ and $0 < \eta_{\alpha_\ell} \leq \alpha$ such that, for every $k \geq k^\alpha_{a,T}$ and $\ell \geq \ell^\alpha_{a,T}$,

$$u^i(t_{\epsilon_m}, x_{\epsilon_m}) \leq u^i_{\varepsilon_{\alpha_k}, \eta_{\alpha_k}}(t_{\epsilon_m}, x_{\epsilon_m} + \alpha) + \alpha \leq \sup_{\substack{\varepsilon_{\alpha_k} \leq \alpha, |s-t| \leq \alpha \\ \eta_{\alpha_\ell} \leq \alpha, |y-x| \leq 2\alpha}} u^i_{\alpha_k, \eta_{\alpha_k}}(s,y) + \alpha.$$

Passing to the limit $m \to +\infty$ and then $\alpha \to 0$, we obtain (3.66).

□

7 Link between sub and super solutions in the general case

Finally, in this section, we give the proof of Theorem 3.1 (iii). Let $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\cdots,d}$ be the solution of (3.7), constructed in Theorem 3.1 (i). As explained in the beginning of the proof of Theorem 3.2, we can extract a subsequence $(u^i_{\varepsilon_n,\eta_n})_{\varepsilon_n,\eta_n}$ satisfying (3.61) with limit u^i that verifies (3.13), (3.14) and (3.15). Moreover, for all $t \in [0,T)$, $u^i(t,\cdot)$ is a right-continuous function on $\mathbb{R}$. It remains to show equality (3.16). For a clear presentation, we will perform this in three steps.

Step 1. (Regularity in time estimate):
Let $T > 0$, $a > 0$, $\beta \in \mathbb{R}$, and set $\gamma = 2(\Lambda + \sqrt{2}\beta + 1)$. First, we will show that there are two bounded and non-decreasing functions $u^{i,1}$ and $u^{i,2}$ satisfying $u^i = u^{i,1} - u^{i,2}$ for every $i = 1,\ldots,d$, and the following inequalities

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\Big(- 2h + u^{i,1}(t,x-h\gamma) - u^{i,2}(t,x+h\gamma)\Big)\int_{-\beta}^{\beta} e^{-y^2}dy - \frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})} \le \\ \underline{u}^i(t+h,x) \le \overline{u}^i(t+h,x) \le \qquad (3.67) \\ \frac{1}{\sqrt{\pi}}\Big(2h + u^{i,1}(t,x+h\gamma) - u^{i,2}(t,x-h\gamma)\Big)\int_{-\beta}^{\beta} e^{-y^2}dy + \frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})},\end{aligned}$$

for all $x \in [-\frac{a}{2},\frac{a}{2}]$, $t \in [0,T)$ and for all $h > 0$ verifying

$$h \le \frac{a}{2(2\Lambda + 2\sqrt{2}\beta + 1)} \quad \text{and} \quad t + h < T. \tag{3.68}$$

We begin with the proof of the right inequality in (3.67), namely,

$$\overline{u}^i(t+h,x) \le \frac{1}{\sqrt{\pi}}\Big(2h+u^{i,1}(t,x+h\gamma)-u^{i,2}(t,x-h\gamma)\Big)\int_{-\beta}^{\beta} e^{-y^2}dy+\frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}. \tag{3.69}$$

Indeed, consider $h > 0$ satisfying (3.68), by the definition of $\overline{u}^i$, we know that there exists a sequence $(\varepsilon_m, \eta_m, t^h_{\varepsilon_m,\eta_m}, x_{\varepsilon_m,\eta_m}) \to (0,0,t+h,x)$, when $m \to +\infty$, such that

$$\overline{u}^i(t+h,x) = \lim_{m\to+\infty} u^i_{\varepsilon_m,\eta_m}(t^h_{\varepsilon_m,\eta_m}, x_{\varepsilon_m,\eta_m}) = \lim_{m\to+\infty} u^i_{d_m}(t^h_{d_m}, x_{d_m}).$$

Now, the convergence $(\varepsilon_m, \eta_m, t^h_{d_m}, x_{d_m}) \to (0,0,t+h,x)$ when $m \to 0$, implies that there exists $m_h > 0$, such that, for all $m \ge m_h$, we have

$$\varepsilon_m \le h, \quad \eta_m \le h, \quad |t^h_{d_m} - t - h| \le h \quad \text{and} \quad |x_{d_m} - x| \le h.$$

Using Lemma 3.4, we have, for all $m \geq m_h$,

$$u^i_{d_m}(t^h_{d_m}, x_{d_m}) \leq \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\eta_m(t^h_{d_m} - t)}} e^{-\frac{y^2}{4\eta_m(t^h_{d_m}-t)}} \max_{|z-(x_{d_m}-y)|\leq \Lambda(t^h_{d_m}-t)} u^i_{d_m}(t,z)dy.$$

By a change of variable, and repeating the same computation done in step 1 of Subsubsection 6.2.2, we obtain

$$u^i_{d_m}(t^h_{d_m}, x_{d_m}) \leq \frac{1}{\sqrt{\pi}} \max_{|z-x|\leq h(2\Lambda+2\sqrt{2}\beta+1)} u^i_{d_m}(t,z) \int_{-\beta}^{\beta} e^{-y^2}dy + \frac{2}{\beta\sqrt{\pi}} \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}. \quad (3.70)$$

Since, for all $t \in [0,T)$, the sequence $u^i_{d_m}(t,\cdot)$ is uniformly bounded in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and converges strongly in $L^1_{loc}(\mathbb{R})$, we can deduce, from Lemma 3.8, that there exists a subsequence $u^i_{d_n}(t,\cdot)$ and a positive constant $n^h_{a,t}$, such that, for all $n \geq n^h_{a,t}$, we have

$$u^i_{d_n}(t,y) \leq 2h + u^{i,1}(t,y+h) - u^{i,2}(t,y-h), \quad \forall y \in [-a,a], \quad (3.71)$$

where $u^{i,1}$ and $u^{i,2}$ are two bounded, right-continuous and non-decreasing functions (with respect to x) satisfying $u^i = u^{i,1} - u^{i,2}$ for every $i = 1,\dots,d$. Collecting (3.70) and (3.71), we obtain that, for all $h > 0$ satisfying (3.68) and for all $n \geq n^h_{a,t}$,

$$u^i_{d_n}(t^h_{d_n}, x_{d_n}) \leq \frac{1}{\sqrt{\pi}}\Big(2h + u^{i,1}(t,x+h\gamma) - u^{i,2}(t,x-h\gamma)\Big)\int_{-\beta}^{\beta} e^{-y^2}dy + \frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}.$$

We pass to the limit $n \to +\infty$ to get (3.69). Similarly, using the finite speed propagation property and the fact that $u^i_{d_m}(t,\cdot)$ is uniformly bounded in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, we can prove the left inequality in (3.67), namely,

$$\frac{1}{\sqrt{\pi}}\Big(-2h + u^{i,1}(t,x-h\gamma) - u^{i,2}(t,x+h\gamma)\Big)\int_{-\beta}^{\beta} e^{-y^2}dy - \frac{2}{\beta\sqrt{\pi}} \leq \underline{u}^i(t+h,x). \quad (3.72)$$

Step 2. (Right and left continuity):
Let $T > 0$ and $t \in [0,T)$. Since $u^{i,1}(t,\cdot)$, $u^{i,2}(t,\cdot)$ are bounded and non-decreasing functions on $\mathbb{R}$ for every $i = 1,\dots,d$, then, from property (ii) of Lemma 3.6, we know that, the right and left limits of these functions exist at every point $x \in \mathbb{R}$. This implies that, for all $\alpha > 0$ and $x \in [-\frac{a}{2}, \frac{a}{2}]$, there exists $h^\alpha_{a,t} > 0$, such that, for all $0 < z \leq h^\alpha_{a,t}$ and $i = 1,\dots,d$, we have

$$\left| \begin{array}{l} u^{i,1}(t,x+z) \leq \frac{\alpha}{4} + u^{i,1}_r(t,x) \\ \\ u^{i,2}(t,x+z) \leq \frac{\alpha}{4} + u^{i,2}_r(t,x) \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} -u^{i,1}(t,x-z) \leq \frac{\alpha}{4} - u^{i,1}_l(t,x) \\ \\ -u^{i,2}(t,x-z) \leq \frac{\alpha}{4} - u^{i,2}_l(t,x) \end{array} \right. \quad (3.73)$$

where $u_r^{i,1}(t,\cdot)$, $u_r^{i,2}(t,\cdot)$, are right-continuous functions on $\mathbb{R}$ and $u_l^{i,1}(t,\cdot)$, $u_l^{i,2}(t,\cdot)$ are left-continuous functions on $\mathbb{R}$. Note that, in here, the choice of the constant $h^\alpha_{a,t}$ does not depend on x, that is a consequence of the Heine-Cantor Theorem.

Now, let $T>0$, $t\in[0,T)$ and $\alpha>0$, we can see that, if we denote

$$\bar{h}^\alpha_{a,t,T}=\min\left(\frac{h^\alpha_{a,t}}{\gamma},\frac{a}{2(2\Lambda+2\sqrt{2}\beta+1)},\frac{\alpha}{4},\frac{T-t}{2}\right),$$

then, for all $0<h\le\bar{h}^\alpha_{a,t,T}$, assumption (3.68) holds. Therefore, we obtain

$$\begin{aligned}\overline{u}^i(t+h,x)&\le\frac{1}{\sqrt{\pi}}\Big(2h+u^{i,1}(t,x+h\gamma)-u^{i,2}(t,x-h\gamma)\Big)\int_{-\beta}^{\beta}e^{-y^2}dy+\frac{2}{\beta\sqrt{\pi}}\left\|u_0^i\right\|_{L^\infty(\mathbb{R})}\\&\le\frac{1}{\sqrt{\pi}}\Big(\alpha+u_r^{i,1}(t,x)-u_l^{i,2}(t,x)\Big)\int_{-\beta}^{\beta}e^{-y^2}dy+\frac{2}{\beta\sqrt{\pi}}\left\|u_0^i\right\|_{L^\infty(\mathbb{R})},\end{aligned}\tag{3.74}$$

where we have used (3.69) in the first inequality and (3.73) in the second one. Similarly, using (3.72) and (3.73), we can prove that, for all $x\in[-\frac{a}{2},\frac{a}{2}]$, $t\in[0,T)$, $\alpha>0$ and $0<h\le\bar{h}^\alpha_{a,t,T}$, we have

$$-\frac{2}{\beta\sqrt{\pi}}\left\|u_0^i\right\|_{L^\infty(\mathbb{R})}+\frac{1}{\sqrt{\pi}}\Big(u_l^{i,1}(t,x)-u_r^{i,2}(t,x)-\alpha\Big)\int_{-\beta}^{\beta}e^{-y^2}dy\le\underline{u}^i(t+h,x).\tag{3.75}$$

Step 3. (Link between $\overline{u}$ and $\underline{u}$):
Let $T>0$, $x\in[-\frac{a}{2},\frac{a}{2}]$, $t\in[0,T)$ and $\alpha>0$. In Step 2, we proved that, there exists a positive constant $\bar{h}^\alpha_{a,t,T}$, such that, for all $i=1,\dots,d$ and $0<h\le\bar{h}^\alpha_{a,t,T}$, we have

$$\begin{aligned}\overline{u}^i(t+h,x)&\le\frac{1}{\sqrt{\pi}}\Big(u_r^{i,1}(t,x)-u_l^{i,2}(t,x)+\alpha\Big)\int_{-\beta}^{\beta}e^{-y^2}dy+\frac{2}{\beta\sqrt{\pi}}\left\|u_0^i\right\|_{L^\infty(\mathbb{R})}\\-\underline{u}^i(t+h,x)&\le-\frac{1}{\sqrt{\pi}}\Big(u_l^{i,1}(t,x)-u_r^{i,2}(t,x)-\alpha\Big)\int_{-\beta}^{\beta}e^{-y^2}dy+\frac{2}{\beta\sqrt{\pi}}\left\|u_0^i\right\|_{L^\infty(\mathbb{R})}.\end{aligned}\tag{3.76}$$

Since $\bigcup_{t\in[0,T)}[t,t+\bar{h}^\alpha_{a,t,T}]$ is a cover of $[0,\frac{T}{2}]$, then there is a finite number N^α_a of ordered intervals, satisfying

$$\left|\begin{array}{l}\bigcup_{0\le j\le N^\alpha_a}[\tau^\alpha_{a,j},\tau^\alpha_{a,j}+\bar{h}^\alpha_{a,\tau^\alpha_{a,j},T}]\supset\left[0,\frac{T}{2}\right]\quad\text{with}\quad\tau^\alpha_{a,0}=0\\[2ex]\text{and}\ \ \tau^\alpha_{a,j+1}=\tau^\alpha_{a,j}+\bar{h}^\alpha_{a,\tau^\alpha_{a,j},T}\text{for}\ \ j=1,\cdots,N^\alpha_a-1\end{array}\right.$$

This expression joint to (3.76) and the fact that $\mathbb{R}=\bigcup_{a\in\mathbb{Q}}[-\frac{a}{2},\frac{a}{2}]$ shows that, for all $x\in\mathbb{R}$, $\tau\in[0,\frac{T}{2}]$, and for all positive constant $\alpha\in\mathbb{Q}$, there exist two indices $a_0\in\mathbb{Q}$ and

$0 \le k \le N^{\alpha}_{a_0}$, such that,

$$\overline{u}^i(\tau,x) \le \frac{1}{\sqrt{\pi}}\Big(u^{i,1}_r(\tau^{\alpha}_{a_0,k},x) - u^{i,2}_l(\tau^{\alpha}_{a_0,k},x) + \alpha\Big)\int_{-\beta}^{\beta} e^{-y^2}dy + \frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}$$

$$-\underline{u}^i(\tau,x) \le -\frac{1}{\sqrt{\pi}}\Big(u^{i,1}_l(\tau^{\alpha}_{a_0,k},x) - u^{i,2}_r(\tau^{\alpha}_{a_0,k},x) - \alpha\Big)\int_{-\beta}^{\beta} e^{-y^2}dy + \frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}. \tag{3.77}$$

Moreover, from property (ii) of Lemma 3.6, we know that, for all positive constants $\alpha, a \in \mathbb{Q}$ and $0 \le j \le N^{\alpha}_a$, the functions $u^{i,1}_r(\tau^{\alpha}_{a,j},\cdot)$, $u^{i,1}_l(\tau^{\alpha}_{a,j},\cdot)$ (resp. $u^{i,2}_r(\tau^{\alpha}_{a,j},\cdot)$, $u^{i,2}_l(\tau^{\alpha}_{a,j},\cdot)$) coincide with $u^{i,1}(\tau^{\alpha}_{a,j},\cdot)$ (resp. $u^{i,2}(\tau^{\alpha}_{a,j},\cdot)$) except on a countable set on $\mathbb{R}$, denoted $D^{\alpha}_{a,j}$. Now, we define the following countable set

$$D = \bigcup_{a,\alpha\in\mathbb{Q}} \bigcup_{0\le j\le N^{\alpha}_a} D^{\alpha}_{a,j}.$$

Thanks to (3.77), we can see that, for all $x \notin D$, $\tau \in [0, \frac{T}{2}]$ and for all positive constant $\alpha \in \mathbb{Q}$, there exist two indices $a_0 \in \mathbb{Q}$ and $0 \le k \le N^{\alpha}_{a_0}$, such that

$$\begin{aligned}
\overline{u}^i(x,\tau) &\le \frac{1}{\sqrt{\pi}}\Big(u^{i,1}(x,\tau^{\alpha}_{a_0,k}) - u^{i,2}(x,\tau^{\alpha}_{a_0,k}) + \alpha\Big)\int_{-\beta}^{\beta} e^{-y^2}dy + \frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}\\
&\le \frac{1}{\sqrt{\pi}}\Big(u^{i}(x,\tau^{\alpha}_{a_0,k}) + \alpha\Big)\int_{-\beta}^{\beta} e^{-y^2}dy + \frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}\\
-\underline{u}^i(x,\tau) &\le -\frac{1}{\sqrt{\pi}}\Big(u^{i,1}(x,\tau^{\alpha}_{a_0,k}) - u^{i,2}(x,\tau^{\alpha}_{a_0,k}) - \alpha\Big)\int_{-\beta}^{\beta} e^{-y^2}dy + \frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}\\
&\le -\frac{1}{\sqrt{\pi}}\Big(u^{i}(x,\tau^{\alpha}_{a_0,k}) - \alpha\Big)\int_{-\beta}^{\beta} e^{-y^2}dy + \frac{2}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Adding the previous inequalities, we deduce that, for all rational number $\alpha > 0$, $x \notin D$ and $\tau \in [0, \frac{T}{2}]$,

$$0 \le \overline{u}^i(\tau,x) - \underline{u}^i(\tau,x) \le \frac{2\alpha}{\sqrt{\pi}}\int_{-\beta}^{\beta} e^{-y^2}dy + \frac{4}{\beta\sqrt{\pi}}\left\|u^i_0\right\|_{L^\infty(\mathbb{R})}.$$

Passing to the limit $\alpha \to 0$ and $\beta \to +\infty$, and replacing T by $2T$, we get

$$\overline{u}^i(\tau,\cdot) = \underline{u}^i(\tau,\cdot), \text{ except at most on a countable set in } \mathbb{R}, \text{ for all } \tau \in [0,T], \text{ and } i = 1,\dots,d. \tag{3.78}$$

This equality allows us to link the sub-solution $\overline{u}^i$ and the super-solution $\underline{u}^i$. It remains to show that

$$u^i(\tau,\cdot) = \overline{u}^i(\tau,\cdot) \quad \text{except at most on a countable set in } \mathbb{R}, \quad \text{for all} \quad \tau \in [0,T].$$

To do this, it is sufficient to use, the right continuity of the functions $u^i(\tau,\cdot)$, $u^{i,1}(\tau,\cdot)$ and Lemma 3.7 (i). Indeed, let $\alpha>0$, the right continuity of the functions $u^i(\tau,\cdot)$, $u^{i,1}(\tau,\cdot)$, implies that, for all $x\in[-\frac{a}{2},\frac{a}{2}]$, there exists $\alpha^1_{a,\tau}>0$, such that, for all $0<\delta\le\alpha^1_{a,\tau}$, we have

$$\begin{aligned} u^i(\tau,x) &\le \alpha+u^i(\tau,x+\delta)\\ &=\alpha+u^{i,1}(\tau,x+\delta)-u^{i,2}(\tau,x+\delta)\\ &\le 2\alpha+u^{i,1}(\tau,x)-u^{i,2}(\tau,x+\delta)\end{aligned} \tag{3.79}$$

where $u^{i,1}$, $u^{i,2}$ are the right-continuous non-decreasing functions, given in (3.71). However, using Lemma 3.7 (i), we know that, for all $0<\delta\le\frac{a}{2}$, there exists $k^\alpha_{a,\tau},\ell^\alpha_{a,\tau}>0$ and two subsequences $0<\epsilon_k\le\delta$, $0<\eta_\ell\le\delta$ such that for every $k\ge k^\alpha_{a,\tau}$, $\ell\ge\ell^\alpha_{a,\tau}$

$$\left|\begin{aligned} &u^{i,1}(x,\tau)\le\tfrac{\delta}{2}+u^{i,1}_{\epsilon_k,\eta_\ell}(\tau,x+\tfrac{\delta}{2})\\ &-u^{i,2}(\tau,x+\delta)\le\tfrac{\delta}{2}-u^{i,2}_{\epsilon_k,\eta_\ell}(\tau,x+\tfrac{\delta}{2}),\end{aligned}\right. \tag{3.80}$$

where the sequences $u^{i,1}_{\epsilon_k,\eta_\ell}$ and $u^{i,2}_{\epsilon_k,\eta_\ell}$ satisfy the following equality $u^i_{\epsilon_k,\eta_\ell}=u^{i,1}_{\epsilon_k,\eta_\ell}-u^{i,2}_{\epsilon_k,\eta_\ell}$. Finally, bringing together the two inequalities (3.79) and (3.80), we can see that, for all $0<\delta\le\min(\frac{a}{2},\alpha^1_{a,\tau})$, $k\ge k^\alpha_{a,\tau}$ and $\ell\ge\ell^\alpha_{a,\tau}$, we have

$$u^i(\tau,x)\le 2\alpha+\delta+u^i_{\epsilon_k,\eta_\ell}(\tau,x+\frac{\delta}{2})\le 2\alpha+\delta+\sup_{\substack{\epsilon_k\le\delta,\ |s-\tau|\le\delta\\ \eta_\ell\le\delta,\ |y-x|\le 2\delta}} u^i_{\epsilon_k,\eta_\ell}(s,y).$$

To complete the proof, we pass to the limit $\delta\to0$ and then $\alpha\to0$, to get $u^i(\tau,x)\le\overline{u}^i(\tau,x)$. Similarly, we can show that $\underline{u}^i(\tau,x)\le u^i(\tau,x)$, which joint to (3.78) proves the desired result.

□

Remark 3.3. Note that, equality (3.16) implies that the functions $\overline{u}^i$, $\underline{u}^i$ also satisfy estimates (3.14) and (3.15).

Remark 3.4. The classical question to be asked after an existence result, is the uniqueness of the constructed solution. In the case of system (3.1), the uniqueness of the solution remains an open question under the conditions considered in this work. Normally, we can apply a Comparison Principle in order to prove the uniqueness of the solution. However, even trying to prove such an argument to (3.1) in an almost everywhere sense is still too difficult. We can try to impose some monotony on the velocities λ^i; for example if we consider that the system is quasi-monotonic in the sense of Ishii, Koike [59], and by taking continuous or maybe piece-wise continuous initial data, perhaps then a Comparison Principle in space can be applied in order to prove the uniqueness of the solution.

4 Existence and uniqueness to a Hamilton-Jacobi system

This chapter is a submitted article that is written in collaboration with Ahmad El Hajj and Mustapha Jazar.

In this work, we study the existence and uniqueness of a non-linear eikonal system in one space dimension. We prove first the existence of a discontinuous viscosity solution by regularizing the problem and passing to the limit as the regularization vanishes. Then, by the means of a Comparison Principle, we show that this discontinuous solution would be continuous if we assume that the initial data are also continuous functions, and under the extra supposition of quasi-monotony we impose on the system. As a consequence, we obtain the uniqueness of this continuous solution. We also present an application to a particular system that models the dynamics of dislocations densities.

Existence and uniqueness results to a system of Hamilton-Jacobi equations

Maryam Al Zohbi, Ahmad El Hajj, Mustapha Jazar

Abstract

We study the existence and uniqueness of a nonlinear system of eikonal equations in one space dimension for any BV initial data. We present two results. In the first one, we prove the existence of a discontinuous viscosity solution without any monotony conditions neither on the velocities nor on the initial data. In the second, we show the continuity of the constructed solution under continuous initial data, and continuous velocities verifying a certain monotony condition. We present an application to a system modeling the dynamics of dislocations densities.

AMS Classification: 35F21, 49L25, 35D40, 34A12, 74H20, 74H25, 35B51, 35F55, 35L40, 35L45.

Key words: Hamilton-Jacobi equations, non-linear eikonal equations, viscosity solution, uniqueness, comprasion principle, dislocations dynamics.

1 Introduction and main results

1.1 Setting of the problem

In this paper, we are interested in a non-linear strongly coupled Hamilton-Jacobi system of the form

$$\begin{cases} \partial_t u^i(t,x) = \lambda^i(t,x,u(t,x))|\partial_x u^i(t,x)| & \text{in } (0,T)\times\mathbb{R}, \\ u^i(0,x) = u_0^i(x) & \text{in } \mathbb{R}, \end{cases} \tag{4.1}$$

where $T>0$ and $i=1,\dots,d$, such that $d\in\mathbb{N}^*$. The functions u^i are real valued, $\partial_t u^i$ and $\partial_x u^i$ represent the time and spatial derivatives of u^i respectively. The velocity λ^i is assumed to satisfy, for all $i=1,\dots,d$, the following assumption

$$\lambda^i \in L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}) \quad \text{for} \quad T>0 \quad \text{and for all compact } \mathcal{K}\subset\mathbb{R}^d. \tag{4.2}$$

Our study of system (4.1) is motivated by the consideration of a model describing the dynamics of dislocations densities (see [45, Section 5] for more details about the model), which is given by

$$\partial_t u^i = \left(\sum_{j=1,\dots,d} A_{ij} u^j \right) |\partial_x u^i| \quad \text{for} \quad i = 1, \dots, d, \tag{4.3}$$

where $(A_{ij})_{i,j=1,\dots,d}$ is a real matrix. This model can be seen as a special case of system (4.1). From another point of view, we remark that system (4.1) can be seen as the "level-set approach" system associated to the motion of the front $\Gamma_t^i := \{x : u^i(t,x) = 0\}$ with a normal velocity $\lambda^j(t,x,u)$ depending on the solution u and affected by $\lambda^j(t,x,u)$ for $i \neq j$ (see for instance Barles *et al.* [12]).

We aim in this work to establish first the existence of a discontinuous viscosity solution assuming we have (4.2) and the following regularity on the initial data

$$u_0^i \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}) \quad \text{for every} \quad i = 1, \dots, d, \tag{4.4}$$

where $BV(\mathbb{R})$ is the space of functions of bounded variations given by

$$BV(\mathbb{R}) = \Big\{ f \in L^1_{loc}(\mathbb{R});\ TV(f) < +\infty \Big\},$$

with $TV(f)$ being the total variation of f defined as

$$TV(f) = \sup \left\{ \int_{\mathbb{R}} f(x)\phi'(x)dx;\ \phi \in C_c^1(\mathbb{R}) \text{ and } \|\phi\|_{L^\infty(\mathbb{R})} \le 1 \right\}.$$

Then, we will show that this discontinuous solution would be continuous if we assume that the system is quasi-monotone in the sense of Ishii, Koike [59, 60], along with (4.4) and

$$u_0^i \in C(\mathbb{R}) \quad \text{for every} \quad i = 1, \dots, d, \tag{4.5}$$

and assuming the velocities verify (4.2) and the following assumptions for every $i = 1, \dots, d$,

$$\left| \begin{array}{l} \lambda^i \in C((0,T) \times \mathbb{R} \times \mathcal{K}) \quad \text{for} \quad T > 0 \quad \text{and for all compact} \quad \mathcal{K} \subset \mathbb{R}^d, \\ \\ \text{there exists } M_1 > 0 \text{ such that, for all } x, y \in \mathbb{R} \text{ and } t \in (0,T), \\ \\ |\lambda^i(t,x,u) - \lambda^i(t,y,u)| \le M_1 |x-y|. \end{array} \right. \tag{4.6}$$

We refer the reader to [9, 35, 37] for a complete overview on viscosity solutions. In the following, we take the space $BV(\mathbb{R})$ endowed with the semi-norm $|f|_{BV(\mathbb{R})} = TV(f)$.

Note that BV functions are integrable functions whose distributional derivative is a finite Radon measure.

