Characterizations of matrix fields, potential matrices and applications to trace operators

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par

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Caractérisations de champs de matrices, potentiels matrices et applications aux opérateurs traces

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A mon petit ange " NAYA TALINE "

Caractérisations de champs de matrices, potentiels matrices et applications aux opérateurs traces

Résumé de thèse

Plusieurs auteurs ont utilisé les champs de contraintes pour résoudre l'équation d'équilibre de la mécanique des milieux continus. Airy (1863) a résolu le cas bidimensionnel, Maxwell (1870) et Morera (1892) ont étudié le cas tridimensionnel. Les solutions obtenues sont des cas particuliers de celles de Beltrami (1892). Gurtin a donné un exemple de solutions ne satisfaisant pas la représentation $\boldsymbol{S} = \mathbf{Curl}\,\mathbf{Curl}\,\mathbf{A}$ de Beltrami, ce qui signifie que la représentation précédente est incomplète. De plus, il a montré que si l'ouvert est régulier, alors elle est complète dans l'espace des champs réguliers de contraintes auto-équilibrés.

Dans cette thèse intitulée "Caractérisations de champs de matrices, potentiels matrices et applications aux opérateurs traces", on s'intéresse à diverses caractérisations de champs de vecteurs, de champs de matrices et spécialement au résultat de Gurtin dans le cas où l'ouvert et les champs de contraintes ne sont pas réguliers.

Cette thèse est décomposée en cinq chapitres. Le premier chapitre expose la problématique de recherche traitée dans cette thèse. Il présente également l'origine du sujet de recherche.

Dans le deuxième chapitre, on étudie l'opérateur **curl** et en particulier l'existence de potentiels vecteurs dans différents cadres fonctionnels.

Dans les chapitres 3 et 4, on va montrer quelques versions de la complétude de la représentation de Beltrami et en déduire des décompositions de Helmholtz pour les champs de matrices.

Le dernier chapitre est consacré à l'étude de l'image de différents opérateurs traces de fonctions $W^{2,p}(\Omega)$, $W^{3,p}(\Omega)$ lorsque Ω est un ouvert borné de $\mathbb{R}^2$ lipschitzien. L'ingrédient essentiel est donné par la fonction d'Airy ou par la représentation de Beltrami.

Mots clés

Champs de contraintes, représentation de Beltrami, potentiels vecteurs, complétude de Beltrami, décomposition de Helmholtz, fonction d'Airy, opérateurs traces.

Thesis abstract

Many authors have used stress fields to solve the equilibrium equation of continuum mechanics. Airy (1863) solved the two-dimensional case, Maxwell (1870) and Morera (1892) solved the three-dimensional case. The above solutions are special cases of those of Beltrami (1892). Gurtin gave an example of solutions that do not have Beltrami's $\boldsymbol{S} = \mathbf{Curl\,Curl\,A}$ representation. He showed that if the domain Ω is regular, then this representation is complete in the class of regular stress fields which are self-equilibrated.

My thesis title is "Characterizations of matrix fields, potential matrices and applications to trace operators". In this work, we are interested by showing many characterizations of vector fields, of matrix fields and especially by generalizing the result of Gurtin in the case when the open set and the stress fields are not regular.

This thesis consists of five chapters. The first chapter presents the research problem addressed in this thesis. It also presents the origin of the subject of research.

In the second chapter, we study the operator **curl**. In particular, the existence of potential vectors in different functional frameworks.

In Chapters 3 and 4, we will show some versions of Beltrami's completeness and we deduce some Helmholtz decomopsitions for symmetric matrix fields.

The last chapter is devoted to the study of the image of different trace operators of functions $W^{2,p}(\Omega)$, $W^{3,p}(\Omega)$ when Ω is a bounded open of $\mathbb{R}^2$ with Lipschitz boundary. The essential ingredient is given by the Airy's function or by the Beltrami representation.

Keywords

Stress fields, Beltrami representation, potential vectors, Beltrami's completeness, Helmholtz decomposition, Airy's function, trace operators.

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Notations and preliminaries

We denote by $|\cdot|$ the euclidean norm in $\mathbb{R}^N$. For $x \in \Omega$ and $d > 0$, we define the ball centred at x with radius d by $B(x, d) = \{y \in \mathbb{R}^N, \ |x - y| < d\}$. The open set Ω is starlike with respect to an open ball $B(x, d)$ if, for each $y \in \Omega$, the convex hull of the set $\{y\} \cup B(x, d)$ is contained in Ω. This amounts to saying that it is starlike with respect to each point of this ball: for each $z \in \Omega$ and $y \in B(x, d)$ the segment $[zy]$ is contained in Ω. With these definitions, we can show that a bounded, starlike open set with respect to an open ball is Lipschitz. Conversely, any bounded and conected open set with Lipschitz-continuous boundary is finite union of bounded and connected open sets, each being starlike with respect to an open ball. We refer here, this property is stated in [7], [21] and proved in [39]. Also, let Ω contained in $\mathbb{R}^3$ be a bounded and connected open set, we recall that Ω is pseudo-Lipschitz if for any point x on the boundary $\partial\Omega$ there exist an integer $r(x)$ equal to 1 or 2 and a strictly positive real number ρ_0 such that for all real numbers ρ with $0 < \rho < \rho_0$, the intersection of Ω with the ball with center x and radius ρ, has $r(x)$ connected components, each one being Lipschitz.

Second, we take the following hypothesis. We do not assume that the boundary of Ω is connected. We denote by Γ_k the connected components of the boundary Γ, $0 \leq k \leq I$, where Γ_0 is the boundary of the unbounded connected component of $\mathbb{R}^3 \backslash \overline{\Omega}$. There exist J connected open surfaces Σ_j, $1 \leq j \leq J$, called 'cuts', contained in Ω, such that

(i) each surface Σ_j is an open part of a smooth manifold $\mathcal{M}_j$,

(ii) the boundary of Σ is contained in Γ for $1 \leq j \leq J$,

(iii) the intersection $\overline{\Sigma_i} \cap \overline{\Sigma_j}$ is empty for $i \neq j$,

(iv) the open set

$$\Omega^\circ = \Omega \backslash \ \cup_{j=1}^{J} \ \Sigma_j$$

is pseudo-Lipschitz and simply-connected.

For J = 2 with I = 5, see for example Fig. 1.

In the following, the vectors, the matrix fields, the vector functions (or distributions), the matrix functions (or distributions) and the spaces of vector-valued functions are represented by bold symbols. For example: $\boldsymbol{\mathcal{D}}(\Omega) := (\mathcal{D}(\Omega))^3$, $\boldsymbol{L}^p(\Omega) := L^p(\Omega)^3$.

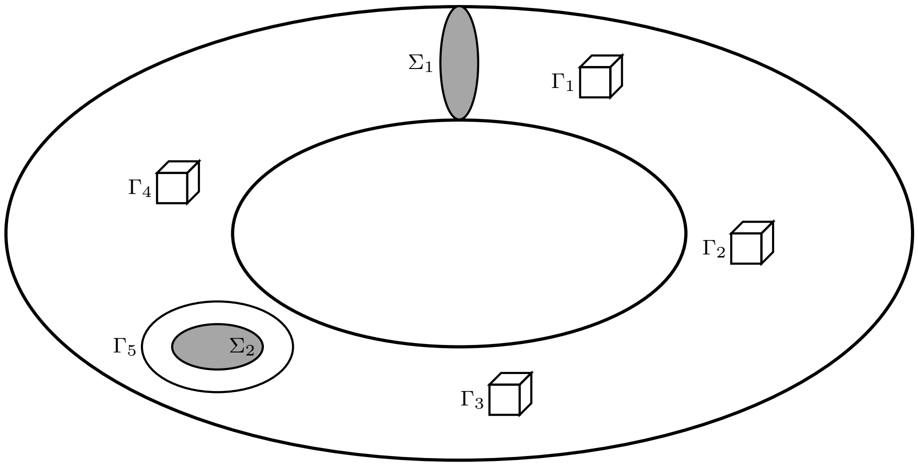

Fig. 1

The orientation tensor (ε_{ijk}) is defined by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } \{i,j,k\} \text{ is an even permutation of } \{1,2,3\}, \\ -1 & \text{if } \{i,j,k\} \text{ is an odd permutation of } \{1,2,3\}, \\ 0 & \text{if at least two indices are equal.} \end{cases}$$

In the rest of this section, Latin indices vary in the set $\{1,2,3\}$ and we use the summation convention with respect to repeated indices.

We use the following vector differential operators throughout the paper: the divergence operator $\operatorname{div} : \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$ is defined by

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \partial_i v_i \quad \text{for any} \quad \boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega).$$

The vector rotational operator $\mathbf{curl} : \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \boldsymbol{\mathcal{D}}'(\Omega)$ is defined by

$$(\mathbf{curl}\, \boldsymbol{v})_i = (\nabla \times \boldsymbol{v})_i = \varepsilon_{ijk}\, \partial_j v_k \quad \text{for any} \quad \boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega).$$

We define the kernel space $\boldsymbol{K}_T(\Omega)$ (or space of harmonic knots) by

$$\boldsymbol{K}_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0},\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \tag{0.0.1}$$

which is of finite dimension and its dimension depends on the geometric of Ω. Its dimension is equal to the second Betti number J, which corresponds to the total genus of the boundary Γ (see for example [12]). We define $\boldsymbol{\mathcal{V}}(\Omega)$ by

$$\boldsymbol{\mathcal{V}}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\Omega),\ \operatorname{div}\boldsymbol{v}=0\quad\text{in }\ \Omega\},$$

and the space $\boldsymbol{V}^{m,p}(\Omega)$ which represents the closure of $\boldsymbol{\mathcal{V}}(\Omega)$ in $\boldsymbol{W}^{m,p}(\Omega)$, by

$$\boldsymbol{V}^{m,p}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{W}_0^{m,p}(\Omega),\ \operatorname{div}\boldsymbol{v}=0\text{ in }\Omega\},$$

for $m\geq 1$ and by

$$\boldsymbol{V}^{0,p}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega),\ \operatorname{div}\boldsymbol{v}=0\,\text{in }\Omega,\ \boldsymbol{v}\cdot\boldsymbol{n}=0\quad\text{on}\quad\partial\Omega\},$$

for $m=0$. We set for $m\in\mathbb{N}$

$$\boldsymbol{U}^{m,p}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{V}^{m,p}(\Omega);\ \int_\Omega\boldsymbol{v}\cdot\boldsymbol{\varphi}\,dx=0\ for\ all\ \boldsymbol{\varphi}\in\boldsymbol{K}_T(\Omega)\}.$$

We have the following equivalence, for any function $\boldsymbol{v}\in\boldsymbol{V}^{m,p}(\Omega)$:

$$\forall\ \boldsymbol{\varphi}\in\boldsymbol{K}_T(\Omega),\ \int_\Omega\boldsymbol{v}\cdot\boldsymbol{\varphi}\,dx=0\Longleftrightarrow\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Sigma_j}=0\ \text{ for }\quad\text{any}\,1\leq j\leq J.$$

We use the following matrix operators. The matrix symmetrized gradient operator $\boldsymbol{\nabla_s}$: $\boldsymbol{\mathcal{D}}'(\Omega)\longrightarrow\mathbb{D}'_s(\Omega)$ is defined by

$$(\boldsymbol{\nabla_s}\,\boldsymbol{v})_{ij}=\frac{1}{2}(\partial_i v_j+\partial_j v_i)\quad\text{for any}\quad\boldsymbol{v}=(v_i)\in\boldsymbol{\mathcal{D}}'(\Omega).$$

For any matrix field $\boldsymbol{S}$, we denote by $\boldsymbol{S}^i$ the ith line of $\boldsymbol{S}$. For any vector field $\boldsymbol{v}$, we define the components of the vector field $\boldsymbol{Sv}$ by

$$(\boldsymbol{Sv})_i=S_{ij}v_j.$$

The vector divergence operator $\mathbf{Div}:\ \mathbb{D}'(\Omega)\longrightarrow\boldsymbol{\mathcal{D}}'(\Omega)$ is defined as follows:

$$\text{for}\quad\text{any}\ \boldsymbol{S}\in\mathbb{D}'(\Omega),\quad(\mathbf{Div}\,\boldsymbol{S})_i=\partial_j S_{ij}.$$

The matrix vector product is defined as follows:

$$(\boldsymbol{v} \times \boldsymbol{S})_{ij} = \varepsilon_{j\ell k} v_\ell S_{ik} \quad \text{for any} \quad \boldsymbol{v} = (v_i) \quad \text{and} \quad \boldsymbol{S} = (S_{ij}),$$

which means that the i$^{\text{th}}$ column of $\boldsymbol{v} \times \boldsymbol{S}$ is the vector product $\boldsymbol{v} \times (\boldsymbol{S}^i)^T$. Also, we define $\boldsymbol{S} \times \boldsymbol{v}$ by $\boldsymbol{S} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{S}$.