In order to prove the existence of a discontinuous solution of (4.1), first we will consider, for every $i = 1, \dots, d$ and $0 < \varepsilon, \eta \leq 1$, the following parabolic regularization

$$\begin{cases} \partial_t u^i_{\varepsilon,\eta}(t,x) = \eta \partial^2_{xx} u^i_{\varepsilon,\eta}(t,x) + \lambda^i_\varepsilon(t,x,u_{\varepsilon,\eta}(t,x))|\partial_x u^i_{\varepsilon,\eta}(t,x)| & \text{in } (0,T)\times\mathbb{R}, \\ u^i_{\varepsilon,\eta}(0,x) = u^i_{0,\varepsilon}(x) & \text{in } \mathbb{R}. \end{cases} \tag{4.7}$$

The functions λ^i_ε and $u^i_{0,\varepsilon}$ are the regularizations of λ^i and u^i_0 by classical convolution, which are defined as

$$u^i_{0,\varepsilon}(x) = u^i_0 \star \rho^1_\varepsilon(x) \ \text{ and } \ \lambda^i_\varepsilon(t,x,w) = \hat{\lambda}^i \star \rho^{d+2}_\varepsilon(t,x,w) \quad \forall\ (t,x,w) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d, \tag{4.8}$$

where $\hat{\lambda}^i$ is an extension of λ^i by 0 for all $i = 1, \dots, d$. Moreover, ρ^n_ε for $n = 1$ and $n = d+2$ are standard mollifiers defined as

$$\rho^n_\varepsilon(\cdot) = \frac{1}{\varepsilon^n}\rho^n\left(\frac{\cdot}{\varepsilon}\right), \text{ such that } \rho^n \in C^\infty_c(\mathbb{R}^n),\ \mathrm{supp}\{\rho^n\} \subseteq B(0,1),\ \rho^n \geq 0, \text{ and } \int_{\mathbb{R}^n} \rho^n = 1.$$

We will show that (4.7) admits a unique Lipschitz solution by the means of a Fixed Point argument. Then, by using stability results of viscosity solutions, we will be able to pass to the limit as $(\varepsilon, \eta) \to (0,0)$, and show that the upper and lower relaxed semi-limits of Barles and Perthame [10, 11], which are defined as

$$\overline{u}^i(t,x) = \limsup{}^\star u^i_{\varepsilon,\eta}(t,x) = \limsup_{\substack{(\varepsilon,\eta)\longrightarrow(0,0)\\(s,y)\longrightarrow(t,x)}} u^i_{\varepsilon,\eta}(s,y), \tag{4.9}$$

and

$$\underline{u}^i(t,x) = \liminf{}_\star u^i_{\varepsilon,\eta}(t,x) = \liminf_{\substack{(\varepsilon,\eta)\longrightarrow(0,0)\\(s,y)\longrightarrow(t,x)}} u^i_{\varepsilon,\eta}(s,y), \tag{4.10}$$

are a couple of discontinuous viscosity sub- and super-solutions of system (4.1) in the sense of discontinuous viscosity solutions introduced by Ishii in [58, Definition 2.1] for Hamilton-Jacobi systems that is recalled below in Definition 4.1. Then, by applying the comparison principle, we will be able to show the continuity of these discontinuous solutions under (4.5) and (4.6), in the case where system (4.1) is quasi-monotone (defined below in (H)). The uniqueness of the continuous solution is obtained again by a comparison principle.

1.2 A brief review of some related literature

Several studies have been made on (4.1), of which we will mention some. In El Hajj *et al.* [44], the global existence of a discontinuous viscosity solution has been obtained, under a certain monotony condition on the velocities. However, without any such monotony conditions, EL Hajj and Oussaily recently proved in [48] the global existence and uniqueness of a continuous viscosity solution, basing on an entropy and a BV estimate. In this work, we treat the case of discontinuous solutions without any monotony conditions.

We note that in the case of nondecreasing solutions, system (4.1) becomes a diagonal hyperbolic system. For such systems, many existence and uniqueness results have been made. We will recall some of the most significant ones.
First, in the case of (2×2) strictly hyperbolic system with nondecreasing initial data, Lax proved in [63], the existence and uniqueness of nondecreasing smooth solutions. Also, assuming that the initial data is nondecreasing, an existence and uniqueness result of a continuous solution was proved in El Hajj, Monneau [46] for a general $(d \times d)$ diagonal strictly hyperbolic system. The proof in [46] is based on a global existence result of a continuous nondecreasing solution established previously in El Hajj, Monneau [45], where the system is assumed to be hyperbolic but not necessarily strictly hyperbolic. Also, in the case of $(d \times d)$ strictly hyperbolic systems, Bianchini and Bressan proved in [17] a global existence and uniqueness result assuming that the initial data had small total variation. This approach is mainly based on a careful analysis of the vanishing viscosity approximation. We can also mention that an existence result has been also obtained by LeFloch, Liu [65] and LeFloch [64, 66], in the non-conservative case.

Many authors have worked on Hamilton-Jacobi equations in several space dimensions with discontinuous coefficients. We mention the work of Camilli and Siconolfi [26, 27], where comparison principles, existence and uniqueness results, stability properties, and representations formulas of viscosity solutions have been made, under the assumptions that the Hamiltonian $H(x, p)$ is measurable with respect to the space variable x and convex in p. We can also point out the work of Chen and Hu [30], where they establish the existence and uniqueness of Lipschitz continuous viscosity solutions, assuming the Hamiltonian is nonnegative and Lipschitz continuous. What is genuine about our work is that we will show the existence of viscosity solutions assuming our Hamiltonians (velocities λ^i) are only bounded functions.

We note that the system understudy in this paper is strongly coupled. However, it is important to mention that the case of weakly coupled Hamilton-Jacobi equations have been widely studied in the literature. For instance, Camilli and Loreti proved in [25] two comparison theorems on the system

$$H_i(x, Du^i) + \sum_{j=1}^{M} c_{ij}(u^i - u^j) = 0, \quad i = 1, \dots, M,$$

by imposing convexity and coercivity conditions on the Hamiltonians H_i. Many other results have been brightened up under such conditions. Loreti and Vergara Caffarelli proved the existence and uniqueness of variational solutions in [70]. Also, the authors in Mitake *et al.* [72] were able to characterize all sub-solutions of their system and represent explicitly some of which enjoy a certain maximality property.

In this paper, we present two results that remain valid whether the system is strictly hyperbolic or not strictly hyperbolic, and without any monotony conditions on the initial data. First, we show the global existence of a discontinuous viscosity solution to (4.1) without any monotony conditions on the velocities for any BV initial data. Second, we prove the continuity of the obtained solution by using the Comparison Principle, assuming that the initial data are continuous and the velocities are Lipschits functions in space verifying certain monotony conditions. Then, as a consequence, we obtain the uniqueness of the solution.

1.3 Main results

In this subsection, we first present, in Theorem 4.1, the global existence of a discontinuous viscosity solution to (4.1). Then, we show, in Theorem 4.2, that this solution is continuous for continuous initial data and under certain monotony conditions on the velocities. Lastly, we present an application to the case of the dynamics of dislocations densities in Theorem 4.3.

Theorem 4.1 (**Existence of a discontinuous viscosity solution to (4.1)**). *Assume that (4.2) and (4.4) are satisfied. Then the following points hold.*

i) Existence and uniqueness to the regularized problem
There exists a unique Lipschitz solution $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\dots,d}$ of (4.7) belonging to the space $(C([0,T); W^{1,\infty}(\mathbb{R})))^d$, and satisfying for all $T > 0$ and $i = 1, \dots, d$, the following uniform

estimates

$$\left\| u^i_{\varepsilon,\eta} \right\|_{L^\infty((0,T)\times\mathbb{R})} \leq \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})}, \tag{4.11}$$

$$\left\| \partial_x u^i_{\varepsilon,\eta} \right\|_{L^\infty((0,T);L^1(\mathbb{R}))} \leq \left\| \partial_x u^i_0 \right\|_{L^1(\mathbb{R})}, \tag{4.12}$$

$$\left\| \partial_t u^i_{\varepsilon,\eta} \right\|_{L^\infty((0,T);W^{-1,1}(\mathbb{R}))} \leq \left(1 + \left\| \lambda^i \right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)}\right) \left| u^i_0 \right|_{BV(\mathbb{R})}, \tag{4.13}$$

where $W^{-1,1}(\mathbb{R})$ is the dual of $W^{1,\infty}(\mathbb{R})$, and

$$\mathcal{K}_0 = \prod_{i=1}^{d} \left[- \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})}, \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})} \right]. \tag{4.14}$$

ii) Sub- and super- solutions of (4.1)

Let $u_{\varepsilon,\eta}$ be the unique solution of (4.7) constructed in (i). Then the upper and lower relaxed semi-limits $\overline{u} = \left(\overline{u}^i\right)_{i=1,\dots,d}$ and $\underline{u} = (\underline{u}^i)_{i=1,\dots,d}$, are a couple of discontinuous discontinuous viscosity sub- and super- solutions of system (4.1) (in the sense of Definition 4.1).

iii) Convergence

Assume that the solution $u^i_{\varepsilon,\eta}$ of (4.7) satisfies (4.11), (4.12) and (4.13) for $i = 1, \dots, d$. Then, up to the extract of a subsequence, the function $u^i_{\varepsilon,\eta}$ converges, as ε and η tend to zero, to a function

$$u^i \in L^\infty\big((0,T)\times\mathbb{R}\big) \cap L^\infty\big((0,T);\, BV(\mathbb{R})\big) \cap C\left([0,T);\, L^1_{loc}(\mathbb{R})\right), \tag{4.15}$$

strongly in $C\left([0,T);\, L^1_{loc}(\mathbb{R})\right)$.
Moreover, u^i satisfies, for all $T > 0$ and for $i = 1, \dots, d$, the following inequalities

$$\left\| u^i \right\|_{L^\infty((0,T)\times\mathbb{R})} \leqslant \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})}, \tag{4.16}$$

$$\left\| u^i \right\|_{L^\infty((0,T);BV(\mathbb{R}))} \leqslant \left| u^i_0 \right|_{BV(\mathbb{R})}, \tag{4.17}$$

and the following equality

$$u^i(t,\cdot) = \overline{u}^i(t,\cdot) = \underline{u}^i(t,\cdot), \text{ except at most on a countable set in } \mathbb{R}, \text{ for all } t \in [0,T). \tag{4.18}$$

Our second result relies on the supposition that system (4.1) is quasi-monotone. This means that the velocities verify the following condition

$$\left| \begin{array}{l} \lambda^j(t,x,s) - \lambda^j(t,x,r) \geq 0 \;\text{ for all vectors } r = (r^i)_{i=1,\dots,d},\, s = (s^i)_{i=1,\dots,d} \text{ such that} \\[2ex] r^j - s^j = \max\limits_{i\in\{1,\dots,d\}} (r^i - s^i) \geq 0. \end{array} \right. \tag{H}$$

Theorem 4.2 (**Existence and uniqueness of a continuous solution to (4.1)**). *Suppose that (4.2), (4.4), (4.5), (4.6) and (H) hold. Then, there exists a unique continuous viscosity solution of (4.1) satisfying (4.16) and (4.17).*

1.3.1 Unique continuous solution for dislocations' dynamics

Now, we present an application of Theorem 4.2 to a nonlinear system that appears in the modeling of the dynamics of dislocations densities in materials.
A dislocation is a linear crystallographic defect or irregularity within a crystal structure that contains an abrupt change in the arrangement of its atoms.
Here, we are interested in a particular 1D model initially proposed in 2D dimensions by Groma and Balogh [53, 54], in order to describe the dynamics of dislocations densities. This 2D model is written in a specific geometry, where the dislocations are considered as points in the plane (x_1, x_2), propagating to the left and to the right, following two Burger's vectors $\pm b = \pm(1, 0)$. In the 1D sub-model, we suppose that the dislocations densities depend only on the variable $x = x_1 + x_2$, which transforms the 2D model into a 1D model. We refer the reader to El Hajj and Forcadel [41, Lemma 3.1] for more details about the modeling.

More precisely, we consider the following system

$$\begin{cases} \partial_t \rho^+(x,t) = -\left((\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy + a(t)\right)\Big|\partial_x \rho^+(x,t)\Big| & \text{in } \mathbb{R} \times (0,T), \\ \partial_t \rho^-(x,t) = \left((\rho^+ - \rho^-)(x,t) + \alpha \displaystyle\int_0^1 (\rho^+ - \rho^-)(y,t)dy + a(t)\right)\Big|\partial_x \rho^-(x,t)\Big| & \text{in } \mathbb{R} \times (0,T), \end{cases} \tag{4.19}$$

where ρ^+, ρ^- are the unknown scalars, that we denote for simplicity by $\rho^\pm$. These two functions, ρ^+ and ρ^-, are respectively the representations of the left-propagating and right-propagating dislocations. Their spatial derivatives $\partial_x \rho^+, \partial_x \rho^-$ represent the dislocations densities of $+, -$ type respectively. The constant α depends on the elastic coefficients and the material size, while the function $a(t)$ represents the exterior strain field.
The initial conditions associated to system (4.19) are defined as follows

$$\rho^\pm(0,x) = \rho_0^\pm(x) = P_0^\pm(x) + a_0 x \quad \text{on } \mathbb{R}, \tag{4.20}$$

where $P_0^\pm$ are 1-periodic functions. In particular, $\rho_0^+ - \rho_0^-$ is 1-periodic. The use of the periodic boundary conditions is a way of regarding what is going on in the interior of the material away from its boundary, assuming the material understudy is made up entirely of small and similar subsets.

Remark 4.1. Mathematically speaking, we could have just considered periodic initial conditions, without the linear part in (4.20). However, the use of periodic plus linear conditions is totally physical, as the dislocations densities are considered to be non-decreasing functions in this model.

Applying Theorem 4.2 to the local case ($\alpha = 0$) of system (4.19) yields the following result, where we note that the set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the $[0, 1)$ periodic interval.

Theorem 4.3 (***Existence and uniqueness of a continuous solution***)**.**
Assume that $\alpha = 0$, and suppose that the functions $P_0^{\pm}$ introduced in (4.20) verify

$$P_0^{\pm} \in C(\mathbb{T}) \cap BV(\mathbb{T}), \tag{4.21}$$

and the function a satisfies

$$a \in C[0, T). \tag{4.22}$$

Then, there exists a unique continuous viscosity solution of system (4.19)-(4.20).

The proof of this theorem derives naturally from Theorem 4.2, since system (4.19) verifies condition (H) in the case where $\alpha = 0$.

1.4 Organization of the paper

This paper is organized as follows: in Section 2, we prove the existence of a discontinuous viscosity solution result, that was announced in Theorem 4.1. Section 3 is devoted to the proof of Theorem 4.2, where we show by the means of a comparison principle, the continuity and the uniqueness of the solution constructed in Theorem 4.1, under certain extra conditions. Finally in Section 4, we present an application of Theorem 4.2 to the dynamics of dislocations, presented in Theorem 4.3.

2 Existence of a discontinuous solution

In this section, we prove the existence of a discontinuous viscosity solution of (4.1), that was announced in Theorem 4.1. First, we recall in the following subsection a well-known compactness lemma, along with the definition of discontinuous viscosity solutions for system (4.1).

2.1 Some useful results

We first recall Simon's Compactness Lemma.

Lemma 4.1. ***(Simon's Lemma [80, Corollary 4])***
Let X, B and Y be three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(\theta_n)_n$ is a sequence uniformly bounded in $L^\infty((0,T);X)$ and $(\partial_t \theta_n)_n$ is uniformly bounded in $L^r((0,T);Y)$ where $r > 1$, then, $(\theta_n)_n$ is relatively compact in $C((0,T);B)$.

Next, we are going to recall the definition of discontinuous viscosity solutions for system (4.1) introduced by Ishii in [58, Definition 2.1].
We denote by $f^\star$ and $f_\star$ the respective upper and lower semi-continuous envelopes of a locally bounded function f defined on an open domain in $\mathbb{R}^n$ and given by

$$f^\star(X) = \limsup_{Y \to X} f(Y) \quad \text{and} \quad f_\star(X) = \liminf_{Y \to X} f(Y) \quad \text{for } X \in \mathbb{R}^n. \tag{4.23}$$

For a vector $u = (u^1, \ldots, u^d)$ locally bounded on $[0,T) \times \mathbb{R}$ for all $T > 0$, we write $u^\star = ((u^1)^\star, \ldots, (u^d)^\star)$ and $u_\star = ((u^1)_\star, \ldots, (u^d)_\star)$.
Given two locally bounded functions $v = (v^i)_{i=1,\ldots,d}$ and $u = (u^i)_{i=1,\ldots,d}$ on $[0,T) \times \mathbb{R}$ such that $(v^i)_\star \leq (u^i)^\star$ for every $i = 1, \ldots, d$, we define the set

$$\mathcal{E}_v^u(t,x) = \prod_{i=1}^d \Big[(v^i)_\star(t,x), (u^i)^\star(t,x)\Big].$$

Definition 4.1. ***(Discontinuous viscosity sub-solution, super-solution and solution)***
Assume that $\lambda = (\lambda^i)_{i=1,\ldots,d}$ is locally bounded on $(0,T) \times \mathbb{R} \times \mathbb{R}^d$ and $u_0 = (u_0^i)_{i=1,\ldots,d}$ is locally bounded on $\mathbb{R}$. Let $v = (v^i)_{i=1,\ldots,d}$, $u = (u^i)_{i=1,\ldots,d}$ be two locally bounded functions on $[0,T) \times \mathbb{R}$ such that $(v^i)_\star \leq (u^i)^\star$ for every $i = 1, \ldots, d$. We say that u and v are a couple of discontinuous viscosity sub- and super- solutions of (4.1) if they satisfy the following two conditions

(i)
- $(u^i)^\star(0,x) \leq (u_0^i)^\star(x)$, for all $i = 1, \ldots, d$ and $x \in \mathbb{R}$.
- $(v^i)_\star(0,x) \geq (u_0^i)_\star(x)$, for all $i = 1, \ldots, d$ and $x \in \mathbb{R}$.

(ii)
- Whenever a test function $\phi \in C^1((0,T) \times \mathbb{R})$, $i = 1, \ldots, d$ and $(u^i)^\star - \phi$ attains a local maximum at $(t_0, x_0) \in (0,T) \times \mathbb{R}$, then we have

$$\min \Big\{ \partial_t \phi(t_0,x_0) - (\lambda^i)^\star(t_0,x_0,r)|\partial_x \phi(t_0,x_0)| : \, r \in \mathcal{E}_v^u(t_0,x_0), \, r^i = (u^i)^\star(t_0,x_0) \Big\} \leq 0. \tag{4.24}$$

- Whenever $\phi \in C^1((0,T)\times\mathbb{R})$, $i = 1,\dots,d$ and $(v^i)_\star - \phi$ attains a local minimum at $(t_0, x_0) \in (0,T)\times\mathbb{R}$, then we have

$$\max\Big\{\partial_t\phi(t_0,x_0) - (\lambda^i)_\star(t_0,x_0,r)|\partial_x\phi(t_0,x_0)| : r \in \mathcal{E}^u_v(t_0,x_0),\ r^i = (v^i)_\star(t_0,x_0)\Big\} \geq 0. \tag{4.25}$$

Finally, we call a function $w = (w^i)_{i=1,\dots,d}$ a discontinuous viscosity solution of (4.1) if $w^\star$ and $w_\star$ verify conditions (i) and (ii).

Noting that the minimum and the maximum in (4.24) and (4.25) are attained, since the sets

$$\Big\{r \in \mathbb{R}^d\ :\ r \in \mathcal{E}^u_v(t_0,x_0),\ r^i = (u^i)^\star(t_0,x_0)\Big\}\ \text{ and }\ \Big\{r \in \mathbb{R}^d\ :\ r \in \mathcal{E}^u_v(t_0,x_0),\ r^i = (u^i)_\star(t_0,x_0)\Big\}$$

are non-empty and compact and moreover $(\lambda^i)^\star$ and $(\lambda^i)_\star$ are upper and lower semi-continuous, respectively.

2.2 Proof of Theorem 4.1

We proceed in three steps.

Step 1. (Proof of (i)):
The proof of Theorem 4.1-(i) is a classic application of the Fixed Point Theorem in Banach spaces. We will explain briefly what we will do. First, we regularize the non-linear term $|\partial_x u^i_{\varepsilon,\eta}|$, in (4.7), by replacing it with the smooth function $\beta_\delta(\partial_x u^i_{\varepsilon,\eta})$, where β_δ is defined as

$$\beta_\delta(x) = \sqrt{x^2+\delta^2}\ \text{ for all } 0 < \delta \leq 1. \tag{4.26}$$

Then, we truncate the function β_δ by plugging in the function

$$\psi_\delta(\cdot) = \phi(\sqrt{\delta}\ \cdot), \tag{4.27}$$

where $\phi \in C^\infty(\mathbb{R})$ is a cut-off function taking values in $[0,1]$, supported by the interval $[-2,2]$ and $\phi(x) \equiv 1$ on $[-1,1]$.
This brings us to consider for all $0 < \delta \leq 1$, $i = 1,\dots,d$, and for $u_{\varepsilon,\eta,\delta} = (u^i_{\varepsilon,\eta,\delta})_{i=1,\dots,d}$, the following problem

$$\begin{cases} \partial_t u^i_{\varepsilon,\eta,\delta}(t,x) - \eta\partial^2_{xx}u^i_{\varepsilon,\eta,\delta}(t,x) = \lambda^i_\varepsilon(t,x,u_{\varepsilon,\eta,\delta})\psi_\delta(x)\beta_\delta(\partial_x u^i_{\varepsilon,\eta,\delta}(t,x)) & \text{on } (0,T)\times\mathbb{R}, \\ u^i_{\varepsilon,\eta,\delta}(0,x) = u^i_{0,\varepsilon}(x) & x \in \mathbb{R}, \end{cases} \tag{4.28}$$

where for all $i = 1, \dots, d$, we have

$$u^i_{0,\varepsilon} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \partial_x u^i_{0,\varepsilon} \in L^p(\mathbb{R}) \text{ for all } 1 \le p \le +\infty, \tag{4.29}$$

and

$$\lambda^i_\varepsilon \in W^{1,\infty}((0,T)\times\mathbb{R}\times\mathcal{K}) \cap C^\infty((0,T)\times\mathbb{R}\times\mathbb{R}^d), \text{ for all compact } \mathcal{K} \subset \mathbb{R}. \tag{4.30}$$

Next, we will write (4.28) in its integral form, for all $i = 1, \dots, d$, as follows

$$u^i_{\varepsilon,\eta,\delta}(t,x) = G_\eta(t) \star u^i_{0,\varepsilon}(x) + \int_0^t G_\eta(t-s) \star \left(\lambda^i_\varepsilon(s,\cdot,u_{\varepsilon,\eta,\delta}(s,\cdot))\psi_\delta(\cdot)\beta_\delta(\partial_x u^i_{\varepsilon,\eta,\delta}(s,\cdot))\right)(x)ds,$$

where $G_\eta(t,x) = \dfrac{1}{\sqrt{4\pi\eta t}} e^{-\frac{x^2}{4\eta t}}$ is the standard heat kernel. In other words, we consider the following problem

$$\begin{cases} u_{\varepsilon,\eta,\delta}(t,x) = (u^i_{\varepsilon,\eta,\delta}(t,x))_{i=1,\dots,d}, \quad u_{\varepsilon,\eta,\delta}(0,x) = u_{0,\varepsilon}(x) = (u^i_{0,\varepsilon}(x))_{i=1,\dots,d}, \\ \\ u_{\varepsilon,\eta,\delta}(t,x) = G_\eta(t) \star u_{0,\varepsilon}(x) + B(u_{\varepsilon,\eta,\delta})(t,x), \end{cases} \tag{4.31}$$

where, for $r(t,x) = (r^i(t,x))_{i=1,\dots,d}$, we have

$$B(r)(t,x) = \int_0^t G_\eta(t-s) \star \begin{pmatrix} \beta_\delta(\partial_x r^1(s,\cdot))\psi_\delta(\cdot) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_\delta(\partial_x r^d(s,\cdot))\psi_\delta(\cdot) \end{pmatrix} \begin{pmatrix} \lambda^1_\varepsilon(s,\cdot,r(s,\cdot)) \\ \lambda^2_\varepsilon(s,\cdot,r(s,\cdot)) \\ \vdots \\ \lambda^d_\varepsilon(s,\cdot,r(s,\cdot)) \end{pmatrix} (x)ds.$$

We introduce, for $T > 0$, the following three Banach spaces

$$X = \left\{ r = (r^i)_{i=1,\dots,d} \in (L^\infty(\mathbb{R}))^d;\ \partial_x r^i \in L^1(\mathbb{R}) \right\}, \tag{4.32}$$

equipped with the norm $\|r\|_X = \sum_{i=1}^d \|r^i\|_{L^\infty(\mathbb{R})} + \sum_{i=1}^d \|\partial_x r^i\|_{L^1(\mathbb{R})}$, and

$$X_T = \left\{ r = (r^i)_{i=1,\dots,d} \in (L^\infty((0,T)\times\mathbb{R}))^d;\ \ \partial_x r^i \in L^\infty((0,T);L^1(\mathbb{R})) \right\},$$

equipped with the norm $\|r\|_{X_T} = \sum_{i=1}^d \|r^i\|_{L^\infty((0,T)\times\mathbb{R})} + \sum_{i=1}^d \|\partial_x r^i\|_{L^\infty((0,T);L^1(\mathbb{R}))}$, and lastly

$$Y_T = \left\{ r \in X_T;\ \ \|r\|_{X_T} \le \|v_0\|_X + 1 \right\}.$$

Then, we define the mapping

$$\begin{aligned} \mathcal{T} : Y_T &\to Y_T \\ r &\to \mathcal{T}(r) = G_\eta(\cdot) \star v_0 + B(r). \end{aligned}$$

We prove that this mapping is well-defined and is a contraction on Y_{T^*} for a certain $T^* > 0$. This enables us to apply the Fixed Point Theorem in order to say that there exists a unique fixed point which is the unique solution of (4.31). Moreover, we can prove that there exists a constant ξ_p^δ depending on $\|u^i_{0,\varepsilon}\|_{L^\infty(\mathbb{R})}$, $\|\partial_x u^i_{0,\varepsilon}\|_{L^p(\mathbb{R})}$, and T^* such that

$$\|\partial_x u^i_{\varepsilon,\eta,\delta}\|_{L^\infty((0,T^*);L^p(\mathbb{R}))} \leq \xi_p^\delta \quad \text{for all} \quad 1 \leq p \leq +\infty. \tag{4.33}$$

Then, by using L^p-regularity of parabolic equations and the fact that β_δ and λ^i_ε are regular, we can show by classical Bootstrap arguments that $u_{\varepsilon,\eta,\delta} \in (C^\infty((0,T^*)\times\mathbb{R}))^d \cap (W^{1,\infty}((0,T^*)\times\mathbb{R}))^d \cap Y_{T^*}$. After that, we can prove via the maximum principle for parabolic equations (see Lieberman [68, Theorem 2.10]) that the smooth solution $u^i_{\varepsilon,\eta,\delta}$ verifies

$$\|u^i_{\varepsilon,\eta,\delta}\|_{L^\infty((0,T^*)\times\mathbb{R})} \leq T^*\delta\widetilde{\Lambda}^i + \|u^i_0\|_{L^\infty(\mathbb{R})} \quad \text{for all} \quad i = 1,\ldots,d, \tag{4.34}$$

where

$$\widetilde{\Lambda}^i = \|\lambda^i\|_{L^\infty((0,T)\times\mathbb{R}\times\widetilde{\mathcal{K}}_0)}, \tag{4.35}$$

with

$$\widetilde{\mathcal{K}}_0 = \prod_{i=1}^d \left[-1 - \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}, \left\|u^i_0\right\|_{L^\infty(\mathbb{R})} + 1\right].$$

Then, using the compactness lemma by Simon, inherited from (4.33) and (4.34), we will be able to pass to the limit as $\delta \to 0$, and prove the existence of (4.7). Estimate (4.11) comes from passing to the limit as $\delta \to 0$ is (4.34). The proof of the BV estimate (4.12) and the time derivative estimate (4.13) can be found in [47]. Finally, using (4.11) and (4.12) we will be able to prove the global in time existence of a solution $u_{\varepsilon,\eta}$ to (4.7) belonging to the space $(C([0,T)); W^{1,\infty}(\mathbb{R}))^d$ for $T > 0$. We refer to [47, Theorem 5.1] for the complete proof of this step.

Step 2. (Proof of (ii)):

We have to prove that the upper and lower relaxed semi-limits $\overline{u}$ and $\underline{u}$ of $u_{\varepsilon,\eta}$ verify the conditions of discontinuous viscosity sub- and super- solutions, given in Definition 4.1, respectively. We introduce the finite speed propagation property, valid on the smooth solutions of (4.7). Indeed, under assumptions (4.2) and (4.4), if $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\ldots,d}$ is the unique solution of (4.7), given by Theorem 4.1-(i), then $u^i_{\varepsilon,\eta}$ satisfies, for all $h \geq 0$, the following estimate

$$\int_{\mathbb{R}} G_\eta(t,y) \min_{|z-(x-y)|\leq\Lambda t} u^i_{\varepsilon,\eta}(h,z)dy \leq u^i_{\varepsilon,\eta}(t+h,x) \leq \int_{\mathbb{R}} G_\eta(t,y) \max_{|z-(x-y)|\leq\Lambda t} u^i_{\varepsilon,\eta}(h,z)dy, \tag{4.36}$$

for all $(t,x) \in [0,T-h) \times \mathbb{R}$, where $\mathcal{K}_0$ is defined in (4.14), and

$$\Lambda = \max_{i\in\{1,\dots,d\}} \left\|\lambda^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)}. \tag{4.37}$$

We refer to [3, Lemma 5.1] for the proof of this estimate.
Now, using estimate (4.36) combined with the property

$$\max_{|x-x_0|\le c} u^i_{0,\varepsilon}(x) \le \max_{|x-x_0|\le c+\varepsilon} u^i_0(x), \qquad \text{where} \ \ c>0,$$

allows us to give sense to the initial data, as it is precised in Definition 4.1 (1)-(i) and (2)-(i). See [3, Section 5] for more details. However, to give meaning to the system, in other words, to prove Definition 4.1 (1)-(ii) and (2)-(ii), we use the stability results of discontinuous viscosity solutions, along with the following properties

$$\overline{\lambda}^i(t,x,r) \le (\lambda^i)^\star(t,x,r) \quad \text{and} \quad (\lambda^i)_\star(t,x,r) \le \underline{\lambda}^i(t,x,r) \quad \text{for all} \quad (t,x,r)\in[0,T)\times\mathbb{R}\times\mathbb{R}^d,$$

where

$$\overline{\lambda}^i(t,x,r) = \limsup_{\substack{\varepsilon\longrightarrow 0\\ (s,y,w)\longrightarrow(t,x,r)}} \lambda^i_\varepsilon(s,y,w), \quad \text{and} \quad \underline{\lambda}^i(t,x,r) = \liminf_{\substack{\varepsilon\longrightarrow 0\\ (s,y,w)\longrightarrow(t,x,r)}} \lambda^i_\varepsilon(s,y,w).$$

We refer the reader also to [3, Section 5] for a complete elaboration of the preceding information.

Step 3. (Proof of (iii)):
We proceed as in [3, Section 6]. Indeed, for $u_{\varepsilon,\eta} = \left(u^i_{\varepsilon,\eta}\right)_{i=1,\dots,d}$ being the solution of (4.7), constructed in Theorem 4.1-(i), and from estimates (4.11), (4.12), (4.13) and Simon's Lemma, we can extract a subsequence, denoted by $(u^i_{(\varepsilon_n,\eta_n)_{K_0}})_{(\varepsilon_n,\eta_n)_{K_0}}$, that converges strongly in $L^\infty((0,T);L^1(K_0))$ to some limit u^i, as $n\to+\infty$. By a standard diagonalization procedure, we can extract a subsequence $(u^i_{\varepsilon_n,\eta_n})_{\varepsilon_n,\eta_n}$ (independent of i and K) that converges to the limit u^i strongly in $C([0,T);L^1(K))$ for all compact $K\subset\mathbb{R}$. Now, thanks to estimates (4.11) and (4.12) we can extract a subsequence, still denoted by $(u^i_{\varepsilon_n,\eta_n})_{\varepsilon_n,\eta_n}$, satisfying the following convergences

$$\left|\begin{array}{llll} u^i_{\varepsilon_n,\eta_n} \longrightarrow u^i, & \text{strongly} & \text{in} & C([0,T);L^1(K)), \ \text{for all compact} \ K\subset\mathbb{R}, \\ u^i_{\varepsilon_n,\eta_n} \longrightarrow u^i, & \text{weakly-}\star & \text{in} & L^\infty((0,T)\times\mathbb{R}), \\ u^i_{\varepsilon_n,\eta_n} \longrightarrow u^i, & \text{weakly-}\star & \text{in} & L^\infty((0,T);BV(\mathbb{R})). \end{array}\right.$$

Taking the lim inf in estimates (4.11), (4.12) and using the lower semi-continuity of $\|\cdot\|_{L^\infty(\mathbb{R})}$ and $|\cdot|_{BV(\mathbb{R})}$, we can prove that u^i satisfies (4.15), (4.16) and (4.17). The proof of (4.18) is based on the Finite Speed Propagation Property (4.36) and the BV estimate (4.12), in particular, the discontinuity of BV functions in dimension one. See [3, Section 7] for more details.

□

3 Unique continuous solution

In this section, we give the proof of Theorem 4.2. We will first show that the discontinuous viscosity sub-solution $\overline{u}$ and super-solution $\underline{u}$, given by Theorem 4.1, are continuous under assumptions (4.2), (4.4), (4.5), (4.6) and (H). We already have $\underline{u}^i \leq \overline{u}^i$ for every $i = 1, \dots, d$, from the definition of upper and lower relaxed semi-limits (see (4.9) and (4.10)). Then, by the use of the comparison principle, we will be able to show that we also have $\underline{u}^i \geq \overline{u}^i$ for every $i = 1, \dots, d$, which proves that the solution u is indeed continuous. Thus, we present the following proposition.

Proposition 4.1.
Assume (4.2), (4.4), (4.5), (4.6) and (H) hold. Let $\overline{u} = (\overline{u}^i)_{i=1,\dots,d}$ and $\underline{u} = (\underline{u}^i)_{i=1,\dots,d}$ be respectively discontinuous viscosity sub- and super solutions of (4.1), in the sense of Definition 4.1, where the functions u^i satisfy

$$\max_{i\in\{1,\dots,d\}} \left(\|u^i\|_{L^\infty(\mathbb{T}\times(0,T))}\right) \leq M_0. \tag{4.38}$$

Then, if $\overline{u}^i(\cdot,0) \leq \underline{u}^i(\cdot,0)$ in $\mathbb{R}$ we get $\overline{u}^i \leq \underline{u}^i$ in $\mathbb{R}\times[0,T)$ for every $i = 1, \dots, d$.

Proof of Proposition 4.1:
Let us denote by

$$M_{sup} = \max_{i\in\{1,\dots,d\}} \sup_{\mathbb{R}\times[0,T]} \left(\overline{V}^i - \underline{V}^i\right),$$

where $\overline{V}^i(x,t) = \overline{u}^i(x,t)e^{-\gamma t}$ and $\underline{V}^i(x,t) = \underline{u}^i(x,t)e^{-\gamma t}$. We remark that $\overline{V}^i$ and $\underline{V}^i$ are respectively discontinuous viscosity sub- and super- solutions of the equation

$$\partial_t V^i = -\gamma V^i + \lambda^i(t,x,u)|\partial_x V^i|. \tag{4.39}$$

It is sufficient to prove that $M_{sup} \leq 0$. Let us suppose by contradiction that $M_{sup} > 0$. We proceed in two steps.

Step 1. (Doubling the variables):
We duplicate the variables by considering, for all ϵ, β, η and α positives

$$\psi(x,y,t,s,i) = \overline{V}^i(x,t) - \underline{V}^i(y,s) - \frac{|x-y|^2}{2\epsilon} - \frac{|t-s|^2}{2\beta} - \frac{\eta}{T-t} - \alpha(|x|^2 + |y|^2).$$

We note that $\psi(\cdot,\cdot,\cdot,\cdot,\cdot)$ is a locally bounded function in $\mathbb{R}^2 \times [0,T)^2 \times \{1,\dots,d\}$.

We can think that the maximum of ψ noted $M(\epsilon,\beta,\alpha,\eta,\gamma) = \sup_{\substack{(x,y)\in\mathbb{R}^2,(t,s)\in[0,T]^2 \\ i\in\{1,\dots,d\}}} \psi(x,y,t,s,i)$, is similar with M_{sup}.
The idea is justified by the following lemma.

Lemma 4.2. *Let $(\overline{x}, \overline{y}, \overline{t}, \overline{s}, \overline{k})$ be a maximum of ψ. If we define $M' = \lim_{(h,k)\to(0,0)} M_h^k$, where $M_h^k = \sup_{\substack{|x-y|,|t-s|\le h \\ |x|,|y|\le \frac{1}{\sqrt{k}}}} \left(\overline{V}^{\overline{i}}(x,t) - \underline{V}^{\overline{i}}(y,s)\right)$, then the following properties hold*

1. $\alpha|\overline{x}|^2, \alpha|\overline{y}|^2 \le 2M_0$.
2. $\lim_{\beta\to 0}|\overline{t}-\overline{s}| = \lim_{\epsilon\to 0}|\overline{x}-\overline{y}| = 0$.
3. $\overline{t} < T - \frac{\eta}{2M_0} < T$.
4. $\liminf_{\alpha\to 0}\liminf_{\epsilon\to 0}\liminf_{\beta\to 0}\liminf_{\eta\to 0}\left(\overline{V}^{\overline{i}}(\overline{x},\overline{t}) - \underline{V}^{\overline{i}}(\overline{y},\overline{s})\right) =$

   $\limsup_{\alpha\to 0}\limsup_{\epsilon\to 0}\limsup_{\beta\to 0}\limsup_{\eta\to 0}\left(\overline{V}^{\overline{i}}(\overline{x},\overline{t}) - \underline{V}^{\overline{i}}(\overline{y},\overline{s})\right) = M'$.
5. $\liminf_{\alpha\to 0}\liminf_{\epsilon\to 0}\liminf_{\beta\to 0}\liminf_{\eta\to 0} M(\epsilon,\beta,\alpha,\eta,\gamma) =$

   $\limsup_{\alpha\to 0}\limsup_{\epsilon\to 0}\limsup_{\beta\to 0}\limsup_{\eta\to 0} M(\epsilon,\beta,\alpha,\eta,\gamma) = M'$.
6. $\liminf_{\alpha\to 0}\liminf_{\epsilon\to 0}\liminf_{\beta\to 0}\liminf_{\eta\to 0}\left(\frac{|\overline{x}-\overline{y}|^2}{2\epsilon} + \frac{|\overline{t}-\overline{s}|^2}{2\beta}\right) =$

   $\limsup_{\alpha\to 0}\limsup_{\epsilon\to 0}\limsup_{\beta\to 0}\limsup_{\eta\to 0}\left(\frac{|\overline{x}-\overline{y}|^2}{2\epsilon} + \frac{|\overline{t}-\overline{s}|^2}{2\beta}\right) = 0$
7. *$\overline{t}, \overline{s}$ are positive if ϵ, β, α and η are sufficiently small.*

Proof of Lemma 4.2:
Using the fact that $\overline{V}^i$ and $\underline{V}^i$ are bounded for every $i \in \{1,\dots,d\}$, we deduce that

$$\lim_{|x|,|y|\to\infty} \psi(x,y,t,s,i) = -\infty,$$

and so the function ψ reaches a maximum at a point $(\overline{x}, \overline{y}, \overline{t}, \overline{s}, \overline{i}) \in \mathbb{R}^2 \times (0,T)^2 \times \{1, \ldots, d\}$. We know that $M_{sup} = \max_{k\in\{1,\ldots,d\}} \sup_{\mathbb{R}\times[0,T]} \left(\overline{V}^i - \underline{V}^i\right) > 0$, then there exists $(x^*, t^*, i^*) \in (0,1) \times (0,T) \times \{1, \ldots, d\}$, such that

$$\overline{V}^{i^*}(x^*, t^*) - \underline{V}^{i^*}(x^*, t^*) > 0.$$

Then, we have

$$\psi(x^*, x^*, t^*, t^*, i^*) \leq M(\epsilon, \beta, \alpha, \eta),$$

which implies

$$\begin{aligned} 0 &< \overline{V}^{i^*}(x^*, t^*) - \underline{V}^{i^*}(x^*, t^*) \leq M(\epsilon, \beta, \alpha, \eta) + \frac{\eta}{T - t^*} + 2\alpha|x^*|^2, \\ 0 &< M(\epsilon, \beta, \alpha, \eta, \gamma) + \frac{\eta}{T - t^*} + 2\alpha|x^*|^2. \end{aligned}$$

Then, for η and α small enough, we get

$$M(\epsilon, \beta, \alpha, \eta, \gamma) > 0. \tag{4.40}$$

We then deduce

$$\alpha(|\overline{x}|^2 + |\overline{y}|^2) < \overline{V}^{\overline{i}}(\overline{x}, \overline{t}) - \underline{V}^{\overline{i}}(\overline{y}, \overline{s}) \leq 2M_0,$$

where we have used (4.38) for the second inequality. Multiplying the previous inequality by α yields (1) and

$$\lim_{\alpha\to 0} \alpha|\overline{x}| = \lim_{\alpha\to 0} \alpha|\overline{y}| = 0.$$

In the same way, we have

$$\frac{|\overline{x} - \overline{y}|^2}{2\epsilon} + \frac{|\overline{t} - \overline{s}|^2}{2\beta} < 2M_0 \tag{4.41}$$

and so

$$\lim_{\epsilon\to 0} |\overline{x} - \overline{y}|^2 = \lim_{\beta\to 0} |\overline{t} - \overline{s}| = 0.$$

We also deduce that

$$\frac{\eta}{T - \overline{t}} < \overline{V}^{\overline{i}}(\overline{x}, \overline{t}) - \underline{V}^{\overline{i}}(\overline{y}, \overline{s}) \leq 2M_0,$$

Which leads to

$$\overline{t} < T - \frac{\eta}{2M_0} < T.$$

Now we prove (4). We recall that $M_h^k = \sup_{\substack{|x-y|,|t-s|\leq h \\ |x|,|y|\leq \frac{1}{\sqrt{k}}}} \left(\overline{V}^{\overline{i}}(x,t) - \underline{V}^{\overline{i}}(y,s)\right)$.

Let $(x_n^{h,k}, y_n^{h,k}, t_n^{h,k}, s_n^{h,k})$ be such that

$$\overline{V}^{\bar{i}}(x_n^{h,k}, t_n^{h,k}) - \underline{V}^{\bar{i}}(y_n^{h,k}, s_n^{h,k}) \geq M_h^k - \frac{1}{n},$$

with $|x_n^{h,k} - y_n^{h,k}|$, $|t_n^{h,k} - s_n^{h,k}| \leq h$ and $|x_n^{h,k}|, |y_n^{h,k}| \leq \dfrac{1}{\sqrt{k}}$. We then have

$$\begin{aligned}
&M_h^k - \frac{1}{n} - \frac{h^2}{2\epsilon} - \frac{h^2}{2\beta} - \frac{\eta}{T - t_n^h} - \frac{2\alpha}{k} \\
&\leq \overline{V}^{\bar{i}}(x_n^{h,k}, t_n^{h,k}) - \underline{V}^{\bar{i}}(y_n^{h,k}, s_n^{h,k}) - \frac{|x_n^{h,k} - y_n^{h,k}|^2}{2\epsilon} - \frac{|t_n^{h,k} - s_n^{h,k}|^2}{2\beta} \\
&- \frac{\eta}{T - t_n^{h,k}} - \alpha\left(|x_n^{h,k}|^2 + |y_n^{h,k}|^2\right) \\
&\leq M(\epsilon, \beta, \eta, \alpha, \gamma) \\
&\leq \overline{V}^{\bar{i}}(\overline{x}, \overline{t}) - \underline{V}^{\bar{i}}(\overline{y}, \overline{s}).
\end{aligned}$$

We pass to the limit infimum and limit supremum in the following order: $\eta \to 0$, $h \to 0$, $\beta \to 0$, $\epsilon \to 0$, $\alpha \to 0$ and lastly $k \to 0$. Thus, using the fact that the function $\overline{V}^{\bar{i}}(x,t) - \underline{V}^{\bar{i}}(y,s)$ is upper semi-continuous and so M_h^k has a limit, we get

$$\begin{aligned}
M' - \frac{1}{n} &\leq \liminf_{\alpha\to 0}\liminf_{\epsilon\to 0}\liminf_{\beta\to 0}\liminf_{\eta\to 0}\left(\overline{V}^{\bar{i}}(\overline{x}, \overline{t}) - \underline{V}^{\bar{i}}(\overline{y}, \overline{s})\right) \\
&\leq \limsup_{\alpha\to 0}\limsup_{\epsilon\to 0}\limsup_{\beta\to 0}\limsup_{\eta\to 0}\left(\overline{V}^{\bar{i}}(\overline{x}, \overline{t}) - \underline{V}^{\bar{i}}(\overline{y}, \overline{s})\right) \\
&\leq \limsup_{\alpha\to 0}\limsup_{\epsilon\to 0}\limsup_{\beta\to 0}\left(\sup_{\substack{|x-y|\leq\sqrt{4\epsilon M_0} \\ |t-s|\leq\sqrt{4\beta M_0} \\ |x|,|y|\leq\sqrt{\frac{2M_0}{\alpha}}}}\left(\overline{V}^{\bar{i}}(x,t) - \underline{V}^{\bar{i}}(y,s)\right)\right) \\
&\leq \lim_{\substack{h'\to 0 \\ k'\to 0}}\left(\sup_{\substack{|x-y|,|t-s|\leq h' \\ |x|,|y|\leq\frac{1}{\sqrt{k'}}}}\left(\overline{V}^{\bar{i}}(x,t) - \underline{V}^{\bar{i}}(y,s)\right)\right) \\
&\leq M',
\end{aligned}$$

for some h', k' such that $\sqrt{4\epsilon M_0}, \sqrt{4\beta M_0} \leq h'$ and $k' \leq \dfrac{\alpha}{2M_0}$, where we have used (4.41) and Lemma 4.2-(1) in the third line. Then, by passing to the limit as $n \to +\infty$, we deduce (4).

In the same way we can prove (5) and (6).

Finally, to prove (7), we suppose for example $\overline{t} = 0$. From (1), we can see that $\overline{x}, \overline{y}, \overline{t}, \overline{s}$ are uniformly bounded with respect to η, β, and ϵ. Then we have, up to the extract of a subsequence

$$\lim_{(\epsilon,\beta,\eta)\to(0,0,0)} (\overline{x}, \overline{y}, \overline{t}, \overline{s}) = (\hat{z}, \hat{z}, \hat{\tau}, \hat{\tau}), \quad \lim_{(\epsilon,\beta,\eta)\to(0,0,0)} \overline{i} = \overline{\overline{i}} = \hat{i}. \tag{4.42}$$

Passing to the limit as $\eta \to 0$, $\beta \to 0$ then $\epsilon \to 0$, and using the fact that $\overline{V}^{\overline{i}}$ and $\underline{V}^{\overline{i}}$ are upper and lower semi-continuous functions respectively, we get

$$\begin{aligned} \limsup_{\epsilon\to 0} \limsup_{\beta\to 0} \limsup_{\eta\to 0} \left(\overline{V}^{\overline{i}}(\overline{x}, 0) - \underline{V}^{\overline{i}}(\overline{y}, \overline{s})\right) \leq \\ \overline{V}^{\hat{i}}(\hat{z}, 0) - \underline{V}^{\hat{i}}(\hat{z}, 0) = \overline{u}^{\hat{i}}(\hat{z}, 0) - \underline{u}^{\hat{i}}(\hat{z}, 0) = 0. \end{aligned} \tag{4.43}$$

The last equality is valid since we have

$$u_0^i(\cdot) = (u_0^i)_\star(\cdot) \leq \underline{u}^i(\cdot, 0) \leq u^i(\cdot, 0) \leq \overline{u}^i(\cdot, 0) \leq (u_0^i)^\star(\cdot) = u_0^i(\cdot) \quad \text{for every } i = 1, \ldots, d,$$

using the fact that $\overline{u}^i$ and $\underline{u}^i$ are discontinuous viscosity sub- and super- solutions of (4.1) respectively. This means that $\overline{u}^i(\cdot, 0) = \underline{u}^i(\cdot, 0) = u_0^i$. Then, from (4.43) we deduce that $M' \leq 0$. However, $M' > 0$ (see Lemma 4.2-(5)) and for $\epsilon, \beta, \eta, \alpha$ small, we get a contradiction.
A similar proof can be made if we consider the case where $\overline{s} = 0$.

Step 2. (Obtaining a contradiction):
We take ϵ, β, α and η small enough such that $\overline{t} > 0$ and $\overline{s} > 0$ (see Lemma 4.2-(7)). We can remark that the function

$$(x,t) \to \overline{V}^{\overline{i}}(x,t) - \left(\underline{V}^{\overline{i}}(\overline{y}, \overline{s}) + \frac{|x-\overline{y}|^2}{2\epsilon} + \frac{|t-\overline{s}|^2}{2\beta} + \frac{\eta}{T-t} + \alpha(|x|^2 + |\overline{y}|^2)\right)$$

reaches a maximum at $(\overline{t}, \overline{x})$. By using the test-function

$$\phi_1(x,t) = \underline{V}^{\overline{i}}(\overline{y}, \overline{s}) + \frac{|x-\overline{y}|^2}{2\epsilon} + \frac{|t-\overline{s}|^2}{2\beta} + \frac{\eta}{T-t} + \alpha(|x|^2 + |\overline{y}|^2),$$

and the fact that $\overline{u} = (\overline{u}^i)_{i=1,\ldots,d}$ is a sub-solution of (4.1) and that λ^i is continuous for every $i = 1, \dots, d$, we deduce that

$$\min\left\{\gamma \overline{V}^{\bar{i}}(\bar{t},\bar{x}) + \frac{(\bar{t}-\bar{s})}{\beta} + \frac{\eta}{(T-\bar{t})^2} - \lambda^{\bar{i}}(\bar{t},\bar{x},r(\bar{t},\bar{x}))\left|\frac{(\bar{x}-\bar{y})}{\epsilon} + 2\alpha\bar{x}\right| : \right.$$
$$\left. r \in \overline{U}(\bar{t},\bar{x}), r^i = \overline{u}^i(\bar{t},\bar{x})\right\} \leq 0.$$

Then we get

$$\gamma \overline{V}^{\bar{i}}(\bar{t},\bar{x}) + \frac{(\bar{t}-\bar{s})}{\beta} + \frac{\eta}{(T-\bar{t})^2} - \lambda^{\bar{i}}(\bar{t},\bar{x},r_1(\bar{t},\bar{x}))\left|\frac{(\bar{x}-\bar{y})}{\epsilon} + 2\alpha\bar{x}\right| \leq 0, \tag{4.44}$$

for some $r_1 = (r_1^j)_{j=1,\dots,d}$ such that $r_1^i(\bar{t},\bar{x}) = \overline{u}^i(\bar{t},\bar{x})$.

On the other hand, we deduce that the function

$$(s,y) \to \underline{V}^{\bar{i}}(s,y) - \left(\overline{V}^{\bar{i}}(\bar{t},\bar{x}) - \frac{|\bar{x}-y|^2}{2\epsilon} - \frac{|\bar{t}-s|^2}{2\beta} - \frac{\eta}{T-\bar{t}} - \alpha(|\bar{x}|^2 + |y|^2)\right)$$

reaches a minimum at $(\bar{s},\bar{y})$. By using the test-function

$$\phi_2(s,y) = \overline{V}^{\bar{i}}(\bar{t},\bar{x}) - \frac{|\bar{x}-y|^2}{2\epsilon} - \frac{|\bar{t}-s|^2}{2\beta} - \frac{\eta}{T-\bar{t}} - \alpha(|\bar{x}|^2 + |y|^2),$$

and the fact that $\underline{u} = (\underline{u}^i)_{i=1,\dots,d}$ is a super-solution of (4.1), we deduce that

$$\max\left\{\gamma \underline{V}^{\bar{i}}(\bar{s},\bar{y}) + \frac{(\bar{t}-\bar{s})}{\beta} - \lambda^{\bar{i}}(\bar{s},\bar{y},r(\bar{s},\bar{y}))\left|\frac{(\bar{x}-\bar{y})}{\epsilon} - 2\alpha\bar{y}\right| : r \in \overline{U}(\bar{s},\bar{y}), r^i = \underline{u}^i(\bar{s},\bar{y})\right\} \geq 0.$$

Then we get

$$\gamma \underline{V}^{\bar{i}}(\bar{s},\bar{y}) + \frac{(\bar{t}-\bar{s})}{\beta} - \lambda^{\bar{i}}(\bar{s},\bar{y},r_2(\bar{s},\bar{y}))\left|\frac{(\bar{x}-\bar{y})}{\epsilon} - 2\alpha\bar{y}\right| \geq 0, \tag{4.45}$$

for some $r_2 = (r_2^j)_{j=1,\dots,d}$ such that $r_2^i(\bar{s},\bar{y}) = \underline{u}^i(\bar{s},\bar{y})$.

By subtracting (4.45) from (4.44), we deduce that

$$\begin{aligned}\gamma(\overline{V}^{\bar{i}}(\bar{t},\bar{x}) - \underline{V}^{\bar{i}}(\bar{s},\bar{y})) - \lambda^{\bar{i}}(\bar{t},\bar{x},r_1(\bar{t},\bar{x}))\left|\frac{(\bar{x}-\bar{y})}{\epsilon} + 2\alpha\bar{x}\right| \\ + \lambda^{\bar{i}}(\bar{s},\bar{y},r_2(\bar{s},\bar{y}))\left|\frac{(\bar{x}-\bar{y})}{\epsilon} - 2\alpha\bar{y}\right| \leq 0.\end{aligned} \tag{4.46}$$

As we have done in (4.42), we can extract a subsequence such that

$$\lim_{(\beta,\eta)\to(0,0)} (\bar{x},\bar{y},\bar{t},\bar{s}) = (\widetilde{x},\widetilde{y},\widetilde{\tau},\widetilde{\tau}), \quad \lim_{(\beta,\eta)\to(0,0)} \bar{i} = \bar{i} = \widetilde{i}$$

Sending $\eta \to 0$ then $\beta \to 0$ in (4.46), we obtain

$$\begin{aligned}\gamma\Big(\liminf_{\beta\to 0} \liminf_{\eta\to 0} \big(\overline{V}^{\tilde{i}}(\bar{t},\bar{x}) - \underline{V}^{\tilde{i}}(\bar{s},\bar{y})\big)\Big) - \lambda^{\tilde{i}}(\tilde{\tau},\tilde{x},r_1(\tilde{\tau},\tilde{x})) \left|\frac{\tilde{x}-\tilde{y}}{\epsilon} + 2\alpha\tilde{x}\right| \\ + \lambda^{\tilde{i}}(\tilde{\tau},\tilde{y},r_2(\tilde{\tau},\tilde{y})) \left|\frac{\tilde{x}-\tilde{y}}{\epsilon} - 2\alpha\tilde{y}\right| \le 0.\end{aligned} \tag{4.47}$$

By adding and subtracting the terms $\lambda^{\tilde{i}}(\tilde{\tau},\tilde{x},r_2(\tilde{\tau},\tilde{y})) \left|\frac{\tilde{x}-\tilde{y}}{\epsilon} + 2\alpha\tilde{x}\right|$ and $\lambda^{\tilde{i}}(\tilde{\tau},\tilde{y},r_2(\tilde{\tau},\tilde{y})) \left|\frac{\tilde{x}-\tilde{y}}{\epsilon} + 2\alpha\tilde{x}\right|$ to inequality (4.47), we get

$$\begin{aligned}&\gamma\Big(\liminf_{\beta\to 0} \liminf_{\eta\to 0} \big(\overline{V}^{\tilde{i}}(\bar{t},\bar{x}) - \underline{V}^{\tilde{i}}(\bar{s},\bar{y})\big)\Big) \\ &+ \Big(\lambda^{\tilde{i}}(\tilde{\tau},\tilde{x},r_2(\tilde{\tau},\tilde{y})) - \lambda^{\tilde{i}}(\tilde{\tau},\tilde{x},r_1(\tilde{\tau},\tilde{x}))\Big) \left|\frac{\tilde{x}-\tilde{y}}{\epsilon} + 2\alpha\tilde{x}\right| \\ &+ \lambda^{\tilde{i}}(\tilde{\tau},\tilde{y},r_2(\tilde{\tau},\tilde{y})) \left(\left|\frac{\tilde{x}-\tilde{y}}{\epsilon} - 2\alpha\tilde{y}\right| - \left|\frac{\tilde{x}-\tilde{y}}{\epsilon} + 2\alpha\tilde{x}\right|\right) \\ &+ \Big(\lambda^{\tilde{i}}(\tilde{\tau},\tilde{y},r_2(\tilde{\tau},\tilde{y})) - \lambda^{\tilde{i}}(\tilde{\tau},\tilde{x},r_2(\tilde{\tau},\tilde{y}))\Big) \left|\frac{\tilde{x}-\tilde{y}}{\epsilon} + 2\alpha\tilde{x}\right| \le 0.\end{aligned} \tag{4.48}$$

From (4.40), we can see that

$$\overline{V}^{\tilde{i}}(\bar{t},\bar{x}) - \underline{V}^{\tilde{i}}(\bar{s},\bar{y}) \ge 0.$$

Thus we have

$$\begin{aligned}0 \le \overline{V}^{\tilde{i}}(\bar{t},\bar{x}) - \underline{V}^{\tilde{i}}(\bar{s},\bar{y}) \le \limsup_{\beta\to 0} \limsup_{\eta\to 0} \Big(\overline{V}^{\tilde{i}}(\bar{t},\bar{x}) - \underline{V}^{\tilde{i}}(\bar{s},\bar{y})\Big) = \overline{V}^{\tilde{i}}(\tilde{\tau},\tilde{x}) - \underline{V}^{\tilde{i}}(\tilde{\tau},\tilde{y}) \\ = e^{-\gamma\tilde{\tau}}\Big(\overline{u}^{\tilde{i}}(\tilde{\tau},\tilde{x}) - \underline{u}^{\tilde{i}}(\tilde{\tau},\tilde{y})\Big) \le \overline{u}^{\tilde{i}}(\tilde{\tau},\tilde{x}) - \underline{u}^{\tilde{i}}(\tilde{\tau},\tilde{y}).\end{aligned}$$

Then by applying (H) in the first line of (4.48), we get

$$\begin{aligned}\gamma\Big(\liminf_{\beta\to 0} \liminf_{\eta\to 0} \big(\overline{V}^{\tilde{i}}(\bar{t},\bar{x}) - \underline{V}^{\tilde{i}}(\bar{s},\bar{y})\big)\Big) + \lambda^{\tilde{i}}(\tilde{\tau},\tilde{y},r_2(\tilde{\tau},\tilde{y})) \left(\left|\frac{\tilde{x}-\tilde{y}}{\epsilon} - 2\alpha\tilde{y}\right| - \left|\frac{\tilde{x}-\tilde{y}}{\epsilon} + 2\alpha\tilde{x}\right|\right) \\ + \Big(\lambda^{\tilde{i}}(\tilde{\tau},\tilde{y},r_2(\tilde{\tau},\tilde{y})) - \lambda^{\tilde{i}}(\tilde{\tau},\tilde{x},r_2(\tilde{\tau},\tilde{y}))\Big) \left|\frac{\tilde{x}-\tilde{y}}{\epsilon} + 2\alpha\tilde{x}\right| \le 0.\end{aligned}$$

Finally, using (4.6), we obtain

$$\gamma\Big(\liminf_{\beta\to 0} \liminf_{\eta\to 0} \big(\overline{V}^{\tilde{i}}(\bar{t},\bar{x}) - \underline{V}^{\tilde{i}}(\bar{s},\bar{y})\big)\Big) - 2\alpha\Lambda^{i}(|\tilde{x}|+|\tilde{y}|) - M_1\frac{|\tilde{x}-\tilde{y}|^2}{\epsilon} - 2\alpha M_1|\tilde{x}||\tilde{x}-\tilde{y}| \le 0.$$

Passing to the limit as $\epsilon \to 0$ then $\alpha \to 0$, we get according to (1), (2), (4) and (6) in Lemma 4.2

$$\gamma M' \le 0,$$

which contradicts our supposition. Hence, we obtain the continuity of the solution.