The matrix rotational operator $\mathbf{Curl} : \mathbb{D}'(\Omega) \longrightarrow \mathbb{D}'(\Omega)$ is defined by

$$(\mathbf{Curl}\, \boldsymbol{S})_{ij} = \varepsilon_{i\ell k}\, \partial_\ell S_{jk} \quad \text{for any} \quad \boldsymbol{S} = (S_{ij}) \in \mathbb{D}'(\Omega).$$

That means that the i$^{\text{th}}$ column of $\mathbf{Curl}\, \boldsymbol{S}$ is the **curl** of the i$^{\text{th}}$ line vector of $\boldsymbol{S}$. Observe that we have the following relation:

$$\mathbf{Curl}\, \boldsymbol{S} = (\boldsymbol{\nabla} \times \boldsymbol{S})^T.$$

It is easy to show that for any matrix field $\boldsymbol{S}$ and any vector field $\boldsymbol{v}$, the following relation holds:

$$((\boldsymbol{S} \times \boldsymbol{v})^T \times \boldsymbol{v})^T = (\boldsymbol{S}^T \times \boldsymbol{v})^T \times \boldsymbol{v}, \tag{0.0.2}$$

which implies that if $\boldsymbol{S}$ is symmetric (resp. anti-symmetric), then the matrix $(\boldsymbol{S} \times \boldsymbol{v})^T \times \boldsymbol{v}$ and $\mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{S}$ are also symmetric (resp. anti-symmetric).

. We define the space of rigid displacements by

$$\boldsymbol{R}(\Omega) = \{\boldsymbol{v} =: \, a_i(\boldsymbol{v})\, \boldsymbol{e}^i + b_i(\boldsymbol{v})\, \boldsymbol{P}^i, \, a_i(\boldsymbol{v}) \in \mathbb{R}^3, \, b_i(\boldsymbol{v}) \in \mathbb{R}^3\},$$

where $\boldsymbol{e}^i$ is the i^{th} vector of the canonical basis of $\mathbb{R}^3$ and $\boldsymbol{P}^i =: -\varepsilon_{ijk} x_k \boldsymbol{e}^j$. The dimension of $\boldsymbol{R}(\Omega)$ is 6 and $\boldsymbol{\nabla}_s \boldsymbol{v} = 0$ for any $\boldsymbol{v} \in \boldsymbol{R}(\Omega)$. A vector field $\boldsymbol{v} = \boldsymbol{rig}$ means that $\boldsymbol{v}$ belongs to $\boldsymbol{R}(\Omega)$.

Spaces of matrix fields are represented by special Roman capitals. Moreover, spaces of symmetric matrix fields are indexed by the Latin letter s. For example, $\mathbb{D}_s(\Omega) = \mathcal{D}(\Omega;\, \boldsymbol{M}^3_{sym})$. We define $\mathbb{V}_{\boldsymbol{s}}(\Omega)$ by

$$\mathbb{V}_s(\Omega) = \{\boldsymbol{S} \in \mathbb{D}_s(\Omega), \, \mathbf{Div}\, \boldsymbol{S} = \mathbf{0} \quad \text{in } \Omega\},$$

and the kernel space $\mathbb{K}_{T,s}(\Omega)$ by

$$\mathbb{K}_{T,s}(\Omega) = \{\boldsymbol{S} \in \mathbb{L}^2_s(\Omega), \, \mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{S} = \mathbf{0}, \, \mathbf{Div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \Omega, \, \boldsymbol{S} \cdot \boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega\},$$

which is of finite dimension and its dimension is dependent on the geometrical properties of Ω. Ciarlet et al in [19] and Geymonat et al in [32] have shown that the dimension of $\mathbb{K}_{T,s}(\Omega)$ is equal to $6J$. As recalled above, we define the space $\mathbb{U}_s^{m,p}(\Omega)$, for $m \geq 1$ by

$$\mathbb{U}_s^{m,p}(\Omega) = \{\boldsymbol{S} \in \mathbb{W}_{0,s}^{m,p}(\Omega),\, \mathbf{Div}\,\boldsymbol{S} = \mathbf{0},\ \langle \boldsymbol{S}{\cdot}\boldsymbol{n}, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \boldsymbol{S}{\cdot}\boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Sigma_j} = 0,\ 1 \leq i \leq 3,\ 1 \leq j \leq J\}, \tag{0.0.3}$$

and for $m = 0$ by

$$\mathbb{U}_s^{0,p}(\Omega) = \{\boldsymbol{S} \in \mathbb{L}_s^p(\Omega),\ \mathbf{Div}\,\boldsymbol{S} = \mathbf{0},\ \boldsymbol{S} \cdot \boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega,\ \langle\ \boldsymbol{S} \cdot \boldsymbol{n},\, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \boldsymbol{S} \cdot \boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Sigma_j} = \mathbf{0}\}. \tag{0.0.4}$$

We also introduce the following space

$$\mathbb{G}_s(\Omega) = \{\boldsymbol{S} \in \mathbb{D}_s(\Omega),\, \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S} = \mathbf{0} \quad \text{in} \quad \Omega\}.$$

For any matrix field $\boldsymbol{S}$ in $\mathbb{L}_s^2(\Omega^\circ)$, we denote by $\widetilde{\boldsymbol{S}}$ its extension in $\mathbb{L}_s^2(\Omega)$.

Now, we suppose $\Omega \subset \mathbb{R}^2$. We use the following operators. The scalar rotational operator curl : $\boldsymbol{\mathcal{D}'}(\Omega) \to \mathcal{D}'(\Omega)$ is defined by

$$\operatorname{curl} \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \quad \text{for} \quad \text{any} \quad \boldsymbol{v} \in \boldsymbol{\mathcal{D}'}(\Omega)$$

The vector rotational operator curl : $\mathcal{D}'(\Omega) \to \boldsymbol{\mathcal{D}'}(\Omega)$ is defined by

$$\mathbf{curl}\,\varphi = \begin{pmatrix} -\dfrac{\partial \varphi}{\partial x_2} \\ \dfrac{\partial \varphi}{\partial x_1} \end{pmatrix} \quad \text{for} \quad \text{any} \quad \varphi \in \mathcal{D}'(\Omega).$$

The Hessian matrix operator Hess : $\mathcal{D}'(\Omega) \to \mathbb{D}_s(\Omega)$ is defined by

$$\operatorname{Hess} \varphi = \begin{pmatrix} \dfrac{\partial^2 \varphi}{\partial x_1^2} & \dfrac{\partial^2 \varphi}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 \varphi}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \varphi}{\partial x_2^2} \end{pmatrix} \quad \text{for} \quad \text{any} \quad \varphi \in \mathcal{D}'(\Omega).$$

For any matrix field

$$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

we define $\boldsymbol{S}^\star$ by

$$\boldsymbol{S}^\star = \begin{pmatrix} S_{22} & -S_{21} \\ -S_{12} & S_{11} \end{pmatrix}.$$

Observe that $(\boldsymbol{S}^\star)^\star = \boldsymbol{S}$ and if $\boldsymbol{S}$ is symmetric, then

$$\mathbf{Div}\, \boldsymbol{S}^\star = \mathbf{0} \Leftrightarrow \mathbf{curl}\, \boldsymbol{S} = \mathbf{0}, \tag{0.0.5}$$

where $\mathbf{curl}\, \boldsymbol{S}$ is the vector field $\begin{pmatrix} \mathbf{curl}\, \boldsymbol{S}_1 \\ \mathbf{curl}\, \boldsymbol{S}_2 \end{pmatrix}$ with $\boldsymbol{S}_i$ is the ith line of the matrix $\boldsymbol{S}$.

We define the functional space $\boldsymbol{L}^p_0(\Omega)$ by

$$\boldsymbol{L}^p_0(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \int_\Omega \boldsymbol{v} \cdot \boldsymbol{r}\, dx = 0,\ \forall \boldsymbol{r} \in \boldsymbol{R}(\Omega)\}, \tag{0.0.6}$$

and $\mathbb{V}^{1,p}_s(\Omega)$ by

$$\mathbb{V}^{1,p}_s(\Omega) = \{\boldsymbol{S} \in \mathbb{W}^{1,p}_{0,s}(\Omega),\ \mathbf{Div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \Omega\}. \tag{0.0.7}$$

Chapter 1

Introduction

The objective of this thesis stem from a desire to start with some new results for vectors fields and we hopefully move on towards to show its analogues and other characterizations of symmetric matrix fields.

We have presented the results of this thesis in three papers. The first article: "On the curl operator and some characterizations of matrix fields in Lipschitz domains", is published in "Journal of Mathematical Analysis and Applications". The second article: "Beltrami's completeness and Beltrami's-type decomposition for $\mathbb{L}^p$-symmetric matrix fields and the third article: "Characterization of the trace of $W^{3,p}(\Omega)$ on Lipschitz domaine of $\mathbb{R}^2$", are submitted.

Bogovskiĭ [13] proved the existence of a continuous right inverse of the divergence operator

$$\operatorname{div}:\ \boldsymbol{W}_0^{1,p}(\Omega)\to L_0^p(\Omega)=\{f\in L^p(\Omega),\ \int_\Omega f\,dx=0\}. \tag{1.0.1}$$

Moreover, he gave an explicit form of the inverse operator in the case when Ω is starlike with respect to an open ball. A detailled proof of Bogovskiĭ's theorem is given by Borchers et al [14] and later by Galdi [27]. Amrouche et al [7] have shown that if $p=2$, then the surjectivity result of the operator (1.0.1) is equivalent with the following results:

(a) **Classical J. L. Lions lemma**:

$$f\in H^{-1}(\Omega)\quad\text{and}\quad \boldsymbol{\nabla} f\in \boldsymbol{H}^{-1}(\Omega)\text{ implies } f\in L^2(\Omega).$$

(b) **J. Neças inequality**: There exists a constant C such that for any $f\in L^2(\Omega)$

$$\|f\|_{L^2(\Omega)}\le C\left(\|f\|_{H^{-1}(\Omega)}+\|\boldsymbol{\nabla} f\|_{\boldsymbol{H}^{-1}(\Omega)}\right).$$

(c) The operator

$$\boldsymbol{\nabla}:\ L_0^2(\Omega)\to \boldsymbol{H}^{-1}(\Omega)$$

has a closed range.

(d) **A coarse version of De Rham's Theorem**: Let $\boldsymbol{h}\in\boldsymbol{H}^{-1}(\Omega)$, there exists a function $p\in L_0^2(\Omega)$ such that

$$\boldsymbol{\nabla} p=\boldsymbol{h}\ \text{in}\ \ \boldsymbol{H}^{-1}(\Omega),$$

if and only if

$${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{h},\,\boldsymbol{v}\rangle_{\boldsymbol{H}_0^1(\Omega)}=0,\ \text{for}\quad\text{all } \boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)\ \text{that}\quad\text{satisfy}\operatorname{div}\boldsymbol{v}=0\text{ in }\Omega.$$

(e) **Approximation lemma**: Assume that Ω is starlike with respect to an open ball. Then, there exist a constant C such that given any function φ in $\mathcal{D}_0(\Omega)=\mathcal{D}(\Omega)\cap L_0^2(\Omega)$, there exists vector fields $\boldsymbol{v}_n\in\mathcal{D}(\Omega)$, $n\ge 1$, such that

$$\|\boldsymbol{v}_n\|_{H^1(\Omega)}\le C\|\varphi\|_{L^2(\Omega)}$$

and

$$\operatorname{div}\boldsymbol{v}_n\to\varphi\quad\text{in}\quad\mathcal{D}(\Omega).$$

(f) **Extension of J. L. Lions lemma**:

$$f \in \mathcal{D}'(\Omega) \quad \text{and} \quad \boldsymbol{\nabla} f \in \boldsymbol{H}^{-1}(\Omega) \text{ implies } \ f \in L^2(\Omega).$$

In a recent work, P. Ciarlet et al [20] have shown a matrix version of the previous equivalence theorem. They proved that the operator

$$\mathbf{Div} : \mathbb{H}^1_{0,s}(\Omega) \to \boldsymbol{L}^2_0(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \int_\Omega \boldsymbol{v} \cdot \boldsymbol{r}\, dx = 0, \forall \boldsymbol{r} \in \boldsymbol{R}(\Omega)\},$$

is equivalent with the following results:

(a′) **Weak version of J. L. Lions lemma**:

$$\boldsymbol{v} \in \boldsymbol{H}^{-1}(\Omega) \quad \text{and} \quad \boldsymbol{\nabla}_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega) \text{ implies } \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

(b′) **Vector version of Neças inequality**: There exists a constant C such that

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq C \left(\|\boldsymbol{v}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\boldsymbol{\nabla}_s \boldsymbol{v}\|_{\mathbb{H}^{-1}(\Omega)} \right), \ \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

(c′) The operator

$$\boldsymbol{\nabla}_s : \boldsymbol{L}^2_0(\Omega) \to \mathbb{H}^{-1}_s(\Omega)$$

has a closed range.

(d′) **Weak Donati's compatibility**: Let $\boldsymbol{E} \in \mathbb{H}^{-1}(\Omega)$, there exists a vector field $\boldsymbol{v} \in \boldsymbol{L}^2_0(\Omega)$ such that

$$\boldsymbol{\nabla}_s \boldsymbol{v} = \boldsymbol{E} \text{ in } \mathbb{H}^{-1}(\Omega),$$

if and only if

$${}_{\mathbb{H}^{-1}(\Omega)}\langle \boldsymbol{E}, \boldsymbol{M} \rangle_{\mathbb{H}^1_0(\Omega)} = 0, \text{ for } \quad \text{all } \boldsymbol{M} \in \mathbb{H}^1_{0,s}(\Omega) \text{ that } \quad \text{satisfy } \mathbf{Div}\, \boldsymbol{M} = 0 \text{ in } \Omega.$$

(e′) **Approximation property**: Assume that the domain Ω is starlike with respect to an open ball. There exists a constant C such that given any vector field $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}_0(\Omega) = \boldsymbol{\mathcal{D}}(\Omega) \cap \boldsymbol{L}^2_0(\Omega)$, there exist matrix fields $\boldsymbol{E}_n \in \mathbb{D}_s(\Omega)$, $n \geq 1$, such that

$$\|\boldsymbol{E}_n\|_{\mathbb{H}^1(\Omega)} \leq C \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^2(\Omega)},$$

and

$$\mathbf{Div}\,\boldsymbol{E}_n \to \boldsymbol{\varphi} \quad \text{in} \quad \boldsymbol{\mathcal{D}}(\Omega).$$

(f′) **Vector version of J. L. Lions lemma**:

$$\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega) \quad \text{and} \quad \boldsymbol{\nabla}_s \boldsymbol{v} \in \mathbb{H}_s^{-1}(\Omega) \quad \text{implies} \quad \boldsymbol{v} \in \boldsymbol{L}^2(\Omega).$$

Observe that the results (a′), (b′), (c′), (d′), (e′) and (f′) are the analogues of (a), (b), (c), (d), (e) and (f) respectively.

Borchers et al [14] proved that for any $\psi \in \mathcal{D}_0(\Omega)$, there exists $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} = \psi$ in Ω and satisfying the estimation

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\psi\|_{\boldsymbol{L}^p(\Omega)},$$

where C depends only on p and Ω. The previous result is more powerfull then result (e) of the first equivalence theorem. It provides us with simple proof the following usual version of De Rham's Theorem: if Ω is any open set of $\mathbb{R}^3$, then, for any $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ satisfying

$$\text{for} \quad \text{all}\, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega), \quad {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0,$$

where $\boldsymbol{\mathcal{V}}(\Omega)$ denotes the subspace of vector fields in $\boldsymbol{\mathcal{D}}(\Omega)$ with divergence free, then there exists a scalar field $\chi \in \mathcal{D}'(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\nabla}\, \chi \quad \text{in} \quad \Omega$.

Analogues properties exist for matrix fields. In 1890, Donati proved that, if Ω is an open subset of $\mathbb{R}^3$ and $\boldsymbol{E} \in \boldsymbol{\mathcal{C}}^2(\Omega)$ is such that

$$\int_\Omega \boldsymbol{E} : \boldsymbol{M} = 0 \quad \text{forall } \boldsymbol{M} \in \mathbb{D}_s(\Omega) \quad \text{such that } \mathbf{Div}\ \boldsymbol{M} = \mathbf{0} \text{ in } \Omega, \tag{1.0.2}$$

then $\boldsymbol{E}$ satisfies the following compatibility equation

$$\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in} \quad \Omega. \tag{1.0.3}$$

A first extension of Donati's Theorem was given in 1974 by Ting [48] as follows:

Theorem 1.0.1. *(Ting's theorem). Let Ω be a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary and $\boldsymbol{E} \in \mathbb{L}^2(\Omega)$. If $\boldsymbol{E}$ satisfies (1.0.2), then there exists $\boldsymbol{v}$ in $\mathbf{H}^1(\Omega)$ such that $\boldsymbol{E} = \boldsymbol{\nabla}_s \boldsymbol{v}$ in Ω.*

Another extension of Donati's Theorem was given in 1979 by Moreau [41] in the case of distributions.

Theorem 1.0.2. *(Moreau's theorem). Let Ω be an arbitrary open set of $\mathbb{R}^3$ and $\boldsymbol{E} \in \mathbb{D}'(\Omega)$.if $\boldsymbol{E}$ satisfies (1.0.2), then there exists $\boldsymbol{v}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$ such that $\boldsymbol{E} = \boldsymbol{\nabla}_s \boldsymbol{v}$ in Ω.*

More recently, using different proofs, some variants of Donati's Theorem have been independently obtained by Geymonat and Krasucki [30] for $\boldsymbol{E} \in \mathbb{W}_s^{-1,p}(\Omega)$, $\boldsymbol{E} \in \mathbb{L}_s^p(\Omega)$ and by Amrouche et al [6] for $\boldsymbol{E} \in \mathbb{L}_s^2(\Omega)$.

Let us observe that Moreau's theorem is the matrix analog of the usual version of De Rham's theorem and Ting's theorem is the matrix analog of the coarse version, here the vector differential operators div and $\boldsymbol{\nabla}$ are replaced by **Div** and $\boldsymbol{\nabla}_s$.

Concerning the operator **curl**, the classical Poincaré's Lemma asserts that if Ω is an arbitrary simply-connected open set of $\mathbb{R}^3$, then for any $\boldsymbol{h} \in \mathcal{C}^1(\Omega)$ which satisfies $\mathbf{curl}\, \boldsymbol{h} = \mathbf{0}$ in Ω, there exists $\chi \in \mathcal{C}^2(\Omega)$ such that $\boldsymbol{h} = \boldsymbol{\nabla}\chi$. This result still true in the case where $\boldsymbol{h} \in \boldsymbol{L}^2(\Omega)$ and in the case $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$ if Ω is a bounded and connected open set with Lipschitz-continuous boundary (see [33] and [18]).

A similar property takes place for matrix fields. Saint-Venant's theorem (1864) announced that if Ω is an arbitrary simply-connected open set of $\mathbb{R}^3$, then for any symmetric matrix in $\boldsymbol{E} = (E_{ij})$ with $E_{ij} \in \mathcal{C}^2(\Omega)$ which satisfies the compatibility equation (1.0.3), there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{C}}^3(\Omega)$ such that

$$\boldsymbol{\nabla}_s \boldsymbol{v} = \boldsymbol{E} \quad \text{in} \quad \Omega. \tag{1.0.4}$$

In fact, the first rigorous proof of the above result was given by Beltrami in 1886. More recently, if in addition Ω is a bounded and connected open set with Lipschitz-continuous boundary, Ciarlet and Ciarlet Jr [18] proved that if $\boldsymbol{E} \in \mathbb{L}^2(\Omega)$ satisfies the compatibility equations (1.0.3), then there exists $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$ such that (1.0.4) holds. A similar result, with $\boldsymbol{E} \in \mathbb{H}^{-1}(\Omega)$ and then $\boldsymbol{v} \in \mathbf{L}^2(\Omega)$, was also obtained by Amrouche et al [6].

Let us observe that the above Saint-Venant's theorem is nothing but only the matrix analog of Poincaré's lemma where the vector differential operators **curl** ***and*** $\boldsymbol{\nabla}$ ***are replaced by the matrix differential operators*** $\mathbf{Curl}\,\mathbf{Curl}$ ***and*** $\boldsymbol{\nabla}_s$***.***

From the above exemples, we can see the analogy between the vector fields results and matrix fields results. In many cases, it suffices to replace the triplet (div, ∇, **curl**) ***by the triplet*** (**Div**, ∇_s, **Curl Curl**) ***to extend fundammental results from the vector case to the matrix case.***

Throughout the rest of manuscript, Ω is a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz continuous boundary (except when we add more hypothesis on the regularity of Ω), p is a real number such that $1 < p < \infty$ and p' is its conjugate.