□

Remark 4.2. Under the same conditions of Proposition 4.1, we can obtain the uniqueness of the solution by proving the following Comparison Principle

$$\text{if } u^i(\cdot,0) \leq v^i(\cdot,0) \ \text{ in } \mathbb{R} \text{ we get that } u^i \leq v^i \text{ in } \mathbb{R}\times[0,T) \text{ for every } i\in\{1,\dots,d\},$$

where $u = (u^i)_{i=1,\dots,d}$ and $v = (v^i)_{i=1,\dots,d}$ are two bounded continuous viscosity solutions of (4.1).

4 Application to dislocations dynamics

In this section, we present an application of the results proven in the previous sections to a system modeling the dynamics of dislocations densities. In other words, we will prove Theorem 4.3.

Let us mention some previous results for system (4.19). In El Hajj [39], basing on an energy estimate, the global existence and uniqueness of a non-decreasing solution in $W^{1,2}_{loc}(\mathbb{R}\times(0,T))$ has been obtained. However, without any monotony assumptions on the initial data, El Hajj and Forcadel proved in [41] the existence and uniqueness of a Lipschitz solution, using the notion of viscosity solutions. They also proposed a convergent numerical scheme and proved a Crandall-Lions error type estimate between the continuous solution and its numerical approximation. Moreover, we can point out the result done by El Hajj and Boudjerada in [19], who were able to prove the global existence of discontinuous viscosity BV solutions for scalar one dimensional nonlinear and nonlocal eikonal equations, including in particular the case $d = 1$ in system (4.1), where the velocity does not contain the solution. This result has been extended to a more general nonlinear (2×2) system in El Hajj *et al.* [43]. Further more, in El Hajj, Oussaily [48], the global existence of a continuous viscosity solution has been presented, which was based on an entropy estimate and under a control on the gradient of the solution.

We omit the proof of the existence to system (4.19), and we refer the reader to El Hajj *et al.* [43], where the global existence of a discontinuous viscosity solution has been obtained under (4.20) and assuming we have the following conditions

$$P_0^{\pm} \in L^\infty(\mathbb{T}) \cap BV(\mathbb{T}), \qquad a \in L^\infty(0,T). \tag{4.49}$$

Proof of Theorem 4.3:

It suffices to prove that system (4.19) satisfies the quasi-monotony condition (H) when $\alpha = 0$. We denote, for $\rho = (\rho^+, \rho^-)$, by

$$\lambda^+(t,x,\rho) = \rho^+(t,x) - \rho^-(t,x) + a(t), \quad \text{and} \quad \lambda^-(t,x,\rho) = -(\rho^+(t,x) - \rho^-(t,x)) + a(t).$$

Let $r = (r^+, r^-)$ and $s = (s^+, s^-)$ be two vectors such that $r^j - s^j = \max\limits_{k \in \{+,-\}} (r^k - s^k) \geq 0$. We have

$$\begin{aligned}
\lambda^j(t,x,r) - \lambda^j(t,x,s) &= j\big(r^+ - r^- + a(t)\big) - j\big(s^+ - s^- + a(t)\big) \\
&= j\left((r^+ - s^+) - (r^- - s^-)\right) \\
&= sign\left((r^+ - s^+) - (r^- - s^-)\right)\left((r^+ - s^+) - (r^- - s^-)\right) \\
&= \left|(r^+ - s^+) - (r^- - s^-)\right| \geq 0.
\end{aligned}$$

This ends the proof.

□

5 Convergent scheme approximating an eikonal system

This chapter is a submitted article that is written in collaboration with Ahmad El Hajj and Mustapha Jazar.

In this work, we propose a finite difference scheme approximating the eikonal system considered in Chapter 4. We show that a certain linear interpolation function of the discrete running points of the scheme verify the same L^∞ and BV estimates proven in the continuous problem. We also prove that this interpolation function converges to a viscosity solution of the continuous problem considered. Finally, we present some numerical illustrations to a particular case of the main eikonal system understudy.

Convergent semi-explicit scheme to a non-linear eikonal system

Maryam Al Zohbi, Ahmad El Hajj, Mustapha Jazar

Abstract

We consider a system of non-linear eikonal equations in one space dimension that describes the evolution of interfaces moving with non-signed strongly coupled velocities. We have recently proven the global existence and uniqueness of viscosity solutions for this system, under a BV estimate. In this paper, we propose a semi-explicit scheme that satisfies the same BV estimate proven in the continuous case, at the discrete level, and we show that a certain linear interpolation of the discrete solution to the scheme converges to a viscosity solution of the main system considered. We also provide some numerical simulations in the case of dislocations dynamics.

AMS Classification: 35A21, 35F50, 65N06, 35D40, 35A23.

Key words: Hamilton-Jacobi equations, non-linear eikonal system, finite difference scheme, viscosity solution, discrete gradient estimates.

1 Introduction and main results

In this paper, we present a convergence result for a semi-explicit scheme considering the framework of non-linear Hamilton-Jacobi equations. Before stating our main results, we recall first in Subsection 1.1 the setting of the continuous problem, along with the existence and uniqueness results that were recently proven in [2].

1.1 The continuous problem

We are going to propose a finite difference scheme approximating non-linear Hamilton-Jacobi systems of the form

$$\begin{cases} \partial_t v^\alpha(t,x) = \lambda^\alpha(t,x,v(t,x))|\partial_x v^\alpha(t,x)| & \text{in } (0,T)\times\mathbb{R}, \\ v^\alpha(0,x) = v_0^\alpha(x) & \text{in } \mathbb{R}, \end{cases} \tag{5.1}$$

where $T > 0$, $v(t,x) = (v^\alpha(t,x))_{\alpha=1,\dots,d}$ with $d \in \mathbb{N}^*$. The functions v^α are real valued, $\partial_t v^\alpha$ and $\partial_x v^\alpha$ represent the time and spatial derivatives of v^α respectively. We have presented in Chapter 4 two results for this system. First, we have proven the global existence of a discontinuous solution under the following condition on the velocities λ^α, for every $\alpha = 1,\dots,d$

$$\lambda^\alpha \in L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}) \quad \text{for} \quad T>0 \quad \text{and for all compact } \ \mathcal{K}\subset\mathbb{R}^d, \tag{5.2}$$

and assuming the initial data v_0^α satisfies

$$v_0^\alpha \in L^\infty(\mathbb{R})\cap BV(\mathbb{R}), \quad \text{for every } \alpha, = 1\dots,d, \tag{5.3}$$

where $BV(\mathbb{R})$ is the space of functions of bounded variations given by

$$BV(\mathbb{R}) = \Big\{ f\in L^1_{loc}(\mathbb{R});\ TV(f) < +\infty \Big\},$$

with $TV(f)$ being the total variation of f defined as

$$TV(f) = \sup\Big\{ \int_{\mathbb{R}} f(x)\phi'(x)dx;\ \phi\in C^1_c(\mathbb{R}) \text{ and } \|\phi\|_{L^\infty(\mathbb{R})}\le 1 \Big\}.$$

We take the space $BV(\mathbb{R})$ endowed with the semi-norm $|f|_{BV(\mathbb{R})} = TV(f)$. We note that BV functions are integrable functions whose distributional derivative is a finite Radon measure.

Second, we proved in Chapter 4 the existence and uniqueness of a continuous solution under (5.2), (5.3), and assuming also the initial data v_0^α are continuous functions on $\mathbb{R}$, for every $\alpha = 1,\dots,d$, and the velocities verify the following conditions

$$\left|\begin{array}{l} \lambda^\alpha \in C((0,T)\times\mathbb{R}\times\mathcal{K}) \quad \text{for} \quad T>0 \quad \text{and for all compact } \ \mathcal{K}\subset\mathbb{R}^d, \\ \\ \text{there exists } M_1>0 \text{ such that, for all } x,y\in\mathbb{R} \text{ and } t\in(0,T), \\ \\ |\lambda^\alpha(t,x,u)-\lambda^\alpha(t,y,u)| \le M_1|x-y|, \\ \\ \lambda^j(t,x,s)-\lambda^j(t,x,r)\ge 0 \ \text{ for all vectors } r=(r^\alpha)_{\alpha=1,\dots,d},\, s=(s^\alpha)_{\alpha=1,\dots,d} \text{ such that} \\ \\ r^j - s^j = \max_{\alpha\in\{1,\dots,d\}}(r^\alpha - s^\alpha)\ge 0. \end{array}\right. \tag{5.4}$$

Now, we restate formally these results in the following subsection.

1.1.1 Recall of previous results

Theorem 5.1 (***Discontinuous viscosity solutions to (5.1), [Chap. 4, Th. 4.1]***). *Assume (5.2) and (5.3) hold. Then, there exists two functions* $v_1 = (v_1^\alpha)_{\alpha=1,\dots,d}$ *and* $v_2 = (v_2^\alpha)_{\alpha=1,\dots,d}$ *which are discontinuous viscosity sub- and super- solutions of (5.1) respectively (in the sense of Definition 5.2 mentioned below). Moreover,* $v_1^\alpha(t,\cdot)$ *and* $v_2^\alpha(t,\cdot)$ *coincide almost everywhere in space, uniformly for all* $t \in [0,T)$*, satisfying the following estimates*

$$\left\| v_j^\alpha \right\|_{L^\infty((0,T)\times\mathbb{R})} \leq \left\| v_0^\alpha \right\|_{L^\infty(\mathbb{R})}, \quad \textit{for } j = 1, 2, \tag{5.5}$$

$$\left\| v_j^\alpha \right\|_{L^\infty((0,T);BV(\mathbb{R}))} \leq \left| v_0^\alpha \right|_{BV(\mathbb{R})}, \quad \textit{for } j = 1, 2. \tag{5.6}$$

Theorem 5.2 (***Existence and uniqueness of a continuous solution to (5.1), [Chap. 4, Th. 4.2]***).
Suppose that v_0^α *are continuous for every* $\alpha = 1,\dots,d$*, and that (5.2), (5.3), and (5.4) hold. Then, there exists a unique continuous viscosity solution of (5.1) satisfying (5.5) and (5.6).*

1.2 The discrete problem

In order to reconstruct the properties of the continuous problem at the discrete level, we consider the following mesh discretization

$$\Xi = \left\{ i\Delta x, i \in \mathbb{Z} \right\}, \quad \Xi_N = \left\{ 0, \dots, (\Delta t)N \right\}, \tag{5.7}$$

where N is an integer such that $\Delta t = T/N$, and Δt, Δx are positive steps of discretization. The discrete running point is (t_n, x_i) with $t_n = n\Delta t$ and $x_i = i\Delta x$. To avoid any ambiguity in notations, we denote by v^α the continuous solution and by $u_i^{\alpha,n}$ the associated discrete solution defined as an approximation of $v^\alpha(n\Delta t, i\Delta x)$. Before introducing our scheme, we first regularize the initial data v_0^α by classical convolution as follows

$$u_{|\varepsilon|}^{\alpha,0}(x) = v_0^\alpha \star \rho_{|\varepsilon|}(x), \quad \forall x \in \mathbb{R}, \tag{5.8}$$

where $\varepsilon = (\Delta t, \Delta x)$, and $\rho_{|\varepsilon|}$ is the standard mollifier defined as

$$\rho_{|\varepsilon|}(\cdot) = \frac{1}{|\varepsilon|}\rho\left(\frac{\cdot}{|\varepsilon|}\right), \text{ such that } \rho \in C_c^\infty(\mathbb{R}),\ \operatorname{supp}\{\rho\} \subseteq B(0,1),\ \rho \geq 0, \text{ and } \int_{\mathbb{R}} \rho = 1.$$

We now can introduce, for $u_i^n = (u_i^{\alpha,n})_{\alpha=1,\dots,d}$, the following numerical scheme of Lax-Friedrichs type

$$\begin{cases} \dfrac{u_i^{\alpha,n+1} - \frac{1}{2}\big(u_{i+1}^{\alpha,n} + u_i^{\alpha,n}\big)}{\Delta t} - \lambda^\alpha(t_{n+1}, x_i, u_i^{n+1})\dfrac{\big|u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\big|}{\Delta x} = 0, \\ u_i^{\alpha,0} = u_{|\varepsilon|}^{\alpha,0}(x_i), \end{cases} \quad \forall \alpha \in \{1,\dots,d\}. \tag{5.9}$$

In the following, we denote by

$$\lambda_i^{\alpha,n+1} = \lambda^\alpha(t_{n+1}, x_i, u_i^{n+1}).$$

We set $\theta_{i+\frac{1}{2}}^{\alpha,n}$ as the discrete approximation of the gradient $\partial_x u^\alpha$, given as

$$\theta_{i+\frac{1}{2}}^{\alpha,n} = \frac{u_{i+1}^{\alpha,n} - u_i^{\alpha,n}}{\Delta x}. \tag{5.10}$$

Our purpose is first to recover the properties of the solution of (5.1) at the discrete level, then to prove the convergence of the discrete solution. First, we will consider a continuous linear interpolation of the discrete points $(u_i^{\alpha,n})_{n,i}$, denoted by $u^{\alpha,\varepsilon}$ for $\varepsilon = (\Delta t, \Delta x)$. Then, we show that this function satisfies the L^∞ and the BV estimates (5.5) and (5.6). These estimates, along with the discrete finite speed propagation property (given below in Lemma 5.1) and the stability, consistency, and monotony of the scheme, allow us to show that the upper and lower relaxed semi-limits of $u^{\alpha,\varepsilon}$ (see Barles and Perthame [10, 11]), which are defined as

$$\overline{u}^\alpha(t,x) = \limsup{}^\star u^{\alpha,\varepsilon}(t,x) = \limsup_{\substack{\varepsilon \longrightarrow 0 \\ (s,y)\longrightarrow(t,x)}} u^{\alpha,\varepsilon}(s,y), \tag{5.11}$$

and

$$\underline{u}^\alpha(t,x) = \liminf{}_\star u^{\alpha,\varepsilon}(t,x) = \liminf_{\substack{\varepsilon \longrightarrow 0 \\ (s,y)\longrightarrow(t,x)}} u^{\alpha,\varepsilon}(s,y), \tag{5.12}$$

are, respectively, discontinuous viscosity sub- and super- solutions of system (5.1) in the sense of discontinuous viscosity solutions introduced by Ishii in [58, Definition 2.1] for Hamilton Jacobi systems that is recalled below in Definition 5.2. Moreover, we will be able to prove that $\overline{u}^\alpha(t,\cdot)$ and $\underline{u}^\alpha(t,\cdot)$ coincide almost everywhere in space and uniformly in time for all $t \in [0,T)$, as in Theorem 5.1. Finally, in the case where system (5.1) verifies the comparison principle, i.e. under the conditions of Theorem 5.2, we will be able to prove that $u^{\alpha,\varepsilon}$ converges to the unique continuous solution of (5.1).

1.3 Main results

We set $u_i^n = (u_i^{\alpha,n})_{\alpha=1,\dots,d}$, $u^n = (u_i^n)_{i\in\mathbb{Z}}$. We introduce the box

$$\mathcal{U} = \prod_{\alpha=1}^{d} \left[- \|v_0^\alpha\|_{L^\infty(\mathbb{R})}, \|v_0^\alpha\|_{L^\infty(\mathbb{R})}\right], \tag{5.13}$$

and we say that $u^n \in \mathcal{U}^{\mathbb{Z}}$ if $u_i^n \in \mathcal{U}$ for all $i \in \mathbb{Z}$. Also, we assume the velocities λ^α verify the following condition

$$\begin{cases} \text{there exists } M > 0 \text{ such that} \\ \displaystyle\sum_{\alpha=1}^{d} |\lambda^\alpha(t,x,u) - \lambda^\alpha(t,x,v)| \leq M|u-v|, \quad \text{for all } u,v \in \mathbb{R}^d, \end{cases} \tag{5.14}$$

where $|w| = \displaystyle\sum_{\alpha=1}^{d} |w^\alpha|$, for $w = (w_1, \dots, w_d)$.

Next, we assume that

$$\frac{\Delta t}{\Delta x} = \min\left(\frac{1}{2\Lambda}, \frac{1}{2M\|v_0\|_{(L^\infty(\mathbb{R}))^d}}\right) = \gamma, \tag{5.15}$$

where $\|v_0\|_{(L^\infty(\mathbb{R}))^d} = \displaystyle\sum_{\alpha=1}^{d} \|v_0^\alpha\|_{L^\infty(\mathbb{R})}$, and

$$\Lambda = \sup_{\alpha\in\{1,\dots,d\}} \|\lambda^\alpha\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{U})}. \tag{5.16}$$

Theorem 5.3 (***Existence of*** BV ***discrete solution***)**.**
Assume (5.2), (5.3), (5.14), and (5.15) hold. Then we have

i) (Existence)
Let $u^n \in \mathcal{U}^{\mathbb{Z}}$. Then there exists a unique solution $u^{n+1} \in \mathcal{U}^{\mathbb{Z}}$ to the semi-explicit scheme (5.9).

(ii) (Discrete BV ***estimate)***
The discrete gradient $\theta_{i+\frac{1}{2}}^{\alpha,n}$, defined in (5.10), verifies the following estimate

$$\sum_{i\in\mathbb{Z}} \left|\theta_{i+\frac{1}{2}}^{\alpha,n+1}\right| \leq \sum_{i\in\mathbb{Z}} \left|\theta_{i+\frac{1}{2}}^{\alpha,n}\right|, \qquad \text{for } n = 0, \dots, N-1. \tag{5.17}$$

Theorem 5.4 (***Convergence of the solution of the numerical scheme***)**.**
Assume (5.2), (5.3), (5.14), and (5.15) are satisfied. Consider the solution $(u^n)_{n=0,\dots,N}$ of

the scheme (5.9) for the time step Δt and the space step Δx. Let us denote by $\varepsilon = (\Delta t, \Delta x)$ and u^ε a continuous linear interpolation function defined as

$$u^\varepsilon(n\Delta t, i\Delta x) = u_i^n, \quad \text{for} \;\; n = 0, \dots, N, \quad i \in \mathbb{Z}.$$

Then the following points hold

***i) Estimates on** u^ε*

The function $u^\varepsilon = (u^{\alpha,\varepsilon})_{\alpha=1,\dots,d}$ verifies

$$\|u^{\alpha,\varepsilon}\|_{L^\infty((0,T)\times\mathbb{R})} \le \|v_0^\alpha\|_{L^\infty(\mathbb{R})}, \tag{5.18}$$

$$\|u^{\alpha,\varepsilon}\|_{L^\infty((0,T);BV(\mathbb{R}))} \le |v_0^\alpha|_{BV(\mathbb{R})}, \tag{5.19}$$

$$\|\partial_t u^{\alpha,\varepsilon}\|_{L^\infty((0,T);L^1(\mathbb{R}))} \le \left(1 + \frac{2}{\gamma} + \Lambda\right) |v_0^\alpha|_{BV(\mathbb{R})}. \tag{5.20}$$

ii) Convergence

The upper and lower relaxed semi-limits of $u^{\alpha,\varepsilon}$, defined in (5.11) and (5.12), are a couple of discontinuous viscosity sub- and super-solutions of (5.1) (in the sense of Definition 5.2).

***iii) Equality between** $\overline{u}^\alpha$ **and** $\underline{u}^\alpha$*

Assume $u^{\alpha,\varepsilon}$ satisfies (5.18), (5.19) and (5.20) for every $\alpha = 1, \dots, d$. Then, up to the extract of a subsequence, the function $u^{\alpha,\varepsilon}$ converges, as $\varepsilon \to 0$, to a function

$$u^\alpha \in L^\infty\big((0,T)\times\mathbb{R}\big) \cap L^\infty\big((0,T);\, BV(\mathbb{R})\big) \cap C\left([0,T);\, L^1_{loc}(\mathbb{R})\right), \tag{5.21}$$

strongly in $C\left([0,T);\, L^1_{loc}(\mathbb{R})\right)$.

Moreover, u^α satisfies, for all $T > 0$ and for $\alpha = 1, \dots, d$, estimates (5.5), (5.6) and the following equality

$$u^\alpha(t,\cdot) = \overline{u}^\alpha(t,\cdot) = \underline{u}^\alpha(t,\cdot), \;\; \text{except at most on a countable set in } \mathbb{R}, \;\; \text{for all } t \in [0,T). \tag{5.22}$$

iv) Unique solution

If (5.4) is also satisfied, and v_0^α are continuous for every $\alpha = 1, \dots, d$, then $u^{\alpha,\varepsilon}$ converges to the unique solution of (5.1).

Remark 5.1. It is possible to prove a similar result, as in Theorem 5.4-(i),(ii),(iii), to the case of transport systems of the form

$$\begin{cases} \partial_t v^\alpha(t,x) = \lambda^\alpha(t,x,v(t,x))\partial_x v^\alpha(t,x) & \text{in } \; (0,T)\times\mathbb{R}, \\ v^\alpha(0,x) = v_0^\alpha(x) & \text{in } \; \mathbb{R}, \end{cases} \tag{5.23}$$

under assumptions (5.2) and (5.3). In other words, if we consider the scheme defined in (5.9) without the absolute value, we can show that it converges to (5.23), and that it also satisfies the same estimates proven on the discontinuous solution of (5.23), which is constructed in Al Zohbi, El Hajj, Jazar [3].

1.4 A brief review of some related literature

Let us mention some numerical results known in the framework of Hamilton-Jacobi equations. In [4, 5], Alvarez et al. proved the convergence of explicit finite difference schemes approximating a non-local eikonal Hamilton-Jacobi equation, using the notion of monotone numerical Hamiltonians introduced by Osher and Sethian [76]. Also, we can refer to the work of Souganidis [81], where convergence of general approximation finite difference schemes to first order Hamilton-Jacobi equations is discussed, provided with explicit error estimates.

We note that in the case of non-decreasing solutions, system (5.1) becomes a hyperbolic system. Numerical schemes for such systems are mainly written in the case where the system is of conservative form, which enables one to recover the correct Rankine-Hugonoit shock relations. We refer to Leveque [67] for a review of the main classes of the existing schemes.
For non-conservative hyperbolic systems, Monasse and Monneau proved in [73] a convergence result of a semi-explicit scheme for a diagonal hyperbolic system assuming that it is strictly hyperbolic, and using a discrete gradient entropy estimate that was proven in the continuous case [46]. This result was done in the framework of vanishing viscosity solutions, introduced by Bianchini and Bressan in [17]. Also, Boudjerada et al. recently shown in [20] the convergence of an implicit scheme, based on some Lipschitz discrete estimates, to the same problem considered in [46], but assuming the system is not necessarily strictly hyperbolic.

It is important to mention that the case where $d = 2$ in system (5.1) can be a model to the dynamics of dislocations densities, which is the study of the movement of microscopic defects in materials. In this framework, El Hajj and Forcadel proved in [41] a convergence result of an explicit scheme to the Lipschitz continuous solution of the system considered. There work was inspired by Alvarez et al. [5], in proving a Crandall-Lions [38] rate of convergence estimate. We also mention that recently El Hajj and Oussaily have shown the convergence of an implicit scheme to the same system considered in [41], assuming the initial data are non-decreasing functions. Their result was based on an L^2 gradient estimate, that was proven in the continuous case, and endowed to the scheme considered.

1.5 Organization of the paper

This paper is organized as follows: in Section 2, we present the discrete estimates on the solution of (5.9) and on the interpolation function of the discrete running points $u_i^{\alpha,n}$, as it was announced in Theorem 5.3 and Theorem 5.4-(i). Then in Section 3, using the discrete finite speed propagation property and the discrete estimates proven on $u^{\alpha,\varepsilon}$, we prove Theorem 5.4-(ii) and (iv). After that, by passing to the limit and using again the discrete finite speed propagation property, we finish the proof of Theorem 5.4-(iii), in Section 4. Finally, in Section 5, we present some numerical simulations to a particular case of (5.1) modeling the dynamics of dislocations densities.

2 Existence of BV discrete solution to (5.9)

In this section, we prove Theorem 5.3 and Theorem 5.4-(i). The proof of Theorem 5.3-(i) is achieved using the Fixed Point Theorem in Banach spaces. Whereas, to prove Theorem 5.3-(ii) and Theorem 5.4-(i), we will first derive an evolution in time equation satisfied by the discrete gradient $\theta_{i+\frac{1}{2}}^{\alpha,n}$, that was defined in (5.10). Indeed, if we consider the same time discretization of $\theta_{i+\frac{1}{2}}^{\alpha,n}$ as we did for $u_i^{\alpha,n}$ in (5.9), we can observe that $\theta_{i+\frac{1}{2}}^{\alpha,n}$ satisfies the following equation

$$\theta_{i+\frac{1}{2}}^{\alpha,n} = \frac{1}{2}\left(\theta_{i+\frac{3}{2}}^{\alpha,n} + \theta_{i+\frac{1}{2}}^{\alpha,n}\right) + \frac{\Delta t}{\Delta x}\lambda_{i+1}^{\alpha,n+1}\left|\theta_{i+\frac{3}{2}}^{\alpha,n}\right| - \frac{\Delta t}{\Delta x}\lambda_i^{\alpha,n+1}\left|\theta_{i+\frac{1}{2}}^{\alpha,n}\right|. \tag{5.24}$$

We now proceed to the proof of Theorem 5.3.

Proof of Theorem 5.3.