In the following, we will present the main results of each chapter. In Chapter 2, we will present a new version of the above theorem that we will call *the rotational version of De Rham's Theorem*. In the case where the open set Ω is star-shaped with respect to an open ball, Costabel et al in [21] and Mitrea in [43] have used the properties of pseudodifferential operators to show that the operator

$$\mathbf{curl} : \boldsymbol{\mathcal{D}}(\Omega) \longrightarrow \boldsymbol{\mathcal{V}}(\Omega), \tag{1.0.5}$$

is onto. In Section 2.1, we will give a new proof of this result by using the theory of singular integrals. Furthermore, we will generalize it in the case where Ω is Lipschitz but not necessarily star-shaped with respect to an open ball. More precisely, we will show that if

$$\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega) \quad \text{satisfies} \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\, dx = 0, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega), \tag{1.0.6}$$

then there exists $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$ satisfying $\mathbf{curl}\, \boldsymbol{\varphi} = \boldsymbol{f}$ in Ω. Next, we deduce a rotational version of De Rham's theorem. The main result of Chapter 2 can be formulated as follows:

Theorem A. (**The rotational version of De Rham's Theorem**). *i) Let m be a nonnegative integer. For any $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$ satisfying (1.0.6), there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that*

$$\mathbf{curl}\, \boldsymbol{\psi} = \boldsymbol{f} \quad \textit{in} \quad \Omega,$$

and there exists a constant C such that

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{m+1,p}(\Omega)} \leq C)\|\boldsymbol{f}\|_{\boldsymbol{W}^{m,p}(\Omega)}.$$

ii) Let $\boldsymbol{f} \in \boldsymbol{U}^{m,p}(\Omega)$*, then there exists* $\boldsymbol{\psi} \in \boldsymbol{W}_0^{m+1,p}(\Omega)$ *such that* $\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f}$.

iii) Let $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ *and satisfies*

$$\textit{for all } \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}(\Omega), \quad {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0,$$

where $\boldsymbol{\mathcal{G}}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega),\ \mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0} \textit{ in } \Omega\}$*. Then, there exists* $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$ *such that*

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f} \quad \textit{in} \quad \Omega.$$

Let us observe that Theorem A is a vector potentials result for divergence-free function in $\boldsymbol{\mathcal{D}}(\Omega)$ and in $\boldsymbol{\mathcal{D}}'(\Omega)$. Amrouche et al [3] have shown some results concerning vector potentials which are associated with a $\boldsymbol{L}^2$-divergence-free function and satisfying some boundary conditions. A generalization for $\boldsymbol{L}^p$ case was given by Amrouche and Seloula [9] . The question that we have posed: *why we do not show the analogues for the symmetric matrix fields ?*

In the absence of body forces the stress equations of equilibrium take the form

$$\mathbf{Div}\,\boldsymbol{S} = \mathbf{0} \quad \text{in} \quad \Omega, \quad \boldsymbol{S} = \boldsymbol{S}^T, \tag{1.0.7}$$

the second order symmetric tensor field being the stress in the reference configuration Ω of an elastic body. The first stress function solution of the equilibrium equation (1.0.7) was presented by Airy in [1] for the two dimensional case. The generalizations for the three dimensional case were obtained by Maxwell in [38], Morera in [42] and Beltrami in [11]. The solutions of Morera and Maxwell are special cases of the Beltrami's solution defined as follows

$$\boldsymbol{S} = \mathbf{Curl\,Curl}\,\boldsymbol{A} \quad \text{for all smooth symmetric second order tensor fields } \boldsymbol{A} \text{ in } \Omega. \tag{1.0.8}$$

Gurtin [35] gave an example of a stress field $\boldsymbol{S}$ satisfying (1.0.7) but which is not given by (1.0.8). So that this representation is incomplete. However the Beltrami solution is complete in the class of smooth stress fields $\boldsymbol{S}$ which are *self-equilibrated*, *i.e.* for each closed regular surface $\mathcal{C}$ contained in Ω, the resultant force and the moment vanish. In other words, $\boldsymbol{S}$ satisfies the following condition:

$$\int_{\mathcal{C}} \boldsymbol{S}\cdot\boldsymbol{n}\,d\sigma = \int_{\mathcal{C}} \boldsymbol{P}^i \times (\boldsymbol{S}\cdot\boldsymbol{n})\,d\sigma = \mathbf{0}, \text{ for all} \quad 1 \le i \le 3,$$

such that $\boldsymbol{P}^i = -\varepsilon_{ijk}x_k\boldsymbol{e}^j$. For more details see [24]. An extension of this result can be found in [31] and in [32] as follows: let Ω be a bounded and connected open set of $\mathbb{R}^3$ with Lipschitz-continuous boundary and $\boldsymbol{S}$ be a symmetric matrix field in $\mathbb{L}^2_s(\Omega)$ and satisfying the following conditions:

$$\textbf{Div}\,\boldsymbol{S} = \boldsymbol{0} \text{ in } \Omega \quad \text{and} \quad \langle \boldsymbol{S}\cdot\boldsymbol{n}, \boldsymbol{e}^i\rangle_{\Gamma_k} = \langle \boldsymbol{S}\cdot\boldsymbol{n}, \cdot\boldsymbol{P}^i\rangle_{\Gamma_k} = 0, 1 \le i \le 3,\ 0 \le k \le I.$$

Then, there exists a symmetric matrix field $\boldsymbol{A} \in \mathbb{H}^2_s(\Omega)$ such that $\textbf{Curl}\,\textbf{Curl}\,\boldsymbol{A} = \boldsymbol{S}$ in Ω. Moreover, P. G Ciarlet et al in [19] stated that if the above symmetric matrix field $\boldsymbol{S}$ satisfies the following conditions:

$$\boldsymbol{S}\cdot\boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega \text{ and } \langle \boldsymbol{S}\cdot\boldsymbol{n}, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \boldsymbol{S}\cdot\boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Sigma_j} = 0, \text{ for all} \quad 1 \le i \le 3,\ 1 \le j \le J,$$

where $\langle\cdot,\cdot\rangle_{\Sigma_j}$ denotes the duality pairing between $\boldsymbol{H}^{-\frac{1}{2}}(\Sigma_j)'$ and $\boldsymbol{H}^{\frac{1}{2}}(\Sigma_j)$, then $\boldsymbol{A} \in \mathbb{H}^2_{0,s}(\Omega)$.

In Chapter 3, we will present a new version of the Beltrami's completeness, in the case when the components of the symmetric matrix $\boldsymbol{S}$ are in $\mathcal{D}(\Omega)$ and we will show the above result of P.G. Ciarlet et al in a general case, when the components of $\boldsymbol{S}$ are in $W^{m,p}_0(\Omega)$, with m a nonnegative integer. Observe that the above versions of Beltrami's completeness are nothing but only the analogues of the vector fields results announced in Theorem A and here we state the main result of Chapter 3.

Theorem B. (**Completeness of the Beltrami Solution**). *i) Let m be a nonnegative integer and $\boldsymbol{S}$ in $\mathbb{V}_s(\Omega)$ satisfies*

$$\int_{\Sigma_j} (\boldsymbol{S}\cdot\boldsymbol{n})\cdot\boldsymbol{e}^i\,d\sigma = \int_{\Sigma_j} (\boldsymbol{S}\cdot\boldsymbol{n})\cdot\boldsymbol{P}^i\,d\sigma = 0, \textit{ for all} \quad 1 \le i \le 3,\ 1 \le j \le J.$$

Then, there exists $\boldsymbol{A} \in \mathbb{D}_s(\Omega)$ such that

$$\textbf{Curl}\,\textbf{Curl}\,\boldsymbol{A} = \boldsymbol{S} \quad in \quad \Omega,$$

and there exists a constant C such that

$$\|\boldsymbol{A}\|_{\mathbb{W}^{m+2,p}(\Omega)} \le C\|\boldsymbol{S}\|_{\mathbb{W}^{m,p}(\Omega)}.$$

ii) Let $\boldsymbol{S} \in \mathbb{U}^{m,p}(\Omega)$*, then there exists* $\boldsymbol{A} \in \mathbb{W}_0^{m+2,p}(\Omega)$ *such that* $\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} = \boldsymbol{S}$*.*
iii) Let $\boldsymbol{S} \in \mathbb{D}'_s(\Omega)$ *and satisfies*

$$\textit{for} \quad \textit{all} \quad \boldsymbol{E} \in \mathbb{G}_s(\Omega), \ {}_{\mathbb{D}'(\Omega)}\langle \boldsymbol{S}, \boldsymbol{E}\rangle_{\mathbb{D}(\Omega)} = 0,$$

then, there exists $\boldsymbol{A} \in \mathbb{D}'_s(\Omega)$ *such that*

$$\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} = \boldsymbol{S} \quad \textit{in} \quad \Omega.$$

Let us introduce the following matrix spaces

$$\mathbb{H}^p_s(\mathbf{Div},\,\Omega) = \{\boldsymbol{S} \in \mathbb{L}^p_s(\Omega),\,\mathbf{Div}\,\boldsymbol{S} \in \boldsymbol{L}^p(\Omega)\},$$

$$\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega) = \{\boldsymbol{S} \in \mathbb{L}^p_s(\Omega),\,\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)\},$$

$$\mathbb{X}^p_s(\Omega) = \mathbb{H}^p_s(\mathbf{Div},\,\Omega) \cap \mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega),$$

$$\mathbb{Y}^p_s(\Omega) = \{\boldsymbol{S} \in \mathbb{X}^p_s(\Omega,\,\mathbf{Div}\,\boldsymbol{S} \in \boldsymbol{W}^{1,p}(\Omega)\},$$

which are equipped with the graph norms.

In Section 4.1, we will show that any matrix field $\boldsymbol{S}$ in $\mathbb{H}^p_s(\mathbf{Div};\,\Omega)$ has a normal trace $\boldsymbol{Sn}$ in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and the following Green' formula holds

$$\forall \boldsymbol{v} \in \boldsymbol{W}^{1,p'}(\Omega), \quad \langle \boldsymbol{Sn}, \boldsymbol{v}\rangle_\Gamma = \int_\Omega \boldsymbol{S} : \boldsymbol{\nabla_s v}\,dx + \int_\Omega \mathbf{Div}\,\boldsymbol{S}\cdot\boldsymbol{v}\,dx.$$

The previous characterization of $\mathbb{H}^p_s(\mathbf{Div};\,\Omega)$ will allows us to present a tangential extension of Beltrami's completeness. We adopt the following notation, if $\boldsymbol{E}(\Omega)$ is a Banach space, we denote by

$$\boldsymbol{E}_T(\Omega) = \{\boldsymbol{S} \in \boldsymbol{E},\,\boldsymbol{Sn} = \mathbf{0} \quad \text{on} \quad \Gamma\}.$$

We will show that if the domain Ω is of class $\mathcal{C}^{1,1}$, then

$$\mathbb{X}^p_{T,s}(\Omega) \hookrightarrow \mathbb{W}^{1,p}_s(\Omega)$$

and if Ω is of class $\mathcal{C}^{2,1}$,

$$\mathbb{Y}^p_{T,s}(\Omega) \hookrightarrow \mathbb{W}^{2,p}_s(\Omega).$$

By using Peetre-Tartar's Theorem, we deduce the following first Friedrich's inequality for every matrix $\boldsymbol{S}$ in $\mathbb{X}^p_{T,s}(\Omega)$:

$$\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} \leq C\big(\|\mathbf{Div}\,\boldsymbol{S}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \sum_{i=1}^{3}\sum_{j=1}^{J}(|\langle \boldsymbol{Sn}, \boldsymbol{e}^i\rangle_{\Sigma_j}| + |\langle \boldsymbol{Sn}, \boldsymbol{P}^i\rangle_{\Sigma_j}|)\big) \quad (1.0.9)$$

We finish this section by showing a new tangential extension of Beltrami's completeness which generalize the version of Geymonat et al (see [31], [32]) in $\mathbb{L}^p_s(\Omega)$. The main result of Section 4.1 is given in the following theorem:

Theorem C. *Assume that Ω is of class $\mathcal{C}^{1,1}$. A matrix $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$ satisfies*

$$\mathbf{Div}\,\boldsymbol{S} = \mathbf{0} \quad \textit{in} \quad \Omega,$$
$$\langle \boldsymbol{Sn}, \boldsymbol{e}^i\rangle_{\Gamma_k} = \langle \boldsymbol{Sn}, \boldsymbol{P}^i\rangle_{\Gamma_k} = 0, \quad 1 \leq i \leq 3 \quad \textit{and} \quad 0 \leq k \leq I,$$

if and only if there exists a matrix $\boldsymbol{A} \in \mathbb{X}^p_s(\Omega)$ such that

$$\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} = \boldsymbol{S} \quad \textit{and} \quad \mathbf{Div}\,\boldsymbol{A} = \mathbf{0} \quad \textit{in} \quad \Omega,$$
$$\boldsymbol{An} = \mathbf{0} \quad \textit{on} \quad \Gamma,$$
$$\langle \boldsymbol{An}, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \boldsymbol{An}, \boldsymbol{P}^i\rangle_{\Sigma_j} = 0, \quad 1 \leq i \leq 3 \quad \textit{and} \quad 1 \leq j \leq J.$$

Moreover $\boldsymbol{A}$ is unique and we have the estimate

$$\|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C_1\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

In addition, if Ω is of class $\mathcal{C}^{2,1}$, then $\boldsymbol{A} \in \mathbb{W}^{2,p}_s(\Omega)$ and we have the estimate

$$\|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C_2\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

In Section 4.2, we will show that if Ω is of class $\mathcal{C}^{1,1}$, then any matrix $\boldsymbol{S}$ in $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$ has a tangential trace $\boldsymbol{S}\times\boldsymbol{n}$ in $\mathbb{W}^{-\frac{1}{p},p}(\Gamma)$. Further, the matrix $\mathbf{Curl}\,\boldsymbol{S}$ (which is not even in $\mathbb{L}^p(\Omega)$) has a tangential trace $\mathbf{Curl}\,\boldsymbol{S}\times\boldsymbol{n}$ in $\mathbb{W}^{-1-\frac{1}{p},p}(\Gamma)$ and the following Green's formula holds for any $\boldsymbol{E} \in \mathbb{W}^{2,p'}_s(\Omega)$:

$$\langle \boldsymbol{S}\times\boldsymbol{n}, \mathbf{Curl}\,\boldsymbol{E}\rangle_\Gamma + \langle \mathbf{Curl}\,\boldsymbol{S}\times\boldsymbol{n},\, \boldsymbol{E}\rangle_\Gamma = \int_\Omega \boldsymbol{S} : \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E}\,dx - \int_\Omega \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S} : \boldsymbol{E}\,dx.$$

The previous result will allows us to present a normal extension of Beltrami's completeness. We adopt the following notation, if $\boldsymbol{E}$ is a Banach space, we denote by

$$\boldsymbol{E}_N(\Omega) = \{\boldsymbol{S} \in \boldsymbol{E},\ \boldsymbol{S} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{and} \quad \mathbf{Curl}\,\boldsymbol{S} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma\}.$$

We will show that if the domain Ω is of class $\mathcal{C}^{1,1}$, then

$$\mathbb{X}^p_{N,s}(\Omega) \hookrightarrow \mathbb{W}^{1,p}_s(\Omega)$$

and if Ω is of class $\mathcal{C}^{2,1}$,

$$\mathbb{Y}^p_{N,s}(\Omega) \hookrightarrow \mathbb{W}^{2,p}_s(\Omega).$$

Using again Peetre-Tartar's theorem, we deduce the following second Friedrich's inequality type for every matrix $\boldsymbol{S}$ in $\mathbb{X}^p_{N,s}(\Omega)$:

$$\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} \leq C\big(\|\mathbf{Div}\,\boldsymbol{S}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \sum_{i=1}^{3}\sum_{k=1}^{I}(|\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{e}^i\rangle_{\Gamma_k}| + |\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Gamma_k}|)\big). \tag{1.0.10}$$

We finish this section by showing a new normal extension of Beltrami's completeness:

Theorem D. *Assume that Ω is of class $\mathcal{C}^{1,1}$. Then a matrix $\boldsymbol{S}$ in $\mathbb{L}^p_s(\Omega)$ satisfies*

$$\begin{aligned} \mathbf{Div}\,\boldsymbol{S} &= \boldsymbol{0} \quad \textit{in} \quad \Omega,\\ \boldsymbol{S}\boldsymbol{n} &= \boldsymbol{0} \quad \textit{on} \quad \Gamma,\\ \langle \boldsymbol{S}\boldsymbol{n},\, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \boldsymbol{S}\boldsymbol{n},\, \boldsymbol{P}^i\rangle_{\Sigma_j} &= 0, \quad 1 \leq i \leq 3 \quad \textit{and} \quad 1 \leq j \leq J, \end{aligned}$$

if and only if there exists a matrix $\boldsymbol{A} \in \mathbb{Y}^p_s(\Omega)$ such that

$$\begin{aligned} \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} = \boldsymbol{S} \ \textit{in} \quad \Omega, &\qquad \mathbf{Div}\,\boldsymbol{A} = \boldsymbol{0} \quad \textit{in} \quad \Omega,\\ \boldsymbol{A} \times \boldsymbol{n} = \boldsymbol{0} \quad \textit{on} \quad \Gamma, &\qquad \mathbf{Curl}\,\boldsymbol{A} \times \boldsymbol{n} = \boldsymbol{0} \quad \textit{on} \quad \Gamma,\\ \langle \boldsymbol{S}\boldsymbol{n},\, \boldsymbol{e}^i\rangle_{\Gamma_k} = \langle \boldsymbol{S}\boldsymbol{n},\, \boldsymbol{P}^i\rangle_{\Gamma_k} = 0, &\qquad 1 \leq i \leq 3, \quad \textit{and} \quad 1 \leq k \leq I. \end{aligned}$$

Moreover, $\boldsymbol{A}$ is unique and we have the estimate

$$\|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C_1\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

In addition, if Ω *is of class* $\mathcal{C}^{2,1}$*, then* $\boldsymbol{A} \in \mathbb{W}_s^{2,p}(\Omega)$ *and we have the estimate*

$$\|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C_2 \|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

In Section 4.3, we will interest by Beltrami's-type decomposition which is the matrix analog of the "well-known" Helmholtz vector decomposition. It describes a symmetric matrix field as the sum of a compatible part (**Curl Curl**-free) and an incompatible part (divergence-free) fields. Magianni et al [36] and Von Goethem [49] have presented a version of the above decomposition for $\mathbb{L}^p$-symmetric matrix fields. They proved that if Ω is simply-connected and of class $\mathcal{C}^\infty$, then for any $\boldsymbol{S} \in \mathbb{L}_s^p(\Omega$, there exists an unique vector $\boldsymbol{v}$ in $\boldsymbol{W}^{1,p}(\Omega)$ and an unique divergence-free matrix field $\boldsymbol{A}$ in $\mathbb{L}_s^p(\Omega)$ such that

$$\boldsymbol{S} = \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl\,Curl}\,\boldsymbol{A}. \tag{1.0.11}$$

Geymonat et al [32] proved a Hodge decomposition of $\mathbb{L}_s^2(\Omega)$ where Ω is only Lipschitz and not necessarily simply-connected. They showed that for any matrix field $\boldsymbol{S}$ of $\mathbb{L}_s^2(\Omega)$, there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, $\boldsymbol{E} \in \mathbb{K}_{T,s}(\Omega)$ and $\boldsymbol{M} \in \mathbb{U}_s^{0,2}(\Omega)$ such that

$$\boldsymbol{S} = \boldsymbol{\nabla}_s \boldsymbol{v} + \boldsymbol{M} + \boldsymbol{E}.$$

In Theorem 3.1.3, we will show that the operator $\mathbf{Curl\,Curl} : \mathbb{H}_{0,s}^2(\Omega) \longrightarrow \mathbb{U}_s^{0,2}(\Omega)$ is onto. Consequently, there exists $\boldsymbol{A} \in \mathbb{H}_{0,s}^2(\Omega)$ such that $\boldsymbol{M} = \mathbf{Curl\,Curl}\,\boldsymbol{A}$ and the following decomposition holds

$$\boldsymbol{S} = \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl\,Curl}\,\boldsymbol{A} + \boldsymbol{E}. \tag{1.0.12}$$

Observe that the decomposition (1.0.12) has a Kernel part $\boldsymbol{E}$ which is due to the fact that Ω is not necessarily simply-connected.

The second aim of Chapter 4 is to show three new versions of Beltrami's-type decomposition for matrix fields in $\mathbb{L}_s^p(\Omega)$ when Ω is not necessarily simply-connected and with boundary of class $\mathcal{C}^{1,1}$. We introduce the spaces

$$\mathbb{W}_{\sigma,s}^{1,p}(\Omega) = \{\boldsymbol{S} \in \mathbb{W}_s^{1,p}(\Omega),\, \mathbf{Div}\,\boldsymbol{S} = \mathbf{0} \quad \text{in} \quad \Omega\},$$

$$\mathbb{W}_{\sigma,s}^{2,p}(\Omega) = \{\boldsymbol{S} \in \mathbb{W}_s^{2,p}(\Omega),\, \mathbf{Div}\,\boldsymbol{S} = \mathbf{0} \quad \text{in} \quad \Omega\}.$$

Theorem E. *Assume that* Ω *is of class* $\mathcal{C}^{1,1}$.

i) Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$, *then there exist* $\boldsymbol{E} \in \mathbb{K}_{T,s}(\Omega)$, $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ *and* $\boldsymbol{A} \in \mathbb{W}^{1,p}_{\sigma,s}(\Omega) \cap \mathbb{Y}^p_{N,s}(\Omega)$ *such that*

$$\boldsymbol{S} = \boldsymbol{E} + \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A},$$

where $\boldsymbol{E}$ *is unique,* $\boldsymbol{v}$ *is unique up to an additive rigid displacement,* $\boldsymbol{A}$ *is unique to an element of* $\mathbb{K}_{N,s}(\Omega)$ *and we have the estimate*

$$\|\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{R}(\Omega)} + \|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)/\mathbb{K}_{N,s}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

** Moreover, if* Ω *is of class* $\mathcal{C}^{2,1}$, *then* $\boldsymbol{A} \in \mathbb{W}^{2,p}_{\sigma,s}(\Omega) \cap \mathbb{Y}^p_{N,s}(\Omega)$ *and we have the estimate*

$$\|\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{R}(\Omega)} + \|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)/\mathbb{K}_{N,s}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

ii) Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$, *then there exist* $\boldsymbol{E} \in \mathbb{K}_{T,s}(\Omega)$, $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ *and* $\boldsymbol{A} \in \mathbb{W}^{2,p}_{0,s}(\Omega)$ *such that*

$$\boldsymbol{S} = \boldsymbol{E} + \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A},$$

where $\boldsymbol{E}$ *is unique,* $\boldsymbol{v}$ *is unique up to an additive rigid displacement and we have the estimate*

$$\|\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{R}(\Omega)} + \|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

iii) Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$, *then there exists* $\boldsymbol{v} \in \boldsymbol{W}^{1,p}_0(\Omega)$, $\boldsymbol{E} \in \mathbb{K}_{N,s}(\Omega)$ *and* $\boldsymbol{A} \in \mathbb{W}^{1,p}_{\sigma,s}(\Omega) \cap \mathbb{X}^p_{T,s}(\Omega)$ *such that*

$$\boldsymbol{S} = \boldsymbol{E} + \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A},$$

where $\boldsymbol{E}$ *and* $\boldsymbol{v}$ *are unique and* $\boldsymbol{A}$ *is unique to an additive element of* $\mathbb{K}_{T,s}(\Omega)$ *and we have the estimate*

$$\|\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)/\mathbb{K}_{T,s}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

** Moreover, if* Ω *is of class* $\mathcal{C}^{2,1}$, *then* $\boldsymbol{A} \in \mathbb{W}^{2,p}_{\sigma,s}(\Omega) \cap \mathbb{X}^p_{T,s}(\Omega)$ *and we have the estimate*

$$\|\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)/\mathbb{K}_{T,s}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

Let Ω be a connected and Lipschitz subset of $\mathbb{R}^N$ whose bounded and orientable boundary is denoted by Γ. A famous result of E. Gagliardo [26] gives, for $m = 1$, the characterization of the range of the restriction $\gamma_0(u) = u_{|\Gamma}$ to Γ. More precisely, Gagliardo proves that the operator γ_0 is linear and continuous from $W^{1,p}(\Omega)$ into $W^{1-\frac{1}{p},p}(\Gamma)$ for $1 \leq p < \infty$ and has a continuous right inverse for $p > 1$.