Proof of (i):

We define for all $v = (v_i^\alpha)_{\alpha\in\{1,\cdots,d\},i\in\mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$, and $\mathcal{U}^{\mathbb{Z}}$ was introduced after (5.13), the mapping

$$F_{u_i^n,u_{i+1}^n} : \mathcal{U}^{\mathbb{Z}} \longrightarrow \mathcal{U}^{\mathbb{Z}}$$

$$v \longrightarrow F_{u_i^n,u_{i+1}^n}(v) = \left(F^\alpha_{u_i^n,u_{i+1}^n}(v)\right)_{\alpha=1,\ldots,d},$$

such that

$$F^\alpha_{u_i^n,u_{i+1}^n}(v) = \frac{1}{2}\left(u_{i+1}^{\alpha,n} + u_i^{\alpha,n}\right) + \frac{\Delta t}{\Delta x}\lambda^\alpha(t_{n+1}, x_i, v_i)\left|u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\right|. \tag{5.25}$$

According to (5.25), we can see that scheme (5.9) can be written as

$$\begin{cases} u_i^{\alpha,n+1} = F^{\alpha}_{u_i^n,u_{i+1}^n}(u^{n+1}), \quad \text{for } i \in \mathbb{Z}, n \in \{0,\dots,N-1\}, \text{ and } u^n \in \mathcal{U}^{\mathbb{Z}}, \\ u_i^{\alpha,0} = u_{|\varepsilon|}^{\alpha,0}(x_i). \end{cases} \tag{5.26}$$

We aim now to show that the mapping $F_{u_i^n,u_{i+1}^n}$ is a well-defined contraction on $\mathcal{U}^{\mathbb{Z}}$. This enables us to deduce, by the Fixed Point Theorem in Banach spaces, the existence of a unique fixed point to $F_{u_i^n,u_{i+1}^n}$ in $\mathcal{U}^{\mathbb{Z}}$, which is the unique solution of (5.9). Thus to prove so, we proceed in two steps.

Step 1. ($F_{u_i^n,u_{i+1}^n}$ is well-defined):
Let $v = (v_i^\alpha)_{\alpha\in\{1,\cdots,d\},i\in\mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$. From (5.25), we can remark that

$$\begin{aligned} F^{\alpha}_{u_i^n,u_{i+1}^n}(v) = &\left(\frac{1}{2} + \frac{\Delta t}{\Delta x}\lambda^\alpha(t_{n+1},x_i,v_i)\,\text{sign}\left(u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\right)\right) u_{i+1}^{\alpha,n} \\ &+ \left(\frac{1}{2} - \frac{\Delta t}{\Delta x}\lambda^\alpha(t_{n+1},x_i,v_i)\,\text{sign}\left(u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\right)\right) u_i^{\alpha,n}. \end{aligned}$$

Using (5.15), we can clearly see that $F_{u_i^n,u_{i+1}^n}(v)$ is a convex combination of u_{i+1}^n and u_i^n, which are both in $\mathcal{U}$. Hence, we deduce that $F_{u_i^n,u_{i+1}^n}(v)$ is in $\mathcal{U}^{\mathbb{Z}}$.

Step 2. ($F_{u_i^n,u_{i+1}^n}$ is a contraction):
We equip $\mathcal{U}^{\mathbb{Z}}$ with the following norm $(l^\infty)^d$

$$\|v\|_{(l^\infty)^d} = \sum_{\alpha=1}^{d} \sup_{i\in\mathbb{Z}} |v_i^\alpha|\,.$$

We remark, for all $\alpha = 1,\dots,d$ and $v = (v_i^\alpha)_{\alpha\in\{1,\cdots,d\},i\in\mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$, $w = (w_i^\alpha)_{\alpha\in\{1,\cdots,d\},i\in\mathbb{Z}} \in \mathcal{U}^{\mathbb{Z}}$, that

$$\begin{aligned} \left\|F_{u_i^n,u_{i+1}^n}(v) - F_{u_i^n,u_{i+1}^n}(w)\right\|_{(l^\infty)^d} &= \sum_{\alpha=1}^{d} \sup_{i\in\mathbb{Z}} \left|F^{\alpha}_{u_i^n,u_{i+1}^n}(v) - F^{\alpha}_{u_i^n,u_{i+1}^n}(w)\right| \\ &\le \frac{\Delta t}{\Delta x}\sum_{\alpha=1}^{d} \sup_{i\in\mathbb{Z}} \left|u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\right| \left|\lambda^\alpha(t_{n+1},x_i,v_i) - \lambda^\alpha(t_{n+1},x_i,w_i)\right| \\ &\le 2\frac{\Delta t}{\Delta x} M \|v_0\|_{(L^\infty(\mathbb{R}))^d} \|v-w\|_{(l^\infty)^d}\,, \end{aligned}$$

where we have used (5.14) in the last line. Thus, under (5.15), we deduce that $F_{u_{i+1}^n,u_i^n}$ is a contraction on $\mathcal{U}^{\mathbb{Z}}$.

Therefore, by the Fixed Point Theorem, we deduce that there exists a unique solution of (5.9) belonging to $\mathcal{U}^{\mathbb{Z}}$.

Proof of (ii):
We will prove estimate (5.17) by recurrence, as we have from (5.6) that $\sum\limits_{i\in\mathbb{Z}} \theta^{\alpha,0}_{i+\frac{1}{2}}$ is finite for every $i \in \mathbb{Z}$. Assume that the sum $\sum\limits_{i\in\mathbb{Z}} \theta^{\alpha,n}_{i+\frac{1}{2}}$ is bounded for every $i \in \mathbb{Z}$. Let $\phi \in C^\infty(\mathbb{R})$ be a cut-off function taking values into $[0,1]$, and supported by the interval $[-2,2]$, with $\phi \equiv 1$ on $[-1,1]$. Multiplying (5.24) by the function $\phi^R_i = \phi^R(x_i) = \phi\left(\frac{x_i}{R}\right)$ for $R>0$, we get

$$\begin{aligned}
\theta^{\alpha,n+1}_i \phi^R_i &= \frac{1}{2}\left(\theta^{\alpha,n}_{i+\frac{3}{2}} + \theta^{\alpha,n}_{i+\frac{1}{2}}\right)\phi^R_i + \frac{\Delta t}{\Delta x}\lambda^{\alpha,n+1}_{i+1}\left|\theta^{\alpha,n}_{i+\frac{3}{2}}\right|\phi^R_i - \frac{\Delta t}{\Delta x}\lambda^{\alpha,n+1}_{i}\left|\theta^{\alpha,n}_{i+\frac{1}{2}}\right|\phi^R_i \\
&= \theta^{\alpha,n}_{i+\frac{3}{2}}\left(\frac{1}{2} + \frac{\Delta t}{\Delta x}\lambda^{\alpha,n+1}_{i+1}\operatorname{sign}\left(\theta^{\alpha,n}_{i+\frac{3}{2}}\right)\right)\phi^R_i + \theta^{\alpha,n}_{i+\frac{1}{2}}\left(\frac{1}{2} - \frac{\Delta t}{\Delta x}\lambda^{\alpha,n+1}_{i}\operatorname{sign}\left(\theta^{\alpha,n}_{i+\frac{1}{2}}\right)\right)\phi^R_i.
\end{aligned}$$

Using (5.15), we can obtain

$$\begin{aligned}
\left|\theta^{\alpha,n+1}_{i+\frac{1}{2}}\phi^R_i\right| &\le \frac{1}{2}\left(\left|\theta^{\alpha,n}_{i+\frac{3}{2}}\phi^R_i\right| + \left|\theta^{\alpha,n}_{i+\frac{1}{2}}\phi^R_i\right|\right) + \frac{\Delta t}{\Delta x}\lambda^{\alpha,n+1}_{i+1}\ \theta^{\alpha,n}_{i+\frac{3}{2}}\ \phi^R_i - \frac{\Delta t}{\Delta x}\lambda^{\alpha,n+1}_{i}\ \theta^{\alpha,n}_{i+\frac{1}{2}}\ \phi^R_i \\
&= \frac{1}{2}\Big(\left|\theta^{\alpha,n}_{i+\frac{3}{2}}\left(\phi^R_i - \phi^R_{i+1} + \phi^R_{i+1}\right)\right| + \left|\theta^{\alpha,n}_{i+\frac{1}{2}}\phi^R_i\right|\Big) - \frac{\Delta t}{\Delta x}\lambda^{\alpha,n+1}_{i}\ \theta^{\alpha,n}_{i+\frac{1}{2}}\ \phi^R_i \\
&+ \frac{\Delta t}{\Delta x}\lambda^{\alpha,n+1}_{i+1}\ \theta^{\alpha,n}_{i+\frac{3}{2}}\ \left(\phi^R_i - \phi^R_{i+1} + \phi^R_{i+1}\right) \\
&\le \frac{1}{2}\left(\left|\theta^{\alpha,n}_{i+\frac{3}{2}}\phi^R_{i+1}\right| + \left|\theta^{\alpha,n}_{i+\frac{1}{2}}\phi^R_i\right|\right) + \frac{1}{2}\left|\theta^{\alpha,n}_{i+\frac{3}{2}}\left(\phi^R_i - \phi^R_{i+1}\right)\right| + \frac{\Delta t}{\Delta x}\lambda^{\alpha,n+1}_{i+1}\ \theta^{\alpha,n}_{i+\frac{3}{2}}\ \left(\phi^R_i - \phi^R_{i+1}\right) \\
&+ \frac{\Delta t}{\Delta x}\left(\lambda^{\alpha,n+1}_{i+1}\ \theta^{\alpha,n}_{i+\frac{3}{2}}\ \phi^R_{i+1} - \lambda^{\alpha,n+1}_{i}\ \theta^{\alpha,n}_{i+\frac{1}{2}}\ \phi^R_i\right).
\end{aligned}$$

Summing over $i \in \mathbb{Z}$ on the right side of the previous inequality and over $i \in I^R$ on the left side, where $I^R = \{i \in \mathbb{Z}/x_i \in [-2R, 2R]\}$, we obtain, as the last term in the previous inequality vanishes, that

$$\sum_{i\in I^R}\left|\theta^{\alpha,n+1}_{i+\frac{1}{2}}\phi^R_i\right| \le \sum_{i\in\mathbb{Z}}\left|\theta^{\alpha,n}_{i+\frac{1}{2}}\phi^R_i\right| + \frac{1}{2}\underbrace{\sum_{i\in\mathbb{Z}}\left|\theta^{\alpha,n}_{i+\frac{3}{2}}\left(\phi^R_i - \phi^R_{i+1}\right)\right|}_{K_1} + \frac{\Delta t}{\Delta x}\underbrace{\sum_{i\in\mathbb{Z}}\lambda^{\alpha,n+1}_{i+1}\ \theta^{\alpha,n}_{i+\frac{3}{2}}\left(\phi^R_i - \phi^R_{i+1}\right)}_{k_2}. \tag{5.27}$$

We wish now to bound K_1 and K_2. By applying the Mean Value Theorem over each interval $[x_i, x_{i+1}]$, we get using (5.5) and the properties of mollifiers that

$$K_1 \le \frac{\Delta x}{R}\,\|\phi'\|_{L^\infty(\mathbb{R})}\sum_{i\in\mathbb{Z}}\left|\theta^{\alpha,n}_{i+\frac{1}{2}}\right|,$$

and

$$K_2 \leq \Lambda \frac{\Delta x}{R} \left\| \phi' \right\|_{L^\infty(\mathbb{R})} \sum_{i \in \mathbb{Z}} \left| \theta^{\alpha,n}_{i+\frac{1}{2}} \right|.$$

Then applying these bounds on K_1 and K_2, and multiplying by Δx gives

$$\sum_{i \in I^R} \left| \theta^{\alpha,n+1}_{i+\frac{1}{2}} \phi^R_i \right| \Delta x \leq \left(1 + \frac{\Delta x}{2R} \left\| \phi' \right\|_{L^\infty(\mathbb{R})} + \Lambda \frac{\Delta t}{R} \left\| \phi' \right\|_{L^\infty(\mathbb{R})} \right) \sum_{i \in \mathbb{Z}} \left| \theta^{\alpha,n}_{i+\frac{1}{2}} \right| \Delta x,$$

where we have also used the fact that $\phi^R \leq 1$.

Therefore, by using Monotone Convergence Theorem, we pass to the limit as $R \to +\infty$ to obtain

$$\sum_{i \in \mathbb{Z}} \left| \theta^{\alpha,n+1}_{i+\frac{1}{2}} \right| \Delta x \leq \sum_{i \in \mathbb{Z}} \left| \theta^{\alpha,n}_{i+\frac{1}{2}} \right| \Delta x.$$

□

In the following subsection, we give the proof of Theorem 5.4-(i) by introducing first a linear interpolation fucntion $u^{\alpha,\varepsilon}$, for $\varepsilon = (\Delta t, \Delta x)$, of the discrete points $u^{\alpha,n}_i$ for every $\alpha \in \{1, \dots, d\}$, $n \in \{0, \dots, N\}$, and $i \in \mathbb{Z}$.

2.1 The Q^1-extension u^ε

Let $(t, x) \in [0, T] \times \mathbb{R}$. Then, there exists $i \in \mathbb{Z}$ and $n \in \{0, \dots, N-1\}$ such that $(t, x) \in [t_n, t_{n+1}] \times [x_i, x_{i+1}]$, where $x_i = i\Delta x$ and $t_n = n\Delta t$. For $\varepsilon = (\Delta t, \Delta x)$, we define the Q^1-extension of the function defined on the grid, for any $(t, x) \in [t_n, t_{n+1}] \times [x_i, x_{i+1}]$, by

$$\begin{aligned} u^\varepsilon(t, x) = & \left(\frac{t - t_n}{\Delta t} \right) \left[\left(\frac{x - x_i}{\Delta x} \right) u^{n+1}_{i+1} + \left(1 - \frac{x - x_i}{\Delta x} \right) u^{n+1}_i \right] \\ & + \left(1 - \frac{t - t_n}{\Delta t} \right) \left[\left(\frac{x - x_i}{\Delta x} \right) u^n_{i+1} + \left(1 - \frac{x - x_i}{\Delta x} \right) u^n_i \right] = \left(u^{\alpha,\varepsilon}(t, x) \right)_{\alpha=1,\dots,d}. \end{aligned} \tag{5.28}$$

In particular, we can see that

$$u^\varepsilon(t_n, x_i) = u^n_i, \quad \text{for } n \in \{0, \dots, N\}, i \in \mathbb{Z}.$$

We shall now proceed to the proof of the first point in Theorem 5.4.

Proof of Theorem 5.4-(i).

- **L^∞ estimate on u^ε:**

From Theorem 5.3-(i), we deduce that the terms u_{i+1}^{n+1}, u_i^{n+1}, u_{i+1}^n, and u_i^n are all in $\mathcal{U}$, for all $i \in \mathbb{Z}$ and for all $n \in \{0, \dots, N\}$. From (5.28), we remark that u^ε is a convex combination of u_{i+1}^{n+1}, u_i^{n+1}, u_{i+1}^n. Then $u^\varepsilon \in \mathcal{U}$, which implies, for every $\alpha = 1, \dots, d$, that

$$\|u^{\alpha,\varepsilon}\|_{L^\infty((0,T)\times\mathbb{R})} \leq \|v_0^\alpha\|_{L^\infty(\mathbb{R})}.$$

• BV **estimate on** u^ε:

We have for $(t, x) \in [t_n, t_{n+1}] \times [x_i, x_{i+1}]$

$$\partial_x u^\varepsilon(t, x) = \left(\frac{t - t_n}{\Delta t}\right) \theta_{i+\frac{1}{2}}^{n+1} + \left(1 - \frac{t - t_n}{\Delta t}\right) \theta_{i+\frac{1}{2}}^n,$$

which implies, using (5.17), the fact that $u_{|\varepsilon|}^{\alpha,0}$ is a smooth function and the properties of mollifiers, that

$$\begin{aligned}
\int_{\mathbb{R}} |\partial_x u^{\alpha,\varepsilon}(t, x)|\, dx &\leq \left(\frac{t - t_n}{\Delta t}\right) \sum_{i\in\mathbb{Z}} \left|\theta_{i+\frac{1}{2}}^{\alpha,n+1}\right| \Delta x + \left(1 - \frac{t - t_n}{\Delta t}\right) \sum_{i\in\mathbb{Z}} \left|\theta_{i+\frac{1}{2}}^{\alpha,n}\right| \Delta x \\
&\leq \sum_{i\in\mathbb{Z}} \left|\theta_{i+\frac{1}{2}}^{\alpha,0}\right| \Delta x \leq \sum_{i\in\mathbb{Z}} \left|u_{|\varepsilon|}^{\alpha,0}(x_{i+1}) - u_{|\varepsilon|}^{\alpha,0}(x_i)\right| \\
&\leq \sum_{i\in\mathbb{Z}} \int_{x_i}^{x_{i+1}} \left|\partial_x u_{|\varepsilon|}^{\alpha,0}(x)\right| dx = \int_{\mathbb{R}} \left|\partial_x u_{|\varepsilon|}^{\alpha,0}(x)\right| dx \\
&= \left|u_{|\varepsilon|}^{\alpha,0}\right|_{BV(\mathbb{R})} \leq |v_0^\alpha|_{BV(\mathbb{R})}.
\end{aligned}$$

Hence, we obtain

$$\|u^{\alpha,\varepsilon}\|_{L^\infty((0,T);BV(\mathbb{R}))} \leq |v_0^\alpha|_{BV(\mathbb{R})}.$$

• **Time regularity for** u^ε:

From (5.28), we get

$$\partial_t u^{\alpha,\varepsilon}(t, x) = \frac{1}{\Delta t}\left(\frac{x - x_i}{\Delta x}\right)\left(u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n+1}\right) - \frac{1}{\Delta t}\left(\frac{x - x_i}{\Delta x}\right)\left(u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\right) + \frac{1}{\Delta t}\left(u_i^{\alpha,n+1} - u_i^{\alpha,n}\right).$$

Inserting (5.9) in this equality yields

$$\begin{aligned}
\partial_t u^{\alpha,\varepsilon}(t, x) &= \frac{1}{\Delta t}\left(\frac{x - x_i}{\Delta x}\right)\left(u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n+1}\right) - \frac{1}{\Delta t}\left(\frac{x - x_i}{\Delta x}\right)\left(u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\right) \\
&\quad + \frac{1}{\Delta x}\lambda_i^{\alpha,n+1}\left|u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\right| + \frac{1}{2}\left(u_{i+1}^{\alpha,n} - u_i^{\alpha,n}\right),
\end{aligned}$$

which implies, using the discrete BV estimate (5.17) and (5.15)

$$\int_{\mathbb{R}} |\partial_t u^{\alpha,\varepsilon}(t, x)| \leq \left(\frac{2}{\gamma} + \Lambda + \frac{\Delta x}{2}\right) \sum_{i\in\mathbb{Z}} \left|\theta_{i+\frac{1}{2}}^{\alpha,0}\right| \Delta x.$$

Finally, using the properties of mollifiers, and the fact that $\frac{\Delta x}{2}$ is very small, we obtain (5.20).

3 Discontinuous viscosity sub and super solutions

In this section, we prove that the upper and lower relaxed semi-limits of u^ε are, respectively, discontinuous viscosity sub- and super-solutions of (5.1), as it was announced in Theorem 5.4-(ii). Also, we will show that in the case where system (5.1) verifies a comparison principle, then u^ε converges to the unique solution of (5.1).
This section is divided into two parts. In Subsection 3.1, we introduce some useful results and definitions. Then, in Subsection 3.2, we give the proof of Theorem 5.4-(ii).

3.1 Some useful results

We begin with the following discrete finite speed propagation property, valid on the interpolation function defined in (5.28).

Lemma 5.1 (***Discrete finite speed propagation property*).**
The function u^ε, defined in (5.28) verifies for all $n_0 \geq 0$, the following estimate

$$\min_{|x_j - x_i| \leq \overline{\gamma} t_{n_0}} u^{\varepsilon,\alpha}(t_n, x_j) \leq u^{\varepsilon,\alpha}(t_{n+n_0}, x_i) \leq \max_{|x_j - x_i| \leq \overline{\gamma} t_{n_0}} u^{\varepsilon,\alpha}(t_n, x_j), \tag{5.29}$$

where $\overline{\gamma} = \dfrac{1}{\gamma}$ and γ is defined in (5.15).

Proof of Lemma 5.1.
We will only prove the right inequality, as the left one can be proved analogously. From (5.9) and (5.15), we have

$$\begin{aligned}
u_i^{\alpha,n+1} &= \left(\frac{1}{2} + \frac{\Delta t}{\Delta x}\lambda_i^{\alpha,n+1}\,\mathrm{sign}\left(u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n}\right)\right) u_{i+1}^{\alpha,n} \\
&+ \left(\frac{1}{2} - \frac{\Delta t}{\Delta x}\lambda_i^{\alpha,n+1}\,\mathrm{sign}\left(u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n}\right)\right) u_i^{\alpha,n} \\
&\leq \left(\frac{1}{2} + \frac{\Delta t}{\Delta x}\lambda_i^{\alpha,n+1}\,\mathrm{sign}\left(u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n}\right)\right) \max_{|j-i|\leq 1} u_j^{\alpha,n} \\
&+ \left(\frac{1}{2} - \frac{\Delta t}{\Delta x}\lambda_i^{\alpha,n+1}\,\mathrm{sign}\left(u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n}\right)\right) \max_{|j-i|\leq 1} u_j^{\alpha,n} \\
&\leq \max_{|j-i|\leq 1} u_j^{\alpha,n}.
\end{aligned}$$

By recurrence, we get

$$u_i^{\alpha,n+n_0} \leq \max_{|j-i|\leq n_0} u_j^{\alpha,n},$$

which is equivalent to

$$u^{\varepsilon,\alpha}(t_n + t_{n_0}, x_i) \leq \max_{|j-i|\leq n_0} u^{\varepsilon,\alpha}(t_n, x_j).$$

Now, as $x_i = i\Delta x$ and $t_n = n\Delta t$, we get

$$u^{\varepsilon,\alpha}(t_n + t_{n_0}, x_i) \leq \max_{|x_j - x_i|\leq \frac{\Delta x}{\Delta t} t_{n_0}} u^{\varepsilon,\alpha}(t_n, x_j).$$

Using (5.15) in the previous inequality yields

$$u^{\varepsilon,\alpha}(t_n + t_{n_0}, x_i) \leq \max_{|x_j - x_i|\leq \frac{1}{\gamma} t_{n_0}} u^{\varepsilon,\alpha}(t_n, x_j).$$

□

Now, we will recall the definitions of monotony, stability, and consistency for scheme (5.9). We denote by $f^\star$ and $f_\star$ the respective upper and lower semi-continuous envelopes of a locally bounded function f defined on an open domain in $\mathbb{R}^n$ and given by

$$f^\star(X) = \limsup_{Y\to X} f(Y) \quad \text{and} \quad f_\star(X) = \liminf_{Y\to X} f(Y) \quad \text{for } X \in \mathbb{R}^n. \tag{5.30}$$

Consider a point (t_n, x_i) in the grid $\Xi \times \Xi_N$, introduced in (5.7). From (5.9) and (5.28), we can define the following operator

$$S_\varepsilon^\alpha(t_{n+1}, x_i, u_i^{n+1}, u^{\varepsilon,\alpha}) := \frac{1}{\Delta t}\left(u_i^{\alpha,n+1} - \frac{1}{2}\left(u^{\alpha,\varepsilon}(t_{n+1} - \Delta t, x_i + \Delta x) + u^{\alpha,\varepsilon}(t_{n+1} - \Delta t, x_i)\right)\right)$$
$$- \lambda^\alpha(t_{n+1}, x_i, u_i^{n+1}))\frac{|u^{\alpha,\varepsilon}(t_{n+1} - \Delta t, x_i + \Delta x) - u^{\alpha,\varepsilon}(t_{n+1} - \Delta t, x_i)|}{\Delta x}. \tag{5.31}$$

We can see then that scheme (5.9) is equivalent to

$$S_\varepsilon^\alpha(t_{n+1}, x_i, u_i^{n+1}, u^{\varepsilon,\alpha}) = 0. \tag{5.32}$$

Definition 5.1 (***Monotony, Stability, and Consistency***).
We say that the numerical scheme (5.9), is

- **Monotone:** if and only if

$$S_\varepsilon^\alpha(t, x, u, \psi_1) \leq S_\varepsilon^\alpha(t, x, u, \psi_2), \quad \text{for every } \psi_1 \geq \psi_2 \;\; (i.e.\ \psi_1(t,x) \geq \psi_2(t,x) \;\; \forall (t,x)),$$
$$\text{such that } \psi_1, \psi_2 \in \left[-\left\|v_0^\alpha\right\|_{L^\infty(\mathbb{R})}, \left\|v_0^\alpha\right\|_{L^\infty(\mathbb{R})}\right].$$

• **Stable:** if and only if $u^\varepsilon = (u^{\alpha,\varepsilon})_{\alpha=1,\ldots,d}$, defined in (5.28), is bounded independently of ε.

• **Consistent:** if and only if the following inequalities are satisfied for every test function $\phi^\alpha \in C^1((0,T)\times\mathbb{R})$, and $\xi = (\xi^\alpha)_{\alpha=1,\ldots,d}$

$$\limsup_{\substack{y\to x, s\to t\\ \xi\to 0,\varepsilon\to 0\\ \psi^j\to r^j\ \forall j\neq\alpha}} S_\varepsilon^\alpha\Big(s,y,\big(\psi^1,\ldots,\psi^{\alpha-1},\phi^\alpha(s,y),\psi^{\alpha+1},\ldots,\psi^d\big)+\xi,\phi^\alpha+\xi^\alpha\Big) \leq$$
$$\partial_t\phi^\alpha(t,x) - (\lambda^\alpha)_\star\Big(t,x,\big(r^1,\ldots,r^{\alpha-1},\phi^\alpha(t,x),r^{\alpha+1},\ldots,r^d\big)\Big)|\partial_x\phi^\alpha(t,x)|, \tag{5.33}$$

$$\liminf_{\substack{y\to x, s\to t\\ \xi\to 0,\varepsilon\to 0\\ \psi^j\to r^j\ \forall j\neq\alpha}} S_\varepsilon^\alpha\Big(s,y,\big(\psi^1,\ldots,\psi^{\alpha-1},\phi^\alpha(s,y),\psi^{\alpha+1},\ldots,\psi^d\big)+\xi,\phi^\alpha+\xi^\alpha\Big) \geq$$
$$\partial_t\phi^\alpha(t,x) - (\lambda^\alpha)^\star\Big(t,x,\big(r^1,\ldots,r^{\alpha-1},\phi^\alpha(t,x),r^{\alpha+1},\ldots,r^d\big)\Big)|\partial_x\phi^\alpha(t,x)|. \tag{5.34}$$

Now, we shall prove in the following lemma that the scheme (5.9) is indeed stable, consistent, and monotone in the sense of Definition 5.1.

Lemma 5.2 (***Monotony, consistency, and stability of*** S_ε^α**).**
Assume (5.15) is satisfied. Then, the scheme defined in (5.9) is monotone, stable and consistent, in the sense of Definition 5.1.

Proof of Lemma 5.2.
The stability of the solution $u_i^{\alpha,n}$ of (5.9) is given by Theorem 5.3-(i). For that reason, we only demonstrate the proofs of the consistency and monotony. We thus proceed in two steps.

Step 1. (Consistency):
Let $\phi^\alpha \in C^1((0,T)\times\mathbb{R})$, and $\xi = (\xi^\alpha)_{\alpha=1,\ldots,d}$. From (5.31), we can see that

$$S_\varepsilon^\alpha\Big(s,y,\big(\psi^1,\ldots,\psi^{\alpha-1},\phi^\alpha(s,y),\psi^{\alpha+1},\ldots,\psi^d\big)+\xi,\phi^\alpha+\xi^\alpha\Big) =$$
$$\frac{1}{\Delta t}\left(\phi^\alpha(s,y)-\frac{1}{2}\Big(\phi^\alpha(s-\Delta t,y+\Delta x)+\phi^\alpha(s-\Delta t,y)\Big)\right)$$
$$-\lambda^\alpha\Big(s,y,\big(\psi^1,\ldots,\psi^{\alpha-1},\phi^\alpha(s,y),\psi^{\alpha+1},\ldots,\psi^d\big)+\xi\Big)\frac{|\phi^\alpha(s-\Delta t,y+\Delta x)-\phi^\alpha(s-\Delta t,y)|}{\Delta x}.$$

Thus, by passing to the lim sup and the lim inf as $y\to x$, $s\to t$, $\xi^\alpha\to 0$, $\varepsilon\to 0$, and $\psi^j\to r^j$ for all $j\neq\alpha$, we can deduce (5.33) and (5.34). Hence, scheme (5.9) is consistent.

Step 2. (Monotony):
The operator S_ε^α, defined in (5.31), can be written as

$$S_\varepsilon^\alpha(t,x,u,\psi) = \frac{u^\alpha - \mathcal{T}_\varepsilon^\alpha\left(\psi(t-\Delta t,x)\right)}{\Delta t},$$

where

$$\mathcal{T}_\varepsilon^\alpha(\psi(t-\Delta t,x)) = \frac{1}{2}\left(\psi(t-\Delta t,x+\Delta x)+\psi(t-\Delta t,x)\right)+ \frac{\Delta t}{\Delta x}\lambda^\alpha(t,x,u)\big|\psi(t-\Delta t,x+\Delta x)-\psi(t-\Delta t,x)\big|.$$

In order to show that S_ε^α is monotonic in the sense of Definition 5.1, it suffices to prove that $\mathcal{T}_\varepsilon^\alpha$ is non-decreasing with respect to $\psi(t-\Delta t,x+\Delta x)$ and $\psi(t-\Delta t,x)$. Under (5.15), we can show that the derivatives of $\mathcal{T}_\varepsilon^\alpha$ with respect to $\psi(t-\Delta t,x+\Delta x)$ and $\psi(t-\Delta t,x)$ are positive, and so $\mathcal{T}_\varepsilon^\alpha$ is non-decreasing. Hence, S_ε^α is non-increasing in the sense of Definition 5.1.

□

3.2 Existence of sub and super solutions to (5.1)

This subsection is devoted to the proof of Theorem 5.4-(ii), (iv). Before illustrating the proof, we recall the definition of discontinuous viscosity solutions for system (5.1), introduced by Ishii in [58, Definition 2.1]. For a complete overview on viscosity solutions, we refer the reader to Barles [9], Crandall and Ishii [35], and to Crandall and Lions [37].

For a vector $u=(u^1,\dots,u^d)$ locally bounded on $[0,T)\times\mathbb{R}$ for all $T>0$, we write $u^\star=((u^1)^\star,\dots,(u^d)^\star)$ and $u_\star=((u^1)_\star,\dots,(u^d)_\star)$.
Given two locally bounded functions $v=(v^\alpha)_{\alpha=1,\dots,d}$ and $u=(u^\alpha)_{\alpha=1,\dots,d}$ on $[0,T)\times\mathbb{R}$ such that $(v^\alpha)_\star\le(u^\alpha)^\star$ for every $\alpha=1,\dots,d$, we define the set

$$\mathcal{E}_v^u(t,x)=\prod_{\alpha=1}^d\Big[(v^\alpha)_\star(t,x),(u^\alpha)^\star(t,x)\Big].$$

Definition 5.2. ***(Discontinuous viscosity sub-solution, super-solution and solution)***
Assume that $\lambda=(\lambda^\alpha)_{\alpha=1,\dots,d}$ is locally bounded on $(0,T)\times\mathbb{R}\times\mathbb{R}^d$ and $v_0=(v_0^\alpha)_{\alpha=1,\dots,d}$ is locally bounded on $\mathbb{R}$. Let $v=(v^\alpha)_{\alpha=1,\dots,d}$, $u=(u^\alpha)_{\alpha=1,\dots,d}$ be two locally bounded functions on $[0,T)\times\mathbb{R}$ such that $(v^\alpha)_\star\le(u^\alpha)^\star$ for every $\alpha=1,\dots,d$. We say that u and v are a couple of discontinuous viscosity sub- and super- solutions of (5.1) if they satisfy the following two conditions

(i) • $(u^\alpha)^\star(0,x) \le (v_0^\alpha)^\star(x)$, for all $\alpha = 1,\dots,d$ and $x \in \mathbb{R}$.

• $(v^\alpha)_\star(0,x) \ge (v_0^\alpha)_\star(x)$, for all $\alpha = 1,\dots,d$ and $x \in \mathbb{R}$.

(ii) • Whenever a test function $\phi^\alpha \in C^1((0,T)\times\mathbb{R})$, $\alpha = 1,\dots,d$ and $(u^\alpha)^\star - \phi^\alpha$ attains a local maximum at $(t_0^\alpha, x_0^\alpha) \in (0,T)\times\mathbb{R}$, then we have

$$\begin{aligned} \min\Big\{&\partial_t\phi^\alpha(t_0^\alpha,x_0^\alpha) - (\lambda^\alpha)^\star(t_0^\alpha,x_0^\alpha,r)|\partial_x\phi^\alpha(t_0^\alpha,x_0^\alpha)| : \\ &r \in \mathcal{E}_v^u(t_0^\alpha,x_0^\alpha),\ r^\alpha = (u^\alpha)^\star(t_0^\alpha,x_0^\alpha)\Big\} \le 0. \end{aligned} \tag{5.35}$$

• Whenever $\phi^\alpha \in C^1((0,T)\times\mathbb{R})$, $\alpha = 1,\dots,d$ and $(v^\alpha)_\star - \phi^\alpha$ attains a local minimum at $(t_0^\alpha, x_0^\alpha) \in (0,T)\times\mathbb{R}$, then we have

$$\begin{aligned} \max\Big\{&\partial_t\phi^\alpha(t_0^\alpha,x_0^\alpha) - (\lambda^\alpha)_\star(t_0^\alpha,x_0^\alpha,r)|\partial_x\phi^\alpha(t_0^\alpha,x_0^\alpha)| : \\ &r \in \mathcal{E}_v^u(t_0^\alpha,x_0^\alpha),\ r^\alpha = (v^i)_\star(t_0^\alpha,x_0^\alpha)\Big\} \ge 0. \end{aligned} \tag{5.36}$$

Finally, we call a function $w = (w^\alpha)_{\alpha=1,\dots,d}$ a discontinuous viscosity solution of (5.1) if $w^\star$ and $w_\star$ verify conditions (i) and (ii).

Now, we can proceed to the proof of Theorem 5.4-(ii).

Proof of Theorem 5.4-(ii).
We only prove the result for the sub-solution case, as the super-solution one can be proved analogously. Let u^ε be the interpolation function defined in (5.28). We have to show that its upper relaxed semi-limit $(\overline{u}^\alpha)^\star = \overline{u}^\alpha$ is a discontinuous viscosity sub-solution of (5.1), in the sense of Definition 5.2. We proceed in two steps.

Step 1. (Meaning of the initial data):
We will show that $\overline{u} = (\overline{u}^1,\dots,\overline{u}^d)$ satisfies the first inequality of Definition 5.2-(i). It is sufficient to prove the following

$$\overline{u}^\alpha(0,x) \le (v_0^\alpha)^\star(x), \quad \text{for all } x \in \mathbb{R}, \quad \alpha = 1,\dots,d. \tag{5.37}$$

From the definition of $\overline{u}^\alpha$, we know that there exists a sequence $(\varepsilon_n, t_{\varepsilon_n}, x_{\varepsilon_n}) \to (0,0,x)$ as $n \to +\infty$, such that

$$\overline{u}^\alpha(0,x) = \lim_{n\to+\infty} u^{\varepsilon_n,\alpha}(t_{\varepsilon_n}, x_{\varepsilon_n}).$$

From the definition of $u^{\alpha,\varepsilon}$ (5.28), we deduce that if $(t_{\varepsilon_n}, x_{\varepsilon_n}) \in [t_{n_0}, t_{n_0+1}] \times [x_i, x_{i+1}]$, then

$$u^{\varepsilon_n,\alpha}(t_{\varepsilon_n},x_{\varepsilon_n}) = \left(\frac{t_{\varepsilon_n}-t_{n_0}}{\Delta t}\right)\left[\left(\frac{x_{\varepsilon_n}-x_i}{\Delta x}\right)u_{i+1}^{\alpha,n_0+1} + \left(1-\frac{x_{\varepsilon_n}-x_i}{\Delta x}\right)u_i^{\alpha,n_0+1}\right]$$
$$+\left(1-\frac{t_{\varepsilon_n}-t_{n_0}}{\Delta t}\right)\left[\left(\frac{x_{\varepsilon_n}-x_i}{\Delta x}\right)u_{i+1}^{\alpha,n_0} + \left(1-\frac{x_{\varepsilon_n}-x_i}{\Delta x}\right)u_i^{\alpha,n_0}\right].$$

Using Lemma 5.1 with $t_n = 0$ on the term u_{i+1}^{α,n_0+1} yields

$$\begin{aligned} u_{i+1}^{\alpha,n_0+1} = u^{\varepsilon_n,\alpha}(t_{n_0+1},x_{i+1}) &\le \max_{|x_j-x_{i+1}|\le\overline{\gamma}t_{n_0+1}} u^{\alpha,\varepsilon_n}(0,x_j) \\ &\le \max_{|x_j-x_{i+1}|\le\overline{\gamma}t_{n_0+1}} \left(\int_{\mathbb{R}} v_0^\alpha(z)\rho_{|\varepsilon_n|}(x_j-z)dz\right) \\ &\le \max_{|x_j-x_{i+1}|\le\overline{\gamma}t_{n_0+1}} \left(\max_{|z-x_j|\le|\varepsilon_n|} v_0^\alpha(z)\right), \end{aligned}$$

where we have used in the second line the definition of the function $u_{|\varepsilon|}^{\alpha,0}$ given in (5.8), and in the third line we used the properties of mollifiers. Furthermore, the convergence as $n \to +\infty$ of $(\varepsilon_n, t_{\varepsilon_n}, x_{\varepsilon_n}) \to (0,0,x)$ implies that for all $\beta > 0$, there exists $n_\beta > 0$, such that, for all $n \ge n_\beta$, we have

$$|\varepsilon_n| \le \beta, \quad |x_{\varepsilon_n} - x| \le \beta, \quad \text{and} \quad t_{\varepsilon_n} \le \beta.$$

Thus, for every $\beta > 0$ and $n \ge n_\beta$, we get as $\Delta x, \Delta t \le |\varepsilon_n| \le \beta$, that

$$u_{i+1}^{\alpha,n_0+1} \le \max_{|z-x|\le\beta(2\overline{\gamma}+3)} v_0^\alpha(z).$$

If we repeat the same process on each of the terms u_i^{α,n_0+1}, u_{i+1}^{α,n_0}, and u_i^{α,n_0}, we obtain

$$u^{\alpha,\varepsilon_n}(t_{\varepsilon_n},x_{\varepsilon_n}) \le \max_{|z-x|\le\beta(2\overline{\gamma}+3)} v_0^\alpha(z).$$

Passing to the limit as $n \to +\infty$ gives

$$\overline{u}^\alpha(0,x) \le \max_{|z-x|\le\beta(2\overline{\gamma}+3)} v_0^\alpha(z).$$

Finally, we pass to the limit as $\beta \to 0$ to obtain (5.37).

Step 2. (Meaning of the equation):
Here, we want to show that $\overline{u}^\alpha$ satisfies the first inequality of Definition 5.2-(ii). This is an adaptation of a result by Barles, Souganidis [14] for systems of equations, using the definition of Ishii, Koike [59, 60] of viscosity solutions. Let $\phi^\alpha \in C^1((0,T)\times\mathbb{R})$ such that

$\overline{u}^\alpha - \phi^\alpha$ attains a local maximum at $(t_0^\alpha, x_0^\alpha) \in (0,T)\times\mathbb{R}$. Up to a slight modification of this test function, we can assume that ϕ^α is tangent from above to $\overline{u}^\alpha$ at $(t_0^\alpha, x_0^\alpha) \in (0,T)\times\mathbb{R}$. By a standard technique used in the theory of viscosity solutions (see Barles [9, Lemma 4.2]), we can say that there exists a subsequence $(\varepsilon_m^\alpha, t_{n_m}^\alpha, x_{i_m}^\alpha) \to (0, t_0^\alpha, x_0^\alpha)$ when $m \to +\infty$, such that $(t_{n_m}^\alpha, x_{i_m}^\alpha)$ is a local maximum of $u^{\alpha,\varepsilon_m^\alpha} - \phi^\alpha$ and

$$\overline{u}^\alpha(t_0^\alpha, x_0^\alpha) = \lim_{m\to+\infty} u^{\alpha,\varepsilon_m^\alpha}(t_{n_m}^\alpha, x_{i_m}^\alpha).$$

We define $\xi_m = (\xi_m^\alpha)_{\alpha=1,\dots,d}$, such that

$$\xi_m^\alpha = u^{\alpha,\varepsilon_m^\alpha}(t_{n_m}^\alpha, x_{i_m}^\alpha) - \phi^\alpha(t_{n_m}^\alpha, x_{i_m}^\alpha).$$

It is then clear that $u^{\alpha,\varepsilon_m^\alpha}(t_{n_m}^\alpha, x_{i_m}^\alpha) = \phi^\alpha(t_{n_m}^\alpha, x_{i_m}^\alpha) + \xi_m^\alpha$, and

$$u^{\alpha,\varepsilon_m^\alpha}(t,x) \le \phi^\alpha(t,x) + \xi_m^\alpha, \quad \forall (t,x) \text{ in the neighborhood of } (t_{n_m}^\alpha, x_{i_m}^\alpha).$$

Let $\psi_m = (\psi_m^\beta)_{\beta=1,\dots,d}$, where

$$\psi_m^\beta = \begin{cases} u^{\beta,\varepsilon_m^\alpha}(t_{n_m}^\alpha, x_{i_m}^\alpha) - \xi_m^\beta & \text{if } \ \beta \neq \alpha \\ \phi^\alpha(t_{n_m}^\alpha, x_{i_m}^\alpha) & \text{if } \ \beta = \alpha. \end{cases}$$

Then using the monotony of the operator $S^\alpha_{\varepsilon_m^\alpha}$, we get

$$S^\alpha_{\varepsilon_m^\alpha}\left(t_{n_m}^\alpha, x_{i_m}^\alpha, \psi_m + \xi_m, \phi^\alpha + \xi_m^\alpha\right) \le S^\alpha_{\varepsilon_m^\alpha}\left(t_{n_m}^\alpha, x_{i_m}^\alpha, \psi_m + \xi_m, u^{\alpha,\varepsilon_m^\alpha}\right)$$

$$= S^\alpha_{\varepsilon_m^\alpha}\left(t_{n_m}^\alpha, x_{i_m}^\alpha, \left(\psi_m^1 + \xi_m^1, \dots, \psi^{\alpha-1} + \xi_m^{\alpha-1}, \phi^\alpha(t_{n_m}^\alpha, x_{i_m}^\alpha) + \xi_m^\alpha, \psi^{\alpha+1} + \xi_m^{\alpha+1}, \dots, \psi_m^d + \xi_m^d\right), u^{\alpha,\varepsilon_m^\alpha}\right)$$

$$= S^\alpha_{\varepsilon_m^\alpha}\left(t_{n_m}^\alpha, x_{i_m}^\alpha, \left(u^{1,\varepsilon_m^\alpha}(t_{n_m}^\alpha, x_{i_m}^\alpha), \dots, u^{\alpha,\varepsilon_m^\alpha}(t_{n_m}^\alpha, x_{i_m}^\alpha), \dots, u^{d,\varepsilon_m^\alpha}(t_{n_m}^\alpha, x_{i_m}^\alpha)\right), u^{\alpha,\varepsilon_m^\alpha}\right)$$

$$= S^\alpha_{\varepsilon_m^\alpha}\left(t_{n_m}^\alpha, x_{i_m}^\alpha, u^{\varepsilon_m^\alpha}(t_{n_m}^\alpha, x_{i_m}^\alpha), u^{\alpha,\varepsilon_m^\alpha}\right) = 0,$$

using (5.32). Now, as ψ_m^β are uniformly bounded for every $\beta = 1, \dots, d$, we can extract a subsequence (independent of β), still denoted ψ_m^β, such that

$$\left|\begin{array}{l} \lim_{m\to+\infty} \psi_m^\beta = r^\beta \quad \text{for} \quad \beta \neq \alpha, \\ \\ \lim_{m\to+\infty} \psi_m^\alpha = r^\alpha = \phi^\alpha(t_0^\alpha, x_0^\alpha) = \overline{u}^\alpha(t_0^\alpha, x_0^\alpha). \end{array}\right. \tag{5.38}$$

Then, for $r = (r^1, \dots, r^{\alpha-1}, \phi^\alpha(t_0^\alpha, x_0^\alpha), r^{\alpha+1}, \dots, r^d)$, we obtain from the consistency of

$S^{\alpha}_{\varepsilon^{\alpha}_m}$, that

$$\partial_t \phi^{\alpha}(t^{\alpha}_0, x^{\alpha}_0) - (\lambda^{\alpha})^{\star}(t^{\alpha}_0, x^{\alpha}_0, r)\,|\partial_x \phi^{\alpha}(t^{\alpha}_0, x^{\alpha}_0)| \leq$$

$$\liminf_{\substack{y\to x^{\alpha}_0, s\to t^{\alpha}_0\\ \xi\to 0, \varepsilon\to 0\\ \psi^j\to r^j \forall j\neq\alpha}} S^{\alpha}_{\varepsilon}\Big(s, y, (\psi^1, \dots, , \psi^{\alpha-1}, \phi^{\alpha}(s,y), \psi^{\alpha+1}, \dots, r^d) + \xi, \phi^{\alpha} + \xi^{\alpha}\Big)$$

$$\leq \liminf_{m\to+\infty} S^{\alpha}_{\varepsilon^{\alpha}_m}\Big(t^{\alpha}_{n_m}, x^{\alpha}_{i_m}, \psi_m + \xi_m, \phi^{\alpha} + \xi^{\alpha}_m\Big)$$

$$\leq \liminf_{m\to+\infty} S^{\alpha}_{\varepsilon^{\alpha}_m}\Big(t^{\alpha}_{n_m}, x^{\alpha}_{i_m}, u^{\varepsilon^{\alpha}_m}(t^{\alpha}_{n_m}, x^{\alpha}_{i_m}), u^{\alpha, \varepsilon^{\alpha}_m}\Big) = 0.$$

Hence, we deduce form the previous inequality that

$$\min\Big\{\partial_t \phi^{\alpha}(t^{\alpha}_0, x^{\alpha}_0) - (\lambda^{\alpha})^{\star}(t^{\alpha}_0, x^{\alpha}_0, r)|\partial_x \phi^{\alpha}(t^{\alpha}_0, x^{\alpha}_0)| : \, r \in \mathcal{E}^{\overline{u}}_{\underline{u}}(t^{\alpha}_0, x^{\alpha}_0), \, r^{\alpha} = \overline{u}^{\alpha}(t^{\alpha}_0, x^{\alpha}_0)\Big\} \leq 0, \tag{5.39}$$

and therefore, $\overline{u} = (\overline{u}^{\alpha})_{\alpha=1,\dots,d}$ is a viscosity sub-solution of (5.1). Similarly, we can verify that $\underline{u} = (\underline{u}^{\alpha})_{\alpha=1,\dots,d}$ satisfies (5.36).

□

Proof of Theorem 5.4-(iv).
Under the suppositions of Theorem 5.2, we know that system (5.1) verifies a comparison principle (see [2, Th 1.2] for more details) and therefore we can obtain that $\overline{u}^{\alpha} \leq \underline{u}^{\alpha}$ for every $\alpha = 1, \dots, d$. Then using the fact that $\underline{u}^{\alpha} \leq \overline{u}^{\alpha}$, for every $\alpha = 1, \dots, d$, we deduce that the function $u^{\alpha,\varepsilon}$ converges to the unique solution of system (5.1).

□

4 Link between sub and super solutions

Finally in this section, we prove Theorem 5.4-(iii). Namely, we will show that the upper and lower relaxed semi-limits $\overline{u}^{\alpha}$ and $\underline{u}^{\alpha}$ of $u^{\alpha,\varepsilon}$ are equal in spaces, for every $\alpha = 1, \dots, d$, except at most on a countable set of points. First, we present some preliminary results in Subsection 4.1, then we demonstrate the proof of Theorem 5.4-(iii) in Subsection 4.2.

4.1 Preliminary results

First we recall some properties valid on bounded $BV(\mathbb{R})$ functions.

Lemma 5.3. ***(Properties of** BV**-functions, [6])***
Let f be a bounded $BV(\mathbb{R})$-function. Then, the following hold
i) f is continuous except at most on a countable set,

ii) The right and left limits

$$f(x^+) = \lim_{\substack{y\to x\\ y>x}} f(y), \qquad f(x^-) = \lim_{\substack{y\to x\\ y<x}} f(y)$$

exists at every point $x \in \mathbb{R}$. Moreover, there exists a unique right-continuous function f_r (resp. left-continuous function f_l) coinciding with f except on a countable set.
iii) There exists a pair of non-decreasing functions f^1, $f^2 \in L^\infty(\mathbb{R})$ such that $f = f^1 - f^2$.

The following lemma shows a local estimate valid on sequences of non-decreasing functions converging locally and strongly in $L^1(\mathbb{R})$.

Lemma 5.4. ***(Sequences of non-decreasing functions)***
i) ***Sequence of non-decreasing functions strongly convergent in*** $L^1_{loc}(\mathbb{R})$
Let $(\phi_\varepsilon)_\varepsilon$ be a sequence of non-decreasing functions defined on $\mathbb{R}$ such that $\phi_\varepsilon \to \phi$ strongly in $L^1_{loc}(\mathbb{R})$, as $\varepsilon \to 0$, with ϕ a non-decreasing function also defined on $\mathbb{R}$. Then, for all $a > 0$ and $0 < \delta \leq \frac{a}{2}$, there exists $\varepsilon^\delta_a > 0$, such that, for every $0 < \varepsilon \leq \varepsilon^\delta_a$, the following estimate holds

$$-\delta + \phi(x-\delta) \leq \phi_\varepsilon(x) \leq \delta + \phi(x+\delta), \quad \forall x \in [-a, a]. \tag{5.40}$$

ii) ***Sequence of non-decreasing functions strongly convergent in*** $C([0,T]; L^1_{loc}(\mathbb{R}))$
Let $(\phi_\varepsilon)_\varepsilon$ be a sequence of functions defined on $[0,T) \times \mathbb{R}$ such that, for all $t \in [0,T)$, the function $\phi_\varepsilon(t,\cdot)$ is non-decreasing on $\mathbb{R}$. Assume, moreover, that $\phi_\varepsilon \to \phi$ strongly in $C([0,T); L^1_{loc}(\mathbb{R}))$, as $\varepsilon \to 0$, with, for all $t \in [0,T)$, the function $\phi(t,\cdot)$ is defined and non-decreasing on $\mathbb{R}$. Then, for all $a > 0$ and $0 < \delta \leq \frac{a}{2}$, there exists $\varepsilon^\delta_{a,T} > 0$, such that, for every $0 < \varepsilon \leq \varepsilon^\delta_{a,T}$, the following estimate holds

$$-\delta + \phi(t, x-\delta) \leq \phi_\varepsilon(t,x) \leq \delta + \phi(t, x+\delta), \quad \forall x \in [-a, a],\ \forall t \in [0,T).$$

Next, we show in the following lemma a local estimate valid on sequences of bounded and BV functions converging locally and strongly in $L^1(\mathbb{R})$.

Lemma 5.5. ***(Sequence of*** $BV(\mathbb{R})$ ***functions)***
Let $(\phi_\varepsilon)_\varepsilon$ be a sequence of functions, defined on $\mathbb{R}$, uniformly bounded in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and strongly convergent to $\phi \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ in $L^1_{loc}(\mathbb{R})$, with ϕ a right-continuous function. Then, there exists a subsequence $(\phi_{\varepsilon'})_{\varepsilon'}$ such that, for all $a > 0$ and for all $0 < \delta \leq \frac{a}{2}$, there exists $\varepsilon^\delta_a > 0$ such that, for all $0 < \varepsilon \leq \epsilon^\delta_a$, the following estimate holds

$$-2\delta + \phi^1(x-\delta) - \phi^2(x-\delta) \leq \phi_{\varepsilon'}(x) \leq 2\delta + \phi^1(x+\delta) - \phi^2(x-\delta), \quad \forall x \in [-a, a] \tag{5.41}$$

where ϕ^1 and ϕ^2 are two bounded, right-continuous and non-decreasing functions on $\mathbb{R}$ satisfying $\phi = \phi^1 - \phi^2$.

For the proof of the three previous lemmas see [19, section 6.1].

We end this subsection with the following compactness lemma.

Lemma 5.6. ***(Simon's Lemma [80, Corollary 4])***
Let X, B and Y be three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(\theta_n)_n$ is a sequence uniformly bounded in $L^\infty((0,T);X)$ and $(\partial_t \theta_n)_n$ is uniformly bounded in $L^r((0,T);Y)$ where $r > 1$, then, $(\theta_n)_n$ is relatively compact in $C((0,T);B)$.

4.2 Proof of Theorem 5.4-(iii)

Let u^ε be the interpolation function defined in (5.28). From estimates (5.18), (5.19), and (5.20), we can say that, for all compact $K_0 \subset \mathbb{R}$, $(u^{\alpha,\varepsilon})_\varepsilon$ is uniformly bounded in $L^\infty((0,T)\times K_0) \cap L^\infty((0,T);BV(K_0))$ and $(\partial_t u^{\alpha,\varepsilon})_\varepsilon$ is uniformly bounded in $L^\infty((0,T);L^1(K_0))$. Using Simon's lemma (Lemma 5.6), in the particular case where $X = BV(K_0)$, $B = Y = L^1(K_0)$, and the following compact embedding $BV(K_0) \hookrightarrow L^1(K_0)$, we can extract a subsequence denoted by $(u^{\alpha,\varepsilon_m}_{K_0})_{\varepsilon_m,K_0}$ that converges strongly in $L^\infty((0,T);L^1(K_0))$ to some limit u^α, as $m \to 0$. By a standard diagonalization procedure, we can extract a subsequence $(u^{\alpha,\varepsilon_m})_{\varepsilon_m}$ (independent of α and K_0) that converges to the limit u^α strongly in $C([0,T);L^1(K))$ for all compact $K \subset \mathbb{R}$. Now, thanks to estimates (5.18) and (5.19) we can extract a subsequence, still denoted by $(u^{\alpha,\varepsilon_m})_{\varepsilon_m}$, satisfying the following convergences

$$\left| \begin{array}{lll} u^{\alpha,\varepsilon_m} \longrightarrow u^\alpha, & \text{strongly in} & C([0,T);L^1(K)), \ \text{for all compact } K \subset \mathbb{R}, \\ u^{\alpha,\varepsilon_m} \longrightarrow u^\alpha, & \text{weakly-}\star \ \text{in} & L^\infty((0,T)\times\mathbb{R}), \\ u^{\alpha,\varepsilon_m} \longrightarrow u^\alpha, & \text{weakly-}\star \ \text{in} & L^\infty((0,T);BV(\mathbb{R})). \end{array} \right. \tag{5.42}$$

Taking the $\liminf$ in estimates (5.18), (5.19) and using the lower semi-continuity of $\|\cdot\|_{L^\infty(\mathbb{R})}$ and $|\cdot|_{BV(\mathbb{R})}$, we can prove that u^α satisfies (5.5) and (5.6). Since, for all $t \in [0,T)$, the function $u^\alpha(t,\cdot) \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, then by property *(ii)* of Lemma 5.3, we know that this function coincides with a right-continuous function almost everywhere in $\mathbb{R}$ and consequently in $L^1_{loc}(\mathbb{R})$. This allows us to consider, in the following, a right-continuous limit with respect to the space variable.

Now we can show that $u^\alpha(t,\cdot)$, $\overline{u}^\alpha(t,\cdot)$, and $\underline{u}^\alpha(t,\cdot)$ verify equality (5.22). For a clear presentation, we'll perform this in three steps.