When $u \in W^{2,p}(\Omega)$, then $\dfrac{\partial u}{\partial x_j} \in W^{1,p}(\Omega)$ for $j = 1, \ldots, N$. Therefore the normal derivative $\gamma_1(u) = \nabla u \cdot \boldsymbol{n} \in L^p(\Gamma)$ since $\boldsymbol{n} = (n_1, \cdots, n_N)$ is defined almost everywhere and belongs to $(L^\infty(\Gamma))^N$. J. Nečas [44] proves that $\gamma_0(u) \in W^{1,p}(\Gamma)$ and that the linear mapping $u \to (\gamma_0(u), \gamma_1(u))$ is continuous from $W^{2,p}(\Omega)$ into $W^{1,p}(\Gamma) \times L^p(\Gamma)$. A natural question is to characterize the range of the mapping (γ_0, γ_1). A first answer has been obtained for polygonal-type domains of $\mathbb{R}^2$ by Kondrat'ev and Grisvard (see e.g. [34] for full references) in terms of compatibility conditions at the corners and then the results have been extended to polyhedral-type domains ($N = 3$). These characterizations have been extensively used in order to give regularity results for different types of boundary-value problems.

For general Lipschitz domains a first characterization of the range of (γ_0, γ_1) has been obtained for $N = 2$ in [29] and if $p = 2$ and extended in [23] for the general case $1 < p < \infty$. This result reads as follows: The range of $(\gamma_0, \gamma_1))$ is the set of $(g_0, g_1) \in W^{1,p}(\Gamma) \times L^p(\Gamma)$ such that:

$$\frac{\partial g_0}{\partial \boldsymbol{t}} \boldsymbol{t} + g_1 \boldsymbol{n} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma). \tag{1.0.13}$$

Let us mention, also, that the generalization for the case $N = 3$ and $1 < p < \infty$ was obtained by Buffa et al (see [17]).

In fact, a more general characterization of the image of the trace operators in $W^{m,p}(\Omega)$, where Ω is a domain in $\mathbb{R}^N$ with Lipschitz boundary, has been obtained for arbitrary m and N, by Maz'ya, Mitrea and Shaposnikova [40]. These authors used an analytical method based on Taylor expansions in Besov and weighted Sobolev spaces.

In Chapter 5, first of all, we will give two applications of the result of Geymonat and Krasucki [29] to solve a boundary value problem for the bi-laplacian equation. The first application concerns a regularity result for the solution to a non homogeneous Dirichlet problem for the homogeneous Bi-Laplacian equation in a lipschitzian domain. This result improve the one obtained in [22]. Up to our knowledge it is the first time that this result is stated in this form. The second application relies on the existence of very weak solution, in Lipschitz domains, to Dirichlet problem for the Bi-Laplacian equation. It is a first time that one can obtain very weak solution in Lipshitz domains.

Next, due to a new representation of the Hessian in $\mathbb{R}^3$, we characterize the range of the trace operator in $W^{3,p}(\Omega)$, more precisely, we would like to characterize the range of the application $(\gamma_0, \gamma_1, \gamma_2)$ defined on $W^{3,p}(\Omega)$ where

$$\begin{array}{rcl}\gamma_2 : W^{3,p}(\Omega) & \rightarrow & L^p(\Omega) \\ u & \rightarrow & \gamma_2(u) = [(\boldsymbol{\nabla}^2\, u)\boldsymbol{n}] \cdot \boldsymbol{n}.\end{array}$$

Necessary conditions are obtained by Geymonat [28].

Even if this result is a particular case of the obtained in [40], our proof is completely new and different from their. Our proof relies on potential matrices which are similar to potential vectors introduced in [9]. We hope that we can extend our proof to $W^{m,p}(\Omega)$ where Ω is a Lipschitz domain.

Chapter 2

Some characterizations of the curl operator

The surjectivity of the operator $\mathrm{div} : \boldsymbol{\mathcal{D}}(\Omega) \longrightarrow \mathcal{D}_0(\Omega)$ is an important tool in the analysis of Stokes equations. This result has been shown by many authors through different techniques (see [21], [27], [43]) and it provides us with a simple proof for the following usual version of De Rham's theorem: let $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ satisfying $\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega), \quad {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0,$ then there exists a scalar field $p \in \mathcal{D}'(\Omega)$ such that $\boldsymbol{f} = \boldsymbol{\nabla}\, p \quad$ in $\quad \Omega.$ The main goal of this chapter is to present some results of vector fields, specially a new extension of the above theorem that we will call the rotational version of De Rham's theorem.

In this chapter, Ω is a bounded and connected open set of $\mathbb{R}^3$ with Lipschitz-continuous boundary.

2.1 Poincaré integral operator

Mitrea [43], Costabel and Macintosch [21] have shown that if Ω is bounded and starlike with respect to an open ball, then the operator

$$\mathbf{curl} : \boldsymbol{\mathcal{D}}(\Omega) \longrightarrow \boldsymbol{\mathcal{V}}(\Omega) \tag{2.1.1}$$

is onto. In this section, we apply the singular integrals theory to give a detailed proof for this result. Then we generalize it for the case where Ω is a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary *i.e.*, we prove that the operator

$$\mathbf{curl} : \boldsymbol{\mathcal{D}}(\Omega) \longrightarrow \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega) \tag{2.1.2}$$

is onto. Here $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ denotes the space of functions $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega)$ such that $\int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi} \, dx = 0$ for all $\boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega)$. This last result is the main key to prove a rotational extension of De Rham's theorem.

Lemma 2.1.1. *Let θ be a function of $\mathcal{D}(\mathbb{R}^3)$ such that*

$$supp\,\theta \subset \Omega \quad and \quad \int_{\mathbb{R}^3} \theta(y)\, dy = 1.$$

Then, for any $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$, the vector field $\boldsymbol{Tf}$ defined by

$$x \in \Omega,\ \boldsymbol{Tf}(x) = \int_\Omega \boldsymbol{f}(y) \times \left((x-y) \int_1^\infty (t-1)t\,\theta(y + t(x-y))\, dt \right) dy, \tag{2.1.3}$$

satisfies

$$\mathbf{curl}\, \boldsymbol{Tf} = \boldsymbol{f}, \qquad \boldsymbol{Tf} \in \boldsymbol{\mathcal{C}}^\infty(\Omega) \tag{2.1.4}$$

and there exists a constant $C_p(\Omega)$ depending only on p and Ω, such that

$$\|\boldsymbol{Tf}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_p(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.1.5}$$

In particular, if Ω is starlike with respect to an open ball B and $supp\,\theta \subset B$, then

$$\boldsymbol{Tf} \in \boldsymbol{\mathcal{D}}(\Omega). \tag{2.1.6}$$

Proof. Note that $\boldsymbol{T}$ is a Poincaré type operator (see [21]). Let $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$, we denote by $\widetilde{\boldsymbol{f}}$ its extension by $\mathbf{0}$ outside of Ω.

Step 1. We start by establishing the two properties of (2.1.4).

We write $\boldsymbol{T f}$ in the form

$$x \in \Omega,\ \boldsymbol{T f}(x) = \int_\Omega \boldsymbol{f}(y) \times \boldsymbol{K}(x,y)\, dy,$$

where

$$\boldsymbol{K}(x,y) = (x-y) \int_1^\infty (t-1)t\, \theta(y + t(x-y))\, dt.$$

We observe that

$$\boldsymbol{T f}(x) = \lim_{\varepsilon \longrightarrow 0} \int_{|x-y| \geq \varepsilon} \widetilde{\boldsymbol{f}}(y) \times \boldsymbol{K}(x,y)\, dy,$$

then

$$\begin{aligned} \mathbf{curl}\,(\boldsymbol{T f})(x) = \nabla_x \times \boldsymbol{T f}(x) \quad &= \quad \lim_{\varepsilon \longrightarrow 0} \int_{|x-y| \geq \varepsilon} \nabla_x \times \Big(\widetilde{\boldsymbol{f}}(y) \times \boldsymbol{K}(x,y)\Big)\, dy \\ &+ \quad \lim_{\varepsilon \longrightarrow 0} \int_{|x-y| = \varepsilon} \frac{(x-y)}{|x-y|} \times \Big(\widetilde{\boldsymbol{f}}(y) \times \boldsymbol{K}(x,y)\Big)\, d\sigma_y \\ &:= \quad \lim_{\varepsilon \longrightarrow 0} (\boldsymbol{A}_\varepsilon + \boldsymbol{B}_\varepsilon). \end{aligned}$$

According to the formula:

$$\mathbf{curl}\,(\boldsymbol{A} \times \boldsymbol{B}) = \nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\nabla \cdot \boldsymbol{B})\boldsymbol{A} - (\nabla \cdot \boldsymbol{A})\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B},$$

we deduce that

$$\boldsymbol{A}_\varepsilon = \int_{|x-y| \geq \varepsilon} \Big[(\nabla_x \cdot \boldsymbol{K}(x,y))\, \widetilde{\boldsymbol{f}}(y) - (\widetilde{\boldsymbol{f}}(y) \cdot \nabla_x)\boldsymbol{K}(x,y)\Big]\, dy := \boldsymbol{A}_1(\varepsilon) - \boldsymbol{A}_2(\varepsilon).$$

Using now the following formula:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}),$$

we have

$$\boldsymbol{B}_\varepsilon = \int_{|x-y| = \varepsilon} \left[\left(\frac{(x-y)}{|x-y|} \cdot \boldsymbol{K}(x,y)\right) \widetilde{\boldsymbol{f}}(y)\ - \left(\frac{(x-y)}{|x-y|} \cdot \widetilde{\boldsymbol{f}}(y)\right) \boldsymbol{K}(x,y)\right]\, d\sigma_y := \boldsymbol{B}_1(\varepsilon) - \boldsymbol{B}_2(\varepsilon).$$

Thus, we can write

$$\mathbf{curl}\,(\boldsymbol{T}\boldsymbol{f})(x) = \lim_{\varepsilon \longrightarrow 0} \left[\boldsymbol{A}_1(\varepsilon) - \boldsymbol{A}_2(\varepsilon) + \boldsymbol{B}_1(\varepsilon) - \boldsymbol{B}_2(\varepsilon)\right].$$

i) Study of $\boldsymbol{A}_1(\varepsilon)$. We have

$$\boldsymbol{A}_1(\varepsilon) \;=\; \int_{|x-y|\geq\varepsilon} \left[(\nabla_x \cdot \boldsymbol{K}_1(x,y)) + (\nabla_x \cdot \boldsymbol{K}_2(x,y))\right] \widetilde{\boldsymbol{f}}(y)\,dy, \tag{2.1.7}$$

where

$$\boldsymbol{K}_1(x,y) = (x-y)\int_1^\infty t^2\theta(y+t(x-y))dt$$

and

$$\boldsymbol{K}_2(x,y) = -(x-y)\int_1^\infty t\,\theta(y+t(x-y))dt. \tag{2.1.8}$$

We remark that $\boldsymbol{K}_1(\cdot,\cdot)$ is the kernel of the Bogovskiî 's operator (see [13]), then

$$\nabla_x \cdot \boldsymbol{K}_1(x,y) = -\theta(x). \tag{2.1.9}$$

It is straightforward to see that

$$\begin{aligned}
\nabla_x \cdot \boldsymbol{K}_2(x,y) \;&=\; -3\int_1^\infty t\,\theta(y+t(x-y))\,dt - \sum_{i=1}^3 (x_i-y_i)\int_1^\infty t^2\partial_i\theta(y+t(x-y))\,dt \\
&=\; -\int_1^\infty t\,\theta(y+t(x-y))\,dt - \int_1^\infty \frac{\partial(t^2\theta)}{\partial t}(y+t(x-y))\,dt \\
&=\; \theta(x) - \int_1^\infty t\,\theta(y+t(x-y))\,dt.
\end{aligned} \tag{2.1.10}$$

Then, from (2.1.7), (2.1.9) and (2.1.10), we obtain

$$\lim_{\varepsilon\longrightarrow 0}\boldsymbol{A}_1(\varepsilon) \;=\; -\int_\Omega \boldsymbol{f}(y)\int_1^\infty t\,\theta(y+t(x-y))\,dt\,dy. \tag{2.1.11}$$

ii) Study of $B_1(\varepsilon)$. It is easy to prove that

$$\lim_{\varepsilon\longrightarrow 0}\int_{|x-y|=\varepsilon}\left(\frac{(x-y)}{|x-y|}\cdot\boldsymbol{K}_1(x,y)\right)\widetilde{\boldsymbol{f}}(y)\,d\sigma_y = \boldsymbol{f}(x).$$

In (2.1.8), we use the change of variable $s = t|x-y|$ to get

$$\boldsymbol{K}_2(x,y) = -\frac{(x-y)}{|x-y|}\int_{|x-y|}^\infty s\,\theta\left(y + s\frac{(x-y)}{|x-y|}\right)\,ds.$$

Then

$$\int_{|x-y|=\varepsilon} \frac{(x-y)}{|x-y|} \cdot \boldsymbol{K}_2(x,y)\widetilde{\boldsymbol{f}}(y)\, d\sigma_y$$

$$= - \sum_{i=1}^{3} \int_{|x-y|=\varepsilon}^{\infty} \widetilde{\boldsymbol{f}}(y)(\frac{x_i-y_i}{|x-y|})^2 \int_{|x-y|}^{\infty} s\,\theta(y+s\frac{(x-y)}{|x-y|})\, ds\, d\sigma_y.$$

Using now the change of variables $z = \dfrac{x-y}{\varepsilon}$ and $s' = s - \varepsilon$, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0} \int_{|x-y|=\varepsilon} \frac{(x-y)}{|x-y|} \cdot \boldsymbol{K}_2(x,y)\widetilde{\boldsymbol{f}}(y) d\sigma_y \\
= & -\lim_{\varepsilon\longrightarrow 0} \sum_{i=1}^{3} \varepsilon \int_{|z|=1} \widetilde{\boldsymbol{f}}(x-\varepsilon z) z_i^2 \int_0^{\infty} (s'+\varepsilon)\,\theta(x+s'z) ds' d\sigma_z \\
= & \ \boldsymbol{0}.
\end{aligned}$$

Consequently,

$$\lim_{\varepsilon\longrightarrow 0} \boldsymbol{B}_1(\varepsilon) = \boldsymbol{f}(x). \tag{2.1.12}$$

iii) Study of $\boldsymbol{A}_2(\varepsilon)+\boldsymbol{B}_2(\varepsilon)$. According to the Stokes formula, we obtain

$$\begin{aligned}
\lim_{\varepsilon\longrightarrow 0}(\boldsymbol{A}_2(\varepsilon)+\boldsymbol{B}_2(\varepsilon)) &= \lim_{\varepsilon\longrightarrow 0} \Big[\int_{|x-y|\geq\varepsilon} (\widetilde{\boldsymbol{f}}(y)\cdot\nabla_x)\boldsymbol{K}(x,y) + \int_{|x-y|=\varepsilon} \Big(\frac{(x-y)}{|x-y|}\cdot \widetilde{\boldsymbol{f}}(y)\Big)\boldsymbol{K}(x,y)\Big] \\
&= \lim_{\varepsilon\longrightarrow 0} \int_{|x-y|\geq\varepsilon} \Big[(\widetilde{\boldsymbol{f}}(y)\cdot\nabla_x)\boldsymbol{K}(x,y) + \boldsymbol{K}(x,y)\,\mathrm{div}\,\widetilde{\boldsymbol{f}}(y) \\
&+ \ (\widetilde{\boldsymbol{f}}(y)\cdot\nabla_y)\,\boldsymbol{K}(x,y)\Big]\ dy \\
&= \lim_{\varepsilon\longrightarrow 0} \int_{|x-y|\geq\varepsilon} \Big[(\widetilde{\boldsymbol{f}}(y)\cdot\nabla_x)\boldsymbol{K}(x,y) \ + (\widetilde{\boldsymbol{f}}(y)\cdot\nabla_y)\,\boldsymbol{K}(x,y)\Big]\ dy.
\end{aligned}$$

For any $1 \leq i, j, \leq 3$, we have

$$\frac{\partial K_j}{\partial x_i}(x,y) = -\frac{\partial K_j}{\partial y_i}(x,y) + (x_j-y_j)\int_1^{\infty} (t-1)t\ \partial_i\theta(y+t(x-y))dt.$$

Then

$$\lim_{\varepsilon\longrightarrow 0}(A_2(\varepsilon)_j + B_2(\varepsilon)_j) = \lim_{\varepsilon\longrightarrow 0} \int_{|x-y|\geq\varepsilon} \Big[\widetilde{\boldsymbol{f}}(y)\cdot \boldsymbol{L}_j(x,y)\Big]\ dy, \tag{2.1.13}$$

where the i^{th} component of $\boldsymbol{L}_j$ is given by

$$((\boldsymbol{L}_j(x,y))_i = (x_j - y_j)\int_1^\infty (t-1)t\ \partial_i\theta(y+t(x-y))\,dt.$$

iv) Verification of curl $\boldsymbol{T}\boldsymbol{f} = \boldsymbol{f}$. From (5.3.1)-(2.1.13), we conclude that for all $1 \le j \le 3$

$$\begin{aligned}(\mathbf{curl}\,\boldsymbol{T}\boldsymbol{f})_j(x) &= f_j(x) - \int_\Omega [\boldsymbol{f}(y)\cdot \boldsymbol{L}_j(x,y)]\ dy - \int_\Omega f_j(y)\int_1^\infty t\theta(y+t(x-y))\,dt\,dy \\ &= f_j(x) + \int_\Omega [\boldsymbol{f}(y)\cdot \boldsymbol{H}_j(x,y)]\ dy,\end{aligned}$$

with

$$\boldsymbol{H}_j(x,y) = -\boldsymbol{L}_j(x,y) - \boldsymbol{e_j}\int_1^\infty t\theta(y+t(x-y))\,dt$$

and where $\boldsymbol{e_j}$ is the j^{th} vector of the canonical basis of $\mathbb{R}^3$. It is easy to verify that

$$\forall x\in\mathbb{R}^3,\ \boldsymbol{H}_j(x,\cdot) = \mathbf{grad}\,\chi_j(x,\cdot),$$

where

$$\chi_j(x,y) = (x_j - y_j)\int_1^\infty t\theta(y+t(x-y))dt.$$

Since $\boldsymbol{f}\in\boldsymbol{\mathcal{V}}(\Omega)$, we deduce

$$\begin{aligned}(\mathbf{curl}\,\boldsymbol{T}\boldsymbol{f})_j(x) &= f_j(x) - \int_\Omega \operatorname{div}\boldsymbol{f}(y)\,\chi_j(x,y)\,dy \\ &= f_j(x).\end{aligned}$$

v) Verification of $\boldsymbol{T}\boldsymbol{f}\in\boldsymbol{\mathcal{C}}^\infty(\Omega)$. In (2.1.3), we use the changes of variables $z = x - y$ and $s = (t-1)|x-y|$, so we obtain

$$x\in\Omega,\ \boldsymbol{T}\boldsymbol{f}(x) = \int_{\mathbb{R}^3}\widetilde{\boldsymbol{f}}(x-z)\times z\int_0^\infty\left(\frac{s^2}{|z|^3}+\frac{s}{|z|^2}\right)\theta(x+s\frac{z}{|z|})ds\,dz. \tag{2.1.14}$$

Then, for any $\alpha\in\mathbb{N}^3$, we have

$$\begin{aligned}\partial^\alpha\boldsymbol{T}\boldsymbol{f}(x) &= \int_{\mathbb{R}^3}\partial^\alpha\widetilde{\boldsymbol{f}}(x-z)\times z\int_0^\infty\left(\frac{s^2}{|z|^3}+\frac{s}{|z|^2}\right)\theta(x+s\frac{z}{|z|})ds\,dz \\ &+ \int_{\mathbb{R}^3}\widetilde{\boldsymbol{f}}(x-z)\times z\int_0^\infty\left(\frac{s^2}{|z|^3}+\frac{s}{|z|^2}\right)\partial^\alpha\theta(x+s\frac{z}{|z|})ds\,dz.\end{aligned}$$

Since $\partial^\alpha \boldsymbol{f}$ and $\partial^\alpha \theta$ are continuous in Ω, then $\partial^\alpha \boldsymbol{Tf}$ is continuous and $\boldsymbol{Tf} \in \boldsymbol{C}^\infty(\Omega)$.

Step 2. Now, we establish the estimate (2.1.5).

Let $f \in \boldsymbol{\mathcal{V}}(\Omega)$ and $1 \le i, j \le 3$, we have

$$\begin{aligned}
\frac{\partial \boldsymbol{Tf}}{\partial x_j}(x) &= \lim_{\epsilon \longrightarrow 0} \left[\int_{|x-y|\ge\epsilon} \frac{\partial}{\partial x_j} \left(\boldsymbol{f}(y) \times \boldsymbol{K}(x,y)\right) dy + \int_{|x-y|=\epsilon} \left(\boldsymbol{f}(y) \times \boldsymbol{K}(x,y)\right) \frac{x_j - y_j}{|x-y|} d\sigma \right] \\
&= \lim_{\epsilon \longrightarrow 0} \left[\int_{|x-y|\ge\epsilon} \left(\boldsymbol{f}(y) \times \frac{\partial \boldsymbol{K_1}}{\partial x_j}(x,y) \right) dy + \int_{|x-y|=\epsilon} \left(\boldsymbol{f}(y) \times \boldsymbol{K_1}(x,y)\right) \frac{x_j - y_j}{|x-y|} d\sigma \right] \\
&+ \lim_{\epsilon \longrightarrow 0} \left[\int_{|x-y|\ge\epsilon} \left(\boldsymbol{f}(y) \times \frac{\partial \boldsymbol{K_2}}{\partial x_j}(x,y) \right) dy + \int_{|x-y|=\epsilon} \left(\boldsymbol{f}(y) \times \boldsymbol{K_2}(x,y)\right) \frac{x_j - y_j}{|x-y|} d\sigma \right].
\end{aligned}$$