Step 1. (Regularity in time estimate):
Let $T > 0$, $a > 0$, and we set $\zeta = 4\overline{\gamma} + 3$, where $\overline{\gamma}$ was defined in Lemma 5.29. First, we will show that there exists two bounded and non-decreasing functions $w^{\alpha,1}$ and $w^{\alpha,2}$ satisfying, for every $\alpha = 1, \dots, d$, the following inequalities

$$\begin{aligned} -2h + w^{\alpha,1}(t, x - h\zeta) - w^{\alpha,2}(t, x + h\zeta) &\leq \underline{u}^\alpha(t + h, x) \\ &\leq \overline{u}^\alpha(t + h, x) \leq 2h + w^{\alpha,1}(t, x + h\zeta) - w^{\alpha,2}(t, x - h\zeta), \end{aligned} \tag{5.43}$$

for all $x \in [\frac{a}{2}, \frac{a}{2}]$, $t \in [0, T)$, and for all $h > 0$ verifying

$$h \leq \frac{a}{4(2\overline{\gamma} + 1)}, \quad \text{and} \quad t + h < T. \tag{5.44}$$

We begin with the proof of the right inequality in (5.43), namely,

$$\overline{u}^\alpha(t + h, x) \leq 2h + w^{\alpha,1}(t, x + h\zeta) - w^{\alpha,2}(t, x - h\zeta). \tag{5.45}$$

Consider $h > 0$ satisfying (5.44). By the definition of $\overline{u}^\alpha$, we know that there exists a sequence $(\varepsilon_m, t^h_{\varepsilon_m}, x_{\varepsilon_m}) \to (0, t + h, x)$ when $m \to +\infty$, such that

$$\overline{u}^\alpha(t + h, x) = \lim_{m \to +\infty} u^{\alpha,\varepsilon_m}(t^h_{\varepsilon_m}, x_{\varepsilon_m}).$$

Now, the convergence $(\varepsilon_m, t^h_{\varepsilon_m}, x_{\varepsilon_m}) \to (0, t + h, x)$ as $m \to +\infty$, implies that there exists $m_h > 0$, such that, for all $m \geq m_h$, we have

$$|\varepsilon_m| \leq h, \quad |x_{\varepsilon_m} - x| \leq h, \quad \text{and} \quad |t^h_{\varepsilon_m} - t - h| \leq h.$$

Assume $t \in [t_n, t_{n+1}]$ and $t + h \in [t_{n_0}, t_{n_0+1}]$ for $n \leq n_0$. As $u^{\alpha,\varepsilon_m}(t^h_{\varepsilon_m}, x_{\varepsilon_m})$ is a function of $u^{\alpha,n_0+1}_{i+1}, u^{\alpha,n_0+1}_i, u^{\alpha,n_0}_{i+1}$, and u^{α,n_0}_i, then applying Lemma 5.1 on each of these terms on $[t_{n_0}, t_{n_0+1}] \times [x_i, x_{i+1}]$ gives, for $\widetilde{n} \in \{n_0, n_0 + 1\}$, $\widetilde{i} \in \{i, i + 1\}$

$$\begin{aligned} u^{\alpha,\widetilde{n}}_{\widetilde{i}} &\leq \max_{|x_j - x_{\widetilde{i}}| \leq \overline{\gamma}(t_{\widetilde{n}} - t_n)} u^{\alpha,\varepsilon_m}(t_n, x_j) \\ &\leq \max_{|x_j - x| \leq h(4\overline{\gamma}+2)} u^{\alpha,\varepsilon_m}(t_n, x_j), \end{aligned}$$

which yields

$$u^{\alpha,\varepsilon_m}(t^h_{\varepsilon_m}, x_{\varepsilon_m}) \leq \max_{|x_j - x| \leq h(4\overline{\gamma}+2)} u^{\alpha,\varepsilon_m}(t_n, x_j).$$

As $t = t_n + c\Delta t$, for $0 \leq c \leq 1$, we introduce the function $w^{\alpha,\varepsilon}$ defined as

$$w^{\alpha,\varepsilon}(t, x) = u^{\alpha,\varepsilon}(t - c\Delta t, x). \tag{5.46}$$

Inserting this function in the previous inequality gives

$$u^{\alpha,\varepsilon_m}(t^h_{\varepsilon_m}, x_{\varepsilon_m}) \le \max_{|x_j - x| \le h(4\overline{\gamma}+2)} w^{\alpha,\varepsilon_m}(t, x_j). \tag{5.47}$$

It is clear that $w^{\alpha,\varepsilon}$ satisfies estimates (5.18), (5.19), and (5.20). Then, since for all $t \in [0,T)$, the sequence $w^{\alpha,\varepsilon_m}(t,\cdot)$ is uniformly bounded in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, we can use the same reasoning as in the beginning of this section, to show that this sequence converges uniformly in $L^1_{loc}(\mathbb{R})$. Thus, we can deduce from Lemma 5.5 that there exists a subsequence $w^{\alpha,\varepsilon_n}(t,\cdot)$ and a positive constant $n^h_{a,t}$, such that, for all $n \ge n^h_{a,t}$, we have

$$w^{\alpha,\varepsilon_n}(t,y) \le 2h + w^{\alpha,1}(t, y+h) - w^{\alpha,2}(t, y-h), \quad \forall\, y \in [-a, a], \tag{5.48}$$

where $w^{\alpha,1}$ and $w^{\alpha,2}$ are two bounded, right-continuous and non-decreasing functions (with respect to x) satisfying $w^\alpha = w^{\alpha,1} - w^{\alpha,2}$, for every $\alpha = 1,\dots,d$. Collecting (5.47) and (5.48), we obtain that, for all $h > 0$ satisfying (5.44) and for all $n \ge n^h_{a,t}$

$$u^{\alpha,\varepsilon_n}(t^h_{\varepsilon_n}, x_{\varepsilon_n}) \le 2h + w^{\alpha,1}(t, x+h\zeta) - w^{\alpha,2}(t, x-h\zeta).$$

We pass to the limit as $n \to +\infty$ to get (5.45). Similarly, using the discrete finite speed propagation property, specifically, the left inequality in (5.29), we can prove the left inequality in (5.43), namely

$$-2h + w^{\alpha,1}(t, x-h\zeta) - w^{\alpha,2}(t, x+h\zeta) \le \underline{u}^\alpha(t+h, x). \tag{5.49}$$

Step 2. (Right and left continuity):
Let $T > 0$ and $t \in [0,T)$. Since $w^{\alpha,1}(t,\cdot)$, $w^{\alpha,2}(t,\cdot)$ are bounded and non-decreasing functions on $\mathbb{R}$ for every $\alpha = 1,\dots,d$, then, from property (ii) of Lemma 5.3, we know that, the right and left limits of these functions exist at every point $x \in \mathbb{R}$. This implies that, for all $\beta > 0$ and $x \in [-\frac{a}{2}, \frac{a}{2}]$, there exists $h^\beta_{a,t} > 0$, such that, for all $0 < z \le h^\beta_{a,t}$ and $i = 1,\dots,d$, we have

$$\left|\begin{array}{l} w^{\alpha,1}(t,x+z) \le \frac{\beta}{4} + w^{\alpha,1}_r(t,x) \\ \\ w^{\alpha,2}(t,x+z) \le \frac{\beta}{4} + w^{\alpha,2}_r(t,x) \end{array}\right. \quad \text{and} \quad \left|\begin{array}{l} -w^{\alpha,1}(t,x-z) \le \frac{\beta}{4} - w^{\alpha,1}_l(t,x) \\ \\ -w^{\alpha,2}(t,x-z) \le \frac{\beta}{4} - w^{\alpha,2}_l(t,x) \end{array}\right. \tag{5.50}$$

where $w^{\alpha,1}_r(t,\cdot)$, $w^{\alpha,2}_r(t,\cdot)$ are right-continuous functions on $\mathbb{R}$ and $w^{\alpha,1}_l(t,\cdot)$, $w^{\alpha,2}_l(t,\cdot)$ are left-continuous functions on $\mathbb{R}$. Note that, here the choice of the constant $h^\beta_{a,t}$ does not depend on x, that is a consequence of the Heine-Cantor Theorem.

Now, let $T>0$, $t\in[0,T)$ and $\beta>0$, we can see that, if we denote

$$\bar{h}^{\beta}_{a,t,T}=\min\left(\frac{h^{\beta}_{a,t}}{\zeta},\frac{a}{4(2\overline{\gamma}+1)},\frac{\beta}{4},\frac{T-t}{2}\right),$$

then, for all $0<h\le\bar{h}^{\beta}_{a,t,T}$, assumption (5.44) holds. Therefore, we obtain

$$\overline{u}^{\alpha}(t+h,x)\le 2h+w^{\alpha,1}(t,x+h\zeta)-w^{\alpha,2}(t,x-h\zeta)\le\beta+w^{\alpha,1}_{r}(t,x)-w^{\alpha,2}_{l}(t,x),\quad(5.51)$$

where we have used (5.45) in the first inequality and (5.50) in the second one. Similarly, using (5.49) and (5.50), we can prove that, for all $x\in[-\frac{a}{2},\frac{a}{2}]$, $t\in[0,T)$, $\beta>0$ and $0<h\le\bar{h}^{\beta}_{a,t,T}$, we have

$$-\beta+w^{\alpha,1}_{l}(t,x)-w^{\alpha,2}_{r}(t,x)\le\underline{u}^{\alpha}(t+h,x).\tag{5.52}$$

Step 3. (Link between $\overline{u}^{\alpha}$ and $\underline{u}^{\alpha}$):
Let $T>0$, $x\in[-\frac{a}{2},\frac{a}{2}]$, $t\in[0,T)$ and $\beta>0$. In Step 2, we proved that, there exists a positive constant $\bar{h}^{\beta}_{a,t,T}$, such that, for all $\alpha=1,\dots,d$ and $0<h\le\bar{h}^{\beta}_{a,t,T}$, we have

$$\begin{aligned}\overline{u}^{\alpha}(t+h,x)&\le\beta+w^{\alpha,1}_{r}(t,x)-w^{\alpha,2}_{l}(t,x)\\-\underline{u}^{\alpha}(t+h,x)&\le\beta-w^{\alpha,1}_{l}(t,x)+w^{\alpha,2}_{r}(t,x).\end{aligned}\tag{5.53}$$

Since $\bigcup_{t\in[0,T)}[t,t+\bar{h}^{\beta}_{a,t,T}]$ is a cover of $[0,\frac{T}{2}]$, then there is a finite number N^{β}_{a} of ordered intervals, satisfying

$$\bigcup_{0\le j\le N^{\beta}_{a}}[\tau^{\beta}_{a,j},\tau^{\beta}_{a,j}+\bar{h}^{\beta}_{a,\tau^{\beta}_{a,j},T}]\supset\left[0,\frac{T}{2}\right]\text{ with }\tau^{\beta}_{a,0}=0\text{ and }\tau^{\beta}_{a,j+1}=\tau^{\beta}_{a,j}+\bar{h}^{\beta}_{a,\tau^{\beta}_{a,j},T}\text{ for }j=1,\dots,N^{\beta}_{a}-1.$$

This expression joint to (5.53) and the fact that $\mathbb{R}=\bigcup_{a\in\mathbb{Q}}[-\frac{a}{2},\frac{a}{2}]$ shows that, for all $x\in\mathbb{R}$, $\tau\in[0,\frac{T}{2}]$, and for all positive constant $\beta\in\mathbb{Q}$, there exist two indices $a_0\in\mathbb{Q}$ and $0\le k_0\le N^{\beta}_{a_0}$, such that,

$$\begin{aligned}\overline{u}^{\alpha}(\tau,x)&\le\beta+w^{\alpha,1}_{r}(\tau^{\beta}_{a_0,k_0},x)-w^{\alpha,2}_{l}(\tau^{\beta}_{a_0,k_0},x)\\-\underline{u}^{\alpha}(\tau,x)&\le\beta-w^{\alpha,1}_{l}(\tau^{\beta}_{a_0,k_0},x)+w^{\alpha,2}_{r}(\tau^{\beta}_{a_0,k_0},x).\end{aligned}\tag{5.54}$$

Moreover, from property (ii) of Lemma 5.3, we know that, for all positive constants $\beta,a\in\mathbb{Q}$ and $0\le k\le N^{\beta}_{a}$, the functions $w^{\alpha,1}_{r}(\tau^{\beta}_{a,k},\cdot)$, $w^{\alpha,1}_{l}(\tau^{\beta}_{a,k},\cdot)$ (resp. $w^{\alpha,2}_{r}(\tau^{\beta}_{a,k},\cdot)$, $w^{\alpha,2}_{l}(\tau^{\beta}_{a,k},\cdot)$) coincide with $w^{\alpha,1}(\tau^{\beta}_{a,k},\cdot)$ (resp. $w^{\alpha,2}(\tau^{\beta}_{a,k},\cdot)$) except on a countable set on $\mathbb{R}$, denoted $D^{\beta,\alpha}_{a,k}$. Now, we define the following countable set

$$D=\bigcup_{\alpha=1}^{d}\bigcup_{a,\beta\in\mathbb{Q}}\bigcup_{0\le k\le N^{\beta}_{a}}D^{\beta,\alpha}_{a,k}.$$

Thanks to (5.54), we can see that, for all $x \notin D$, $\tau \in [0, \frac{T}{2}]$ and for all positive constant $\beta \in \mathbb{Q}$, there exist two indices $a_0 \in \mathbb{Q}$ and $0 \leq k_0 \leq N^{\beta}_{a_0}$, such that

$$\overline{u}^{\alpha}(\tau, x) \leq \beta + w^{\alpha,1}(\tau^{\beta}_{a_0,k_0}, x) - w^{\alpha,2}(\tau^{\beta}_{a_0,k_0}, x) \leq \beta + w^{\alpha}(\tau^{\beta}_{a_0,k_0}, x)$$
$$-\underline{u}^{\alpha}(\tau, x) \leq \beta - w^{\alpha,1}(\tau^{\beta}_{a_0,k_0}, x) + w^{\alpha,2}(\tau^{\beta}_{a_0,k_0}, x) \leq \beta - w^{\alpha}(\tau^{\beta}_{a_0,k_0}, x).$$

Adding the previous inequalities, we deduce that, for all rational number $\alpha > 0$, $x \notin D$ and $\tau \in [0, \frac{T}{2}]$,

$$0 \leq \overline{u}^{\alpha}(\tau, x) - \underline{u}^{\alpha}(\tau, x) \leq 2\beta.$$

Passing to the limit $\beta \to 0$, and replacing T by $2T$, we get

$$\overline{u}^{\alpha}(\tau, \cdot) = \underline{u}^{\alpha}(\tau, \cdot), \text{ except at most on a countable set in } \mathbb{R}, \text{ for all } \tau \in [0, T], \ \alpha = 1, \dots, d. \tag{5.55}$$

This equality allows us to link the sub-solution $\overline{u}^{\alpha}$ and the super-solution $\underline{u}^{\alpha}$. It remains to show that

$$u^{\alpha}(\tau, \cdot) = \overline{u}^{\alpha}(\tau, \cdot) \quad \text{except at most on a countable set in } \mathbb{R}, \quad \text{for all} \quad \tau \in [0, T].$$

To do this, it is sufficient to use, the right continuity of the function $u^{\alpha}(\tau, \cdot)$ and Lemma 5.4 (i). Indeed, let $\beta > 0$, the right continuity of the functions $u^{\alpha}(\tau, \cdot)$ implies that, for all $x \in [-\frac{a}{2}, \frac{a}{2}]$, there exists $\beta^1_{a,\tau} > 0$, such that, for all $0 < \delta \leq \beta^1_{a,\tau}$, we have

$$\begin{aligned} u^{\alpha}(\tau, x) &\leq \beta + u^{\alpha}(\tau, x + \delta) = \beta + u^{\alpha,1}(\tau, x + \delta) - u^{\alpha,2}(\tau, x + \delta) \\ &\leq 2\beta + u^{\alpha,1}(\tau, x) - u^{\alpha,2}(\tau, x + \delta) \end{aligned} \tag{5.56}$$

where $u^{\alpha,1}$, $u^{\alpha,2}$ are the right-continuous non-decreasing functions, given in (5.48). However, using Lemma 5.4 (i), we know that, for all $0 < \delta \leq \frac{a}{2}$, there exists $k^{\beta}_{a,\tau} > 0$ and a subsequence $0 < \varepsilon_k \leq \delta$, such that for every $k \geq k^{\alpha}_{a,\tau}$,

$$\left| \begin{aligned} & u^{\alpha,1}(\tau, x) \leq \tfrac{\delta}{2} + u_1^{\alpha,\varepsilon_k}(\tau, x + \tfrac{\delta}{2}) \\ & -u^{\alpha,2}(\tau, x + \delta) \leq \tfrac{\delta}{2} - u_2^{\alpha,\varepsilon_k}(\tau, x + \tfrac{\delta}{2}), \end{aligned} \right. \tag{5.57}$$

where the sequences $u_1^{\alpha,\varepsilon_k}$ and $u_2^{\alpha,\varepsilon_k}$ satisfy the following equality $u^{\alpha,\varepsilon_k} = u_1^{\alpha,\varepsilon_k} - u_2^{\alpha,\varepsilon_k}$. Finally, bringing together the two inequalities (5.56) and (5.57), we can see that, for all $0 < \delta \leq \min(\frac{a}{2}, \beta^1_{a,\tau})$, $k \geq k^{\alpha}_{a,\tau}$, we have

$$u^{\alpha}(\tau, x) \leq 2\beta + \delta + u^{\alpha,\varepsilon_k}(\tau, x + \frac{\delta}{2}) \leq 2\beta + \delta + \sup_{\substack{\epsilon_k \leq \delta, |s-\tau| \leq \delta \\ |y-x| \leq 2\delta}} u^{\alpha,\varepsilon_k}(s, y).$$

To complete the proof, we pass to the limit $\delta \to 0$ and then $\beta \to 0$, to get $u^\alpha(\tau, x) \leq \overline{u}^\alpha(\tau, x)$. Similarly, we can show that $\underline{u}^\alpha(\tau, x) \leq u^\alpha(\tau, x)$, which joint to (5.55) proves the desired result.

□

5 Numerical simulations

We consider in this section a simplified model of (5.1) that describes the dynamics of dislocations densities, where a dislocation is a linear crystallographic defect or irregularity within a material. This particular model was initially proposed in 2D dimensions by Groma and Balogh [53, 54], where the dislocations are considered as points moving in the plane (x_1, x_2), propagating to the left and to the right, following two vectors $\pm(1, 0)$. In a specific geometry, we assume that the dislocations densities depend on one variable $x = x_1 + x_2$, which reduces the 2D model into a 1D one. We refer to El Hajj, Forcadel [41] for more details about the modeling.
More precisely, this 1D model can be expressed as

$$\begin{cases} \partial_t v^1(t,x) = -\left((v^1 - v^2)(t,x) + \beta \displaystyle\int_0^1 (v^1 - v^2)(t,y)dy + a(t)\right)\left|\partial_x v^1(t,x)\right| & \text{in } (0,T)\times\mathbb{R}, \\ \partial_t v^2(t,x) = \left((v^1 - v^2)(t,x) + \beta \displaystyle\int_0^1 (v^1 - v^2)(t,y)dy + a(t)\right)\left|\partial_x v^2(t,x)\right| & \text{in } (0,T)\times\mathbb{R}, \end{cases} \tag{5.58}$$

where v^1, v^2 are the scalar functions, representing respectively the right and left propagating dislocations. Their spatial derivatives $\partial_x v^1, \partial_x v^2$ represent the dislocations densities corresponding to each type. The constant β depends on the elastic coefficients and the material size, while the function $a(t)$ represents the exterior shear stress.
It is clear that in the particular case where $\beta = 0$, the model (5.58) reduces to the following system

$$\begin{cases} \partial_t v^1(t,x) = -\big((v^1 - v^2)(t,x) + a(t)\big)\left|\partial_x v^1(t,x)\right| & \text{in } (0,T)\times\mathbb{R}, \\ \partial_t v^2(t,x) = \big((v^1 - v^2)(t,x) + a(t)\big)\left|\partial_x v^2(t,x)\right| & \text{in } (0,T)\times\mathbb{R}, \end{cases} \tag{5.59}$$

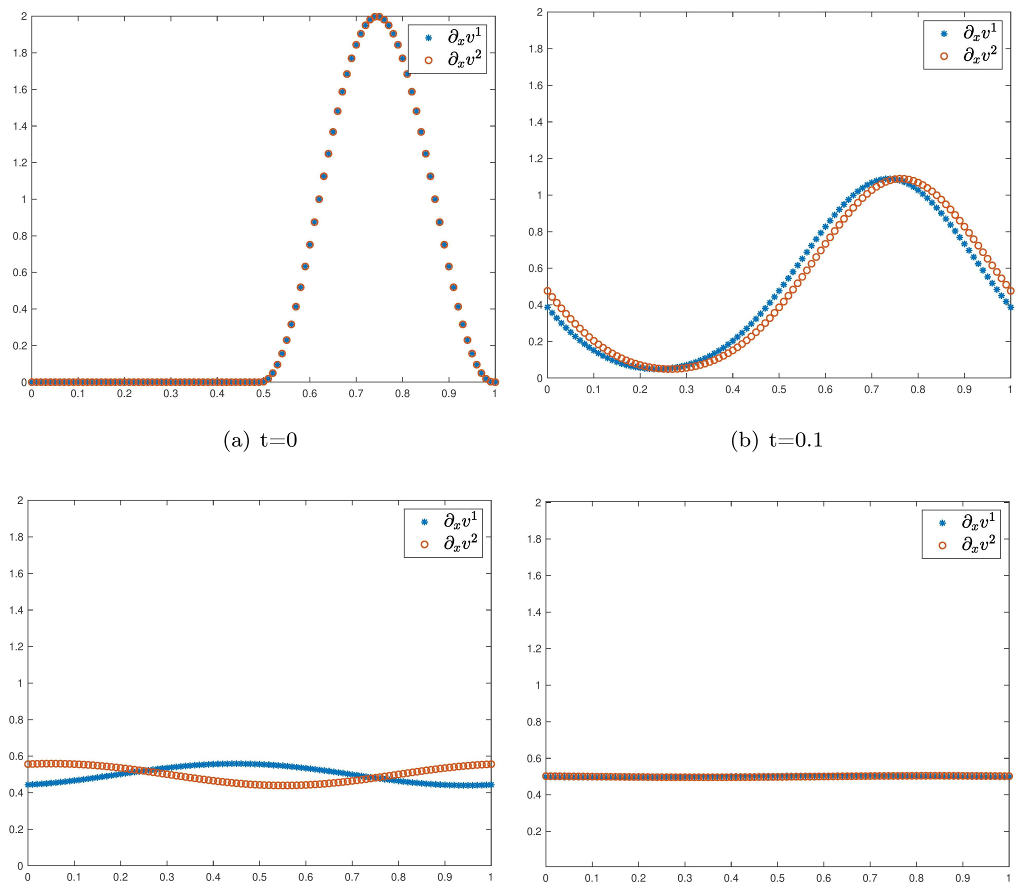
which is indeed in the form of system (5.1).
We equip system (5.59) with non-decreasing initial data of the form

$$v_0^1(x) = v_0^2(x) = v_0(x) = v^{per}(x) + L_0 x,$$

where v^{per} are periodic functions of period 1. The use of periodic plus linear boundary conditions is a way of regarding what is going on inside the material, away from the boundaries. We thus model a periodic distribution for the two dislocations types, with a spatial period of length 1.

Now, we will demonstrate numerical simulations of system (5.59), using scheme (5.9) and choosing discretization parameters $\Delta t, \Delta x$ that satisfy (5.15), under a stress of $a(t) = 3t$, and $L_0 = 0.5$. As it is represented in Figure 5.1(a), we assumed that the dislocations densities are not uniformly distributed in space at $t = 0$, in other words, there exists regions with concentrated dislocations, and others without any dislocations at all. We remark here that if $a(t) = 0$, then the dislocations will not propagate. However, when we exert an exterior stress, we notice that the dislocations densities begin to diffuse inside the material (Figure 5.1(b),5.1(c)), to reach a constant density that is equal to $L_0 = 0.5$, and fill the entire material at $t = 1$, as we can see in Figure 5.1(d).

We remark that when $a(t)$ is non-stationary, system (5.59) behaves as a diffusion equation (see Briani, Monneau [23] for further details).



(a) t=0 (b) t=0.1

(c) t=0.5 (d) t=1

Figure 5.1: Dislocations densities distribution $\partial_x v^1(\cdot,t), \partial_x v^2(\cdot,t)$ at several instants.

6 New contraction to the evolutionary p-Laplacian equation

This chapter is in collaboration with Mustapha Jazar and Ghada Chamycem.

In this work, we present a new contraction result to the positive solutions of the evolutionary p-Laplacian equation by introducing a new distance that depends on certain paths constructed between the solutions of the given equation, and showing that this distance is decreasing on a certain domain.

We have some primary outcomes of a similar result in the case where the solutions can change sign.

New contraction result to the positive solutions of the evolutionary p-Laplacian equation

Maryam Al Zohbi, Ghada Chmaycem, Mustapha Jazar

Abstract

For $\Omega \subset \mathbb{R}^d$ an open connected subset with smooth boundary, and $p > 1$, we prove a new contraction result to the positive solutions of the evolutionary p-Laplacian equation, by introducing a new distance between the solutions. This distance also enables us to obtain, as a consequence, the well known result which states that the solutions of the p-Laplacian evolutionary equation have non-decreasing $L^r(\Omega)$ norm for $r \geq 1$.

AMS Classification: 47H20, 47H35, 47H09.

Key words: Non-linear operators, p-Laplace operator, contraction distance.

1 Introduction and main result

We consider, for $\Omega \subset \mathbb{R}^d$ an open connected subset with smooth boundary and $p > 1$, the following p-Laplacian problem

$$\begin{cases} \partial_t u(t,x) = \Delta_p u(t,x) \quad \text{in} \quad Q_T = (0,T) \times \Omega, \\ u(t,x) = 0 \quad \text{on} \quad \partial(0,T) \times \Omega, \\ u(0,\cdot) = u_0 \quad \text{in} \quad \Omega. \end{cases} \tag{6.1}$$

The initial condition u_0 satisfies

$$u_0 \geq 0 \quad \text{in} \quad \Omega. \tag{6.2}$$

It is known that for $\Omega \subset \mathbb{R}^d$ bounded domain with Lipschitz continuous boundary, the energy functional

$$\varphi(u) = \int_\Omega |\nabla u|^p dx, \tag{6.3}$$

associated with the p-Laplacian operator $\Delta_p u = \nabla \cdot (\nabla u |\nabla u|^{p-2})$, gives rise to strongly continuous nonlinear (linear if $p = 2$) semigroups of contractions on $L^2(\Omega)$ (see for example [32, Th. 4.8]). A classical result by Minty [71] (see also Evans [49, Ch. 9], or the monograph by Brézis [22]) shows that every convex, lower semi-continuous functional φ on a Hilbert space H generates a strongly continuous semigroup (nonlinear in general) of contractions on $\overline{D(\varphi)}$.

Moreover on the contraction results known for (6.1), it is easy to show that $\|u(t)\|_{W^{1,p}(\Omega)} \leq \|u_0\|_{W^{1,p}(\Omega)}$ for all t, since

$$\frac{d}{dt}\left(\frac{1}{p}\int_\Omega |\nabla u|^p dx\right) = \int_\Omega \nabla(\partial_t u)\nabla u|\nabla u|^{p-2}dx = -\int_\Omega (\partial_t u)^2 dx \leq 0.$$

Similarly,

$$\frac{d}{dt}\left(\frac{1}{p}\int_\Omega |u|^p dx\right) = -(p-1)\int_\Omega (\nabla u)^2|\nabla u|^{p-2}|u|^{p-2}dx \leq 0.$$

In this work, we introduce a new pseudo-metric between positive solutions of (6.1), in order to design a new family of contraction to this system. To that end, we announce our main result in Theorem 6.1.

In what follows, we denote by $(f)_+$ the positive part of a scalar function f.

Theorem 6.1.

Let u and v be two solutions of (6.1), belonging to the space $C^2(Q_T)$, with initial data u_0 and v_0 respectively, both satisfying (6.2). For $p > 1$, $q > 1$, and $\alpha \in \mathbb{R}$ such that $0 < \alpha_- < \alpha < 1$, where

$$\alpha_- := \frac{4(q-1)(p-1)}{4q(q-1)(p-1) - p^2q^2} + 1, \tag{6.4}$$

we have

$$\int_\Omega (v^\alpha - u^\alpha)_+^q dx \leq \int_\Omega ((v_0)^\alpha - (u_0)^\alpha)_+^q \, dx. \tag{6.5}$$

As a consequence of this theorem, we obtain that the solutions of (6.1) have non-increasing $L^r(\Omega)$ norm for $r \geq 1$, which is a quite known result for the p-Laplacian operator (See [16]). Thus, we have the following corollary.

Corollary 6.1.

Under the conditions of Theorem 6.1, the solution v satisfies, for all $r \geq 1$, the following estimate

$$\|v(t)\|_{L^r(\Omega)} \leq \|v_0\|_{L^r(\Omega)}, \quad \text{for all } t > 0. \tag{6.6}$$

The proof of this corollary is obvious if Theorem 6.1 is valid. The following section is devoted to the proof of the Theorem 6.1. It is divided into two Sections. First in Section 2, we illustrate the distance which will be used in the contraction argument. Then in Section 3, we show that this distance admits indeed a domain of contraction for the positive solutions of system (6.1), hence, proving Theorem 6.1, and as a consequence we obtain Corollary 6.1.

2 Construction of a distance

Our main tool in the proof of the contraction is a new distance constructed between the solutions. Let u and v be two solutions of (6.1), belonging to the space $C^2(Q_T)$, with positive initial data u_0 and v_0 respectively. We introduce the function

$$\begin{aligned} w_0 : [0,1] &\to C^2(\Omega) \\ s &\mapsto w_0(s) := w_0^{(s)}, \end{aligned}$$

where $w_0 \in C^2\big([0,1]; C^2(\Omega)\big)$ is a path joining u_0 to v_0. In other words, we have $w_0^{(0)} = u_0$ and $w_0^{(1)} = v_0$.
Next, we construct the function

$$\begin{aligned} w : [0,1] \times Q_T &\to \mathbb{R} \\ (s,t,x) &\mapsto w(s,t,x), \end{aligned}$$

where for all $s \in [0,1]$, we have $w(s,\cdot,\cdot) =: w^{(s)}(\cdot,\cdot)$ is a solution of (6.1) with initial data $w_0^{(s)}$. It is clear that $w^{(0)} = u$ and $w^{(1)} = v$.
Now we define the set

$$\mathcal{W}_v^u = \Big\{ w \in C^2\big([0,1] \times Q_T\big) : w^{(0)} = u \text{ and } w^{(1)} = v \Big\}, \tag{6.7}$$

which is a set of paths connecting u to v.

Definition 6.1 (A pseudo-distance)**.**
Given two $C^2(Q_T)$-solutions u and v of (6.1), we define the following pseudo-distance

$$d(u,v) := \inf_{w \in \mathcal{W}_v^u} \mathcal{A}(w) \quad \text{with} \quad \mathcal{A}(w) := \int_0^1 ds \int_\Omega w_+^\gamma \frac{(w')_+^q}{q} dx, \tag{6.8}$$

where the set $\mathcal{W}_v^u$ was defined in (6.7), $\gamma \in \mathbb{R}$, and $q \in \mathbb{R}_+^*$.

Remark 6.1. (i) It is known that the p-Laplacian operator generates a positive semi-group; meaning that a solution u is positive whenever its initial data u_0 is positive.

(ii) As we only consider positive solutions, it is meaningful then to consider only the positive paths w joining u to v in the definition of the set $\mathcal{W}_v^u$, and thus

$$\mathcal{A}(w) = \int_0^1 ds \int_\Omega w^\gamma \frac{(w')_+^q}{q} dx. \tag{6.9}$$

What is interesting about this distance is that it can be explicitly expressed in terms of the solutions. To this end, we introduce the following lemma.

Lemma 6.1.
For $\alpha = 1 + \frac{\gamma}{q}$, where $q \in \mathbb{R}_+^*$ and $\gamma \in \mathbb{R}$, the distance defined in Definition 6.1 can be explicitly written as follows

$$d(u,v) = \inf_{w \in \mathcal{W}_v^u} \int_0^1 ds \int_\Omega w^\gamma \frac{(w')_+^q}{q} dx = \frac{1}{q\alpha^q} \int_\Omega (v^\alpha - u^\alpha)_+^q dx. \tag{6.10}$$

Proof of Lemma 6.1.
The proof of this Lemma is done in two steps.

Step 1.
We will prove the following inequality

$$\inf_{w \in \mathcal{W}_v^u} \mathcal{A}(w(t)) \geq \frac{1}{q\alpha^q} \int_\Omega (v^\alpha - u^\alpha)_+^q dx. \tag{6.11}$$

We have

$$\begin{aligned}\mathcal{A}(w) &= \int_0^1 ds \int_\Omega w^\gamma \frac{(w')_+^q}{q} dx \\ &= \frac{1}{q\alpha^q} \int_\Omega dx \int_0^1 \left(\frac{\partial}{\partial s} w^\alpha\right)_+^q ds.\end{aligned}$$

Now we apply Jensen's inequality. Thus we get

$$\begin{aligned}\mathcal{A}(w) = \frac{1}{q\alpha^q} \int_\Omega dx \int_0^1 \left(\frac{\partial}{\partial s} w^\alpha\right)_+^q ds &\geq \frac{1}{q\alpha^q} \int_\Omega dx \left(\int_0^1 \frac{\partial}{\partial s} w^\alpha ds\right)_+^q \\ &= \frac{1}{q\alpha^q} \int_\Omega \left(\Big[w^\alpha\Big]_{s=0}^{s=1}\right)_+^q dx \\ &= \frac{1}{q\alpha^q} \int_\Omega (v^\alpha - u^\alpha)_+^q dx.\end{aligned}$$

So we get

$$\mathcal{A}(w) \geq \frac{1}{q\alpha^q} \int_\Omega (v^\alpha - u^\alpha)_+^q dx \quad \text{for any } w \in \mathcal{W}_v^u.$$

Hence we obtain inequality (6.11).

Step 2.
Now we shall prove the following inequality

$$\inf_{w \in \mathcal{W}_v^u} \mathcal{A}(w) \leq \frac{1}{q\alpha^q} \int_\Omega (v^\alpha - u^\alpha)_+^q dx. \tag{6.12}$$

Consider the path $\widetilde{w} \in \mathcal{W}_v^u$ defined by $\widetilde{w} := ((v^\alpha - u^\alpha)s + u^\alpha)^{\frac{1}{\alpha}}$.
We have

$$d(u,v) = \inf_{w \in \mathcal{W}_v^u} \mathcal{A}(w) \leq \mathcal{A}(\widetilde{w}) = \int_0^1 ds \int_\Omega \widetilde{w}^\gamma \frac{(\widetilde{w}')_+^q}{q} dx = \frac{1}{q\alpha^q} \int_\Omega dx \int_0^1 \left(\frac{\partial}{\partial_s} \widetilde{w}^\alpha\right)_+^q ds$$

$$= \frac{1}{q\alpha^q} \int_\Omega (v^\alpha - u^\alpha)_+^q dx.$$

Consequently, we obtain inequality (6.12).

Therefore, inequalities (6.11) and (6.12) lead to (6.10).

□

3 Contraction of the distance

In this section, we present the proof of Theorem 6.1, and we show how we can easily obtain Corollary 6.1.

Proof of Theorem 6.1.
Denote by $\Omega_+ =: \{x \in \Omega : w'(x,t,s) \geq 0\}$. We poceed in two steps.

Step 1. (Derivation of $\mathcal{A}$ with respect to time)
Differentiating the first equation in (6.1) with respect to the variable s, we deduce that the equation satisfied by w' is

$$\partial_t w' = (p-1) \operatorname{div} \left(\nabla w' |\nabla w|^{p-2}\right). \tag{6.13}$$

Using (6.13) in the derivative of the quantity $A(w)$ (defied in (6.9)), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{A}(w) &= \frac{d}{dt}\left(\frac{1}{q}\int_0^1 ds\int_{\Omega_+} w^\gamma (w')^q dx\right)\\ &= \frac{1}{q}\int_0^1 ds\int_{\Omega_+}\Big((\gamma w^{\gamma-1}\partial_t w(w')^q + q(w')^{q-1}\partial_t w' w^\gamma\Big)dx\\ &= \frac{1}{q}\int_0^1 ds\int_{\Omega_+}\Big(\gamma w^{\gamma-1}(w')^q \operatorname{div}\left(\nabla w|\nabla w|^{p-2}\right)\\ &+ q(p-1)(w')^{q-1}w^\gamma \operatorname{div}\left(\nabla w'|\nabla w'|^{p-2}\right)\Big)dx.\end{aligned}$$

Applying Green's formula we get

$$\begin{aligned}\frac{d}{dt}\mathcal{A}(w) =& -\frac{1}{q}\int_0^1 ds\int_{\Omega_+}\Big(\gamma\nabla\left(w^{\gamma-1}(w')^q\right)\nabla w|\nabla w|^{p-2}\\ &+ q(p-1)\nabla\left((w')^{q-1}w^\gamma\right)\nabla w'|\nabla w|^{p-2}\Big)dx\\ =& -\frac{1}{q}\int_0^1 ds\int_{\Omega_+}\Big(\gamma\Big((\gamma-1)w^{\gamma-2}\nabla w(w')^q + q(w')^{q-1}\nabla w' w^{\gamma-1}\Big)\nabla w|\nabla w|^{p-2}\\ &+ q(p-1)\Big((q-1)(w')^{q-2}\nabla w' w^\gamma\\ &+ \gamma w^{\gamma-1}\nabla w(w')^{q-1}\Big)\nabla w'|\nabla w|^{p-2}\Big)dx\\ =& -\frac{1}{q}\int_0^1 ds\int_{\Omega_+} w^\gamma (w')^q|\nabla w|^{p-2}\Big(\gamma(\gamma-1)w^{-2}|\nabla w|^2 + \gamma q w^{-1}(w')^{-1}\nabla w\nabla w'\\ &+ q(q-1)(p-1)(w')^{-2}|\nabla w'|^{-2} + \gamma q(p-1)w^{-1}(w')^{-1}\nabla w\nabla w'\Big)dx\\ =& -\frac{1}{q}\int_0^1 ds\int_{\Omega_+} w^\gamma (w')^q|\nabla w|^{p-2}\Big(\gamma(\gamma-1)|\nabla\ln w|^2 + \gamma q p\nabla\ln w\nabla\ln w'\\ &+ q(q-1)(p-1)|\nabla\ln w'|\Big)dx\\ =& -\int_0^1 ds\int_{\Omega_+} w^\gamma (w')^q|\nabla w|^{p-2}\begin{pmatrix}\nabla\ln w\\ \nabla\ln w'\end{pmatrix}^t M\begin{pmatrix}\nabla\ln w\\ \nabla\ln w'\end{pmatrix}dx,\end{aligned}$$

where

$$M = \begin{pmatrix} \dfrac{\gamma(\gamma-1)}{q} & \dfrac{\gamma p}{2} \\ \\ \dfrac{\gamma p}{2} & (q-1)(p-1) \end{pmatrix}.$$

Since M is a real symmetric square matrix, and $(q-1)(p-1) > 0$, then M is positive definite if and only if $\det M > 0$, which is equivalent to having the condition $\gamma_- < \gamma < 0$, where

$$\gamma_- := \frac{4(p-1)(q-1)}{4(p-1)(q-1) - p^2 q}.$$

Then we get $\dfrac{d}{dt}\mathcal{A}(w) \leq 0$ on Ω_+ whenever $\gamma \in (\gamma_-, 0)$, or equivalently, whenever $\alpha \in (\alpha_-, 1)$.

Step 2. (Verification of inequality (6.5))
By Lemma 6.1, we have

$$\frac{1}{q\alpha^q}\int_\Omega (v^\alpha - u^\alpha)_+^q dx = \inf_{w \in \mathcal{W}_v^u} \mathcal{A}(w(t)) \leq \mathcal{A}(w(t)) \leq \mathcal{A}(w(0)),$$

for any $w \in \mathcal{W}_v^u$ and $\alpha \in (\alpha_-, 1)$ by step 1.
Thus

$$\inf_{w \in \mathcal{W}_v^u} \mathcal{A}(w(t)) \leq \inf_{w \in \mathcal{W}_v^u} \mathcal{A}(w(0)) = d(u_0, v_0) = \frac{1}{q\alpha^q}\int_\Omega (v_0^\alpha - u_0^\alpha)_+^q dx.$$

Hence, we have proved inequality (6.5).

□

Proof of Corollary 6.1.
Applying Theorem 6.1 for $u \equiv 0$, inequality (6.5) reduces to

$$\int_\Omega v_+^{\alpha q} dx \leq \int_\Omega v_{0+}^{\alpha q} dx, \text{ i.e., } \int_\Omega v^{\alpha q} dx \leq \int_\Omega v_0^{\alpha q} dx,$$

since we are working with positive solutions only. Hence, we get (6.6) for any $r := \alpha q = q + \gamma \in (\gamma_- + q, q)$. Notice that the contraction result holds for all $q > 1$, therefore inequality (6.6) is valid here for all $r \geq 1$.

□

Another application of the distance defined in Definition 6.1 can be found in [33].

Conclusion and perspectives

As a conclusion, we have proven 4 main results in this thesis dissertation, of which 3 are motivated by the dynamics of dislocations, and the fourth is a contraction result to the evolutionary p-Laplacian equation.

The first main result was a global in time existence of a discontinuous viscosity solution to a diagonal hyperbolic system of transport equations in one space dimension. The proof of this result was accomplished by considering a parabolic regularization of the main system and then passing to the limit when the regularization vanishes. The same technique was then employed in order to prove a similar result on an eikonal system, which is in fact a generalization of the hyperbolic system studied in the first result. The second main result was proving the existence and uniqueness of a continuous solution to the same eikonal system. This result was based on a Comparison Principle. The third and final main result in this field of study, was proving that a certain semi-explicit finite difference scheme approximating the eikonal system converges to a discontinuous viscosity solution of this system. We have applied these last two results to a particular periodic case of the main eikonal system. In addition, we have provided our final result with some numerical simulations.

On a different approach, our fourth result in this dissertation was creating a new family of contracting positive solutions to the evolutionary p-Laplacian equation. This result was possible due to the construction of a certain distance between the solutions of the p-Laplacian equation considered.

After this thesis, we are planning first to apply the result of Chapter 3 to a system that models the dynamics of isentropic gas, which is also of diagonal hyperbolic type, in one space dimension. We will show that this system admits a discontinuous viscosity solution in distributional sense.

We also plan on studying different models describing the dynamics of dislocations, with the aim of better understanding their mechanism and effect to materials. Of these models we mention one by Acharya [1], where building on some fundamental laws of kinematics or conservation, a full time-dependent 3D system of equations governing the rate of plastic distortion tensor is formulated. We are also interested in the 2D Frenkel–Kontorova model [21, 61] which describes a chain of classical particles with nearest neighbor interactions. This model was originally introduced to describe the structure and dynamics of a crystal lattice near a dislocation core.
From a numerical point of view, we will try to prove an error estimate between the continuous solution and its numerical approximation as a continuation of the work presented in Chapter 5.
In addition, after this thesis we are planning to carry out a homogenization study of the hyperbolic system studied in Chapter 3, with the aim of applying the results to the dynamics dislocations. Due to the fact that dislocations are disordered microscopic defects, it is natural to try homogenizing their behavior or effect to the macroscopic level.

As for the theory of differential contraction, we aim in improving our result on the p-Laplacian evolutionary equation, in order to include all solutions, whether positive or negative. We already have primary results on how to enhance the distance used. Moreover, we will try to modify our distance so that it can applied on equations that are not necessarily of divergence type.

Bibliography

[1] A. Acharya, *New inroads in an old subject: plasticity, from around the atomic to the macroscopic scale*, J. Mech. Phys. Solids, 58 (2010), pp. 766–778.

[2] M. Al Zohbi, A. El hajj, and M. Jazar, *Existence and uniqueness results to a system of hamilton-jacobi equations*, submitted, (2021).

[3] M. Al Zohbi, A. El Hajj, and M. Jazar, *Global existence to a diagonal hyperbolic system for any BV initial data*, Nonlinearity, 34 (2021), pp. 54–85.

[4] O. Alvarez, E. Carlini, R. Monneau, and E. Rouy, *Convergence of a first order scheme for a non-local eikonal equation*, Appl. Numer. Math., 56 (2006), pp. 1136–1146.

[5] ——, *A convergent scheme for a non local Hamilton Jacobi equation modelling dislocation dynamics*, Numer. Math., 104 (2006), pp. 413–444.

[6] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[7] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.

[8] G. Barles, *Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: a guided visit*, Nonlinear Anal., 20 (1993), pp. 1123–1134.

[9] ——, *Solutions de viscosité des équations de Hamilton-Jacobi*, vol. 17 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer-Verlag, Paris, 1994.

[10] G. Barles and B. Perthame, *Exit time problems in optimal control and vanishing viscosity method*, SIAM J. Control Optim., 26 (1988), pp. 1133–1148.

[11] ——, *Comparison principle for Dirichlet-type Hamilton-Jacobi equations and singular perturbations of degenerated elliptic equations*, Appl. Math. Optim., 21 (1990), pp. 21–44.

[12] G. Barles, H. M. Soner, and P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim., 31 (1993), pp. 439–469.

[13] ——, *Front propagation and phase field theory*, SIAM J. Control Optim., 31 (1993), pp. 439–469.

[14] G. Barles and P. E. Souganidis, *Convergence of approximation schemes for fully nonlinear second order equations*, Asymptotic analysis, 4 (1991), pp. 271–283.

[15] J. Benedikt, P. Girg, L. Kotrla, and P. Takac, *Origin of the p-laplacian and a. missbach*, Electronic Journal of Differential Equations, 2018 (2018), pp. 1–17.

[16] P. Bénilan and M. Crandall, *Completely accretive operators*, Semigroups Theory and Evolution Equations, Ph. Clement et al. editors, Marcel Dekker, (1991), pp. 41–76.

[17] S. Bianchini and A. Bressan, *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Ann. of Math. (2), 161 (2005), pp. 223–342.

[18] W. Bollmann, *Interference effects in the electron microscopy of thin crystal foils*, Physical Review, 103 (1956), p. 1588.

[19] R. Boudjerada and A. El Hajj, *Global existence results for eikonal equation with BV initial data*, NoDEA Nonlinear Differential Equations Appl., 22 (2015), pp. 947–978.

[20] R. Boudjerada, A. El Hajj, and A. Oussaily, *Convergence of an implicit scheme for diagonal non-conservative hyperbolic systems*, ESAIM Math. Model. Numer. Anal., 55 (2021), pp. S573–S591.

[21] O. M. Braun and Y. S. Kivshar, *Nonlinear dynamics of the Frenkel-Kontorova model*, Phys. Rep., 306 (1998), p. 108.

[22] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, Elsevier, 1973.

[23] A. Briani and R. Monneau, *Time-homogenization of a first order system arising in the modelling of the dynamics of dislocation densities*, Comptes Rendus Mathematique, 347 (2009), pp. 231–236.

[24] J. Burgers, *Geometrical considerations concerning the structural irregularities to be assumed in a crystal*, Proceedings of the Physical Society (1926-1948), 52 (1940), p. 23.

[25] F. Camilli and P. Loreti, *Comparison results for a class of weakly coupled systems of eikonal equations*, Hokkaido Math. J., 37 (2008), pp. 349–362.

[26] F. Camilli and A. Siconolfi, *Hamilton-Jacobi equations with measurable dependence on the state variable*, Adv. Differential Equations, 8 (2003), pp. 733–768.

[27] ——, *Time-dependent measurable Hamilton-Jacobi equations*, Comm. Partial Differential Equations, 30 (2005), pp. 813–847.

[28] P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, vol. 58, Springer Science & Business Media, 2004.

[29] M. Cannone, A. El Hajj, R. Monneau, and F. Ribaud, *Global existence for a system of non-linear and non-local transport equations describing the dynamics of dislocation densities*, Arch. Ration. Mech. Anal., 196 (2010), pp. 71–96.

[30] X. Chen and B. Hu, *Viscosity solutions of discontinuous Hamilton-Jacobi equations*, Interfaces Free Bound., 10 (2008), pp. 339–359.

[31] Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom., 33 (1991), pp. 749–786.

[32] R. Chill, D. Hauer, and J. Kennedy, *Nonlinear semigroups generated by j-elliptic functionals*, Journal de Mathématiques Pures et Appliquées, 105 (2016), pp. 415–450.

[33] G. Chmaycem, M. Jazar, and R. Monneau, *A new contraction family for porous medium and fast diffusion equations*, Archive for Rational Mechanics and Analysis, 221 (2016), pp. 805–815.

[34] M. G. Crandall, L. C. Evans, and P.-L. Lions, *Some properties of viscosity solutions of hamilton-jacobi equations*, Transactions of the American Mathematical Society, 282 (1984), pp. 487–502.

[35] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), 27 (1992), pp. 1–67.

[36] M. G. Crandall and P.-L. Lions, *Viscosity solutions of hamilton-jacobi equations*, Transactions of the American mathematical society, 277 (1983), pp. 1–42.

[37] M. G. Crandall and P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), pp. 1–42.

[38] M. G. Crandall and P.-L. Lions, *Two approximations of solutions of Hamilton-Jacobi equations*, Math. Comp., 43 (1984), pp. 1–19.

[39] A. El Hajj, *Well-posedness theory for a nonconservative Burgers-type system arising in dislocation dynamics*, SIAM J. Math. Anal., 39 (2007), pp. 965–986.

[40] A. El Hajj, *Short time existence and uniqueness in Hölder spaces for the 2D dynamics of dislocation densities*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), pp. 21–35.

[41] A. El Hajj and N. Forcadel, *A convergent scheme for a non-local coupled system modelling dislocations densities dynamics*, Math. Comp., 77 (2008), pp. 789–812.

[42] A. El Hajj, H. Ibrahim, and V. Rizik, *Global BV solution for a non-local coupled system modeling the dynamics of dislocation densities*, Journal of Differential Equations, 264 (2018), pp. 1750–1785.

[43] ——, *Global BV solution for a non-local coupled system modeling the dynamics of dislocation densities*, J. Differential Equations, 264 (2018), pp. 1750–1785.

[44] ——, *BV solution for a non-linear Hamilton-Jacobi system*, Discrete & Continuous Dynamical Systems, 41 (2021), p. 3273.

[45] A. El Hajj and R. Monneau, *Global continuous solutions for diagonal hyperbolic systems with large and monotone data*, J. Hyperbolic Differ. Equ., 7 (2010), pp. 139–164.

[46] ——, *Uniqueness results for diagonal hyperbolic systems with large and monotone data*, J. Hyperbolic Differ. Equ., 10 (2013), pp. 461–494.

[47] A. El Hajj and A. Oussaily, *Continuous solution for a non-linear eikonal system*, Communications on Pure & Applied Analysis, (2021).

[48] A. El Hajj and A. Oussaily, *Existence and Uniqueness of Continuous Solution for a Non-local Coupled System Modeling the Dynamics of Dislocation Densities*, J. Nonlinear Sci., 31 (2021), p. 20.

[49] L. Evans, *Partial Differential Equations*, vol. 19 of Graduate studies in mathematics, American Mathematical Society, 2010.

[50] L. C. Evans, *On solving certain nonlinear partial differential equations by accretive operator methods*, Israel Journal of Mathematics, 36 (1980), pp. 225–247.

[51] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature. I*, J. Differential Geom., 33 (1991), pp. 635–681.

[52] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math., 18 (1965), pp. 697–715.

[53] I. Groma and P. Balogh, *Link between the individual and continuum approaches of the description of the collective behavior of dislocations*, Materials Science and Engineering: A, 234 (1997), pp. 249–252.

[54] ——, *Investigation of dislocation pattern formation in a two-dimensional self-consistent field approximation*, Acta Materialia, 47 (1999), pp. 3647–3654.

[55] P. Hirsch, R. Horne, and M. Whelan, *Direct observations of the arrangement and motion of dislocations in aluminium*, Philosophical Magazine, 86 (2006), pp. 4553–4572.

[56] J. P. Hirth, J. Lothe, and T. Mura, *Theory of dislocations*, 1983.

[57] D. Hull and D. J. Bacon, *Introduction to dislocations*, Butterworth-Heinemann, 2001.

[58] H. Ishii, *Perron's method for monotone systems of second-order elliptic partial differential equations*, Differential Integral Equations, 5 (1992), pp. 1–24.

[59] H. Ishii and S. Koike, *Viscosity solutions for monotone systems of second-order elliptic PDEs*, Comm. Partial Differential Equations, 16 (1991), pp. 1095–1128.

[60] ——, *Viscosity solutions of a system of nonlinear second-order elliptic PDEs arising in switching games*, Funkcial. Ekvac., 34 (1991), pp. 143–155.

[61] Y. S. Kivshar, H. Benner, and O. M. Braun, *Nonlinear models for the dynamics of topological defects in solids*, in Nonlinear science at the dawn of the 21st century, vol. 542 of Lecture Notes in Phys., Springer, Berlin, 2000, pp. 265–291.

[62] S. Koike, *A beginner's guide to the theory of viscosity solutions*, vol. 13 of MSJ Memoirs, Mathematical Society of Japan, Tokyo, 2004.

[63] P. D. Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.

[64] P. LeFloch, *Entropy weak solutions to nonlinear hyperbolic systems under nonconservative form*, Comm. Partial Differential Equations, 13 (1988), pp. 669–727.

[65] P. LeFloch and T.-P. Liu, *Existence theory for nonlinear hyperbolic systems in nonconservative form*, Forum Math., 5 (1993), pp. 261–280.

[66] P. G. LeFloch, *Graph solutions of nonlinear hyperbolic systems*, J. Hyperbolic Differ. Equ., 1 (2004), pp. 643–689.

[67] R. J. LeVeque, *Finite volume methods for hyperbolic problems*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.

[68] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[69] P.-L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, vol. 69, London Pitman, 1982.

[70] P. Loreti and G. Vergara Caffarelli, *Variational solutions of coupled Hamilton-Jacobi equations*, Appl. Math. Optim., 41 (2000), pp. 9–24.

[71] G. J. Minty, *Monotone (nonlinear) operators in hilbert space*, Duke Mathematical Journal, 29 (1962), pp. 341–346.

[72] H. Mitake, A. Siconolfi, H. V. Tran, and N. Yamada, *A Lagrangian approach to weakly coupled Hamilton-Jacobi systems*, SIAM J. Math. Anal., 48 (2016), pp. 821–846.

[73] L. Monasse and R. Monneau, *Gradient entropy estimate and convergence of a semi-explicit scheme for diagonal hyperbolic systems*, SIAM J. Numer. Anal., 52 (2014), pp. 2792–2814.

[74] F. Nabarro, *Steady-state diffusional creep*, Philosophical Magazine, 16 (1967), pp. 231–237.

[75] E. Orowan, *Zur kristallplastizität. i-iii*, Zeitschrift für Physik, 89 (1934), pp. 605–613.

[76] S. Osher and J. A. Sethian, *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys., 79 (1988), pp. 12–49.

[77] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

[78] M. Polanyi, *Über eine art gitterstörung, die einen kristall plastisch machen könnte*, Zeitschrift für Physik, 89 (1934), pp. 660–664.

[79] D. Serre, *Systems of conservation laws. I, II*, Cambridge University Press, Cambridge, 1999-2000. Geometric structures, oscillations, and initial-boundary value problems, Translated from the 1996 French original by I. N. Sneddon.

[80] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), 146 (1987), pp. 65–96.

[81] P. E. Souganidis, *Approximation schemes for viscosity solutions of Hamilton-Jacobi equations*, J. Differential Equations, 59 (1985), pp. 1–43.

[82] G. I. Taylor, *The mechanism of plastic deformation of crystals. part i.—theoretical*, Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character, 145 (1934), pp. 362–387.

[83] H. V. Tran, *Hamilton-jacobi equations: viscosity solutions and applications*, 2019.

[84] V. Volterra, *Sur l'équilibre des corps élastiques multiplement connexes*, Ann. Sci. École Norm. Sup. (3), 24 (1907), pp. 401–517.

[85] S. Yefimov, *Discrete dislocation and nonlocal crystal plasticity modelling*, University Library Groningen][Host], 2004.

[86] S. Yefimov and E. Van der Giessen, *Multiple slip in a strain-gradient plasticity model motivated by a statistical-mechanics description of dislocations*, International Journal of Solids and Structures, 42 (2005), pp. 3375–3394.