Since we have shown

$$\lim_{\epsilon \longrightarrow 0} \int_{|x-y|=\epsilon} \left(\boldsymbol{f}(y) \times \boldsymbol{K_2}(x,y)\right) \frac{x_j - y_j}{|x-y|} d\sigma = 0,$$

then

$$\begin{aligned}
\frac{\partial \boldsymbol{Tf}}{\partial x_j}(x) &= \int_\Omega \left(\boldsymbol{f}(y) \times \frac{\partial \boldsymbol{K_1}}{\partial x_j}(x,y) \right) dy + \int_\Omega \left(\boldsymbol{f}(y) \times \frac{\partial \boldsymbol{K_2}}{\partial x_j}(x,y) \right) dy \\
&+ \left(\boldsymbol{f}(x) \times \int_\Omega (x-y) \left(\theta(y) \frac{x_j - y_j}{|x-y|} \right) dy \right) \\
&:= \boldsymbol{J_1 f}(x) + \boldsymbol{J_2 f}(x) + \boldsymbol{J_3 f}(x).
\end{aligned}$$

Also

$$\begin{cases}
(\boldsymbol{J_1 f})_{ij}(x) = \displaystyle\int_\Omega \left(\varepsilon_{imn} f_m(y) \frac{\partial K_{1_n}}{\partial x_j}(x,y) \right) dy \\
(\boldsymbol{J_2 f})_{ij}(x) = \displaystyle\int_\Omega \left(\varepsilon_{imn} f_m(y) \frac{\partial K_{2_n}}{\partial x_j}(x,y) \right) dy \\
(\boldsymbol{J_3 f})_{ij}(x) = \varepsilon_{imn} f_m(x) \displaystyle\int_\Omega K_{1_n}(x,y) \frac{x_j - y_j}{|x-y|^2} dy.
\end{cases}$$

There exists a constant $C(\Omega)$ (see page 166 of [27]) such that

$$\forall \varphi \in \mathcal{D}(\Omega), \; \| \int_\Omega \varphi(y) \frac{\partial K_{1_n}}{\partial x_j}(\cdot, y)\, dy \|_{L^p(\Omega)} \le C(\Omega) \|\varphi\|_{L^p(\Omega)}.$$

Then, there exists a constant $C_1(\Omega)$ such that

$$\|\boldsymbol{J_1 f}\|_{L^p(\Omega)} \le C_1(\Omega) \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.1.15}$$

Besides, we have

$$\forall \varphi \in \mathcal{D}(\Omega), \int_\Omega \varphi(y) \frac{\partial K_{2_n}}{\partial x_j}(x,y)dy = \frac{1}{2}\int_\Omega \varphi(y) G_{nj}(x,y)dy.$$

Thus, there exists a constant $C_2(\Omega)$ such that (see page 165 of [27])

$$\|\boldsymbol{J_2 f}\|_{L^p(\Omega)} \leq C_2(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.1.16}$$

For the last estimate, since for each $x \in \Omega$

$$|(\boldsymbol{J_3 f})_i(x)| \leq \sum_{k=1}^{3} |f_k(x)|,$$

we deduce that

$$\|\boldsymbol{J_3 f}\|_{L^p(\Omega)} \leq C_3(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.1.17}$$

Finally, from (2.1.15)-(2.1.17), there exists a constant $C_p(\Omega)$ such that

$$\|\boldsymbol{Tf}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_p(\Omega)\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}.$$

Step 3. Now, we suppose that Ω is starlike with respect to an open ball B and that supp $\theta \subset B$. We will prove the property (2.1.6).

Indeed, in what follows we take

$$A = \{z \in \Omega; z = \lambda z_1 + (1-\lambda)z_2,\ z_1 \in \text{ supp}\,\boldsymbol{f},\ z_2 \in \overline{B},\ \lambda \in [0,1]\}.$$

Since Ω is starlike with respect to an open ball B, the compact set A is included in Ω. Fixing any $x \in \Omega \setminus A$, for any $y \in \text{supp}\,\boldsymbol{f}$ and $t \geq 1$ we have $y + t(x-y) \notin \overline{B}$. According to (2.1.3), we deduce that $\boldsymbol{Tf}(x) = \boldsymbol{0}$ and $\text{supp}\,\boldsymbol{Tf} \subset A \subset \Omega$. Consequently $\boldsymbol{Tf} \in \boldsymbol{\mathcal{D}}(\Omega)$. □

The following corollary generalizes the estimate (2.1.5) in the case where we replace the Lebesgue space $\boldsymbol{L}^p(\Omega)$ by the Sobolev space $\boldsymbol{W}^{m,p}(\Omega)$, for any positive integer m.

Corollary 2.1.2. *Let $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$ and $\boldsymbol{T}$ the operator defined in* (2.1.3). *Then, for any real number $1 < p < \infty$ and for any integer $m \geq 1$, there exists a constant C depending only on p, m and Ω such that*

$$\|\boldsymbol{Tf}\|_{\boldsymbol{W}^{m+1,p}(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{W}^{m,p}(\Omega)}. \tag{2.1.18}$$

Proof. All the constants which appear in the inequalities below are noted by a generic letter C. We consider the case $m = 1$ and we write the operator $\boldsymbol{T}$ as in (2.1.14). So for any $i, j = 1, \cdots, N$, we have

$$\begin{aligned}\partial_j \boldsymbol{T}\boldsymbol{f}(x) &= \int_{\mathbb{R}^3} \partial_j \widetilde{\boldsymbol{f}}(x-z) \times z \int_0^\infty \left(\frac{s^2}{|z|^3} + \frac{s}{|z|^2}\right) \theta(x + s\frac{z}{|z|}) ds\, dz \\ &+ \int_{\mathbb{R}^3} \widetilde{\boldsymbol{f}}(x-z) \times \int_0^\infty \left(\frac{s^2}{|z|^3} + \frac{s}{|z|^2}\right) \partial_j\theta(x + s\frac{z}{|z|}) ds\, dz \\ &:= \boldsymbol{h}_1(x) + \boldsymbol{h}_2(x).\end{aligned}$$

Estimate of $\|\partial_k \boldsymbol{h}_1\|_{L^p(\Omega)}$. We observe that $\boldsymbol{h}_1 = \partial_j \boldsymbol{T}\boldsymbol{f}$, then Lemma 2.1.1 implies

$$\|\partial_k \boldsymbol{h}_1\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\partial_j \boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{2.1.19}$$

Estimate of $\|\partial_k \boldsymbol{h}_2\|_{L^p(\Omega)}$. We remark that the function $\boldsymbol{h}_2$ has the same form as the function $\boldsymbol{T}\boldsymbol{f}$ with θ replaced by $\partial_j\theta$. Note that, we find the estimate of the point without the need of the property $\int_\Omega \theta(x)dx = 1$. This means that by the same method, we obtain

$$\|\partial_k \boldsymbol{h}_2\|_{L^p(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{2.1.20}$$

From (2.1.19) and (2.1.20), we deduce the existence of a constant C depending only on m, p and Ω such that (2.1.18) holds. For $m > 1$, we proceed by induction and so we apply the same approach as for the case $m = 1$. □

We have shown that if Ω is a starlike open set with respect to an open ball, then the rotational operator is onto from $\boldsymbol{\mathcal{D}}(\Omega)$ into $\boldsymbol{\mathcal{V}}(\Omega)$. This result can be extended to a bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. For that we need the following lemma (see [10]).

Lemma 2.1.3. *Let Ω be a bounded and connected open set of $\mathbb{R}^N$ with a Lipschitz-continuous boundary. Then, there exist connected open sets Ω_j of $\mathbb{R}^N$, $j \geq 1$, with the following properties:*
i) $\partial\Omega_j$ is of classe $\mathcal{C}^\infty$.
ii) $\overline{\Omega_j} \subset \Omega_{j+1} \subset \Omega$ for each $j \geq 1$, and $\Omega = \cup_{j=1}^\infty \Omega_j$.

Theorem 2.1.4. *For any $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$, there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$ such that*

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f} \quad \text{in} \quad \Omega.$$

Moreover, for any $1 < p < \infty$ and for any nonnegative integer m, there exists a constant C such that

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{m+1,p}(\Omega)} \leq C\|\boldsymbol{f}\|_{\boldsymbol{W}^{m,p}(\Omega)}. \tag{2.1.21}$$

Proof. Let $\boldsymbol{f} \in \boldsymbol{\mathcal{V}}(\Omega)$ satisfying the condition (1.0.6). Lemma 2.1.1 and Corollary 2.1.2 imply that $\boldsymbol{Tf} \in \boldsymbol{\mathcal{C}}^\infty(\Omega)$, $\mathbf{curl}\,(\boldsymbol{Tf}) = \boldsymbol{f}$ in Ω with the estimate (2.1.21). Thanks to Lemma 2.1.3 there exists an open set Ω_{j_0} which is connected and of class $\mathcal{C}^\infty$, such that $\text{supp}\boldsymbol{f} \subset \overline{\Omega_{j_0}} \subset \Omega$. Define the open set $\Omega' = \Omega \setminus \overline{\Omega_{j_0}}$, which is bounded and connected open set of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. Setting now $\boldsymbol{\psi'} = \boldsymbol{Tf}|_{\Omega'}$, it follows from Lemma 2.1.1 that $\mathbf{curl}\,\boldsymbol{\psi'} = \mathbf{0}$ in Ω' and by Corollary 2.1.2 that $\boldsymbol{\psi'} \in \bigcap_{\substack{1<p<\infty,\\ m\in\mathbb{N}}} \boldsymbol{W}^{m,p}(\Omega)$. The compatibility condition $\displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx = 0$ for all $\boldsymbol{\varphi} \in \boldsymbol{K}_T(\Omega)$ implies that for any curves γ_j^* inside Ω' and surrounding Σ_j, we have $\displaystyle\int_{\gamma_j^*} \boldsymbol{\psi'}\cdot\boldsymbol{t} = \int_{\Sigma_j} \boldsymbol{f}\cdot\boldsymbol{n} = 0$. Hence $\boldsymbol{\psi'}$ has no circulations in Ω'. Then, there exists χ' satisfying $\chi' \in \bigcap_{\substack{1<p<\infty,\\ m\in\mathbb{N}}} W^{m,p}(\Omega)$, such that $\mathbf{grad}\,\chi' = \boldsymbol{\psi'}$ in Ω' (see Corollary 1 page 199 in [46]) and with the estimate

$$\|\chi'\|_{W^{m+2,p}(\Omega')} \leq C\|\boldsymbol{\psi'}\|_{\boldsymbol{W}^{m+1,p}(\Omega')}.$$

Theorem 1.4.3.1 of [34] implies that there exists $\widetilde{\chi} \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that $\widetilde{\chi}|_{\Omega'} = \chi'$ and

$$\|\widetilde{\chi}\|_{W^{m+2,p}(\mathbb{R}^3)} \leq C\|\chi'\|_{W^{m+2,p}(\Omega')} \leq C\|\boldsymbol{\psi'}\|_{\boldsymbol{W}^{m+1,p}(\Omega')}.$$

Setting now $\chi = \widetilde{\chi}|_\Omega$ and $\boldsymbol{\psi} = \boldsymbol{Tf} - \mathbf{grad}\,\chi$, we have $\boldsymbol{\psi}|_{\Omega'} = \mathbf{0}$, and then $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$. Furthermore, it is clear that $\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f}$ in Ω and for any $1 < p < \infty$ and $m \in \mathbb{N}$ the estimate (2.1.21) holds. □

2.2 The rotational version of De Rham's theorem

In this sction, we will use the surjectivity of the rotational operator (2.1.2) to show a rotational extension of De Rham's theorem.

Theorem 2.2.1. *Let* $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ *satisfying the following condition:*

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}(\Omega), \quad {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0. \tag{2.2.1}$$

Then, there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$ *such that*

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f} \quad \textit{in} \quad \Omega.$$

Remark 2.2.2. The converse is obvious, because for any $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$ and any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}(\Omega)$, we have

$${}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \mathbf{curl}\,\boldsymbol{\psi}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{\psi}, \mathbf{curl}\,\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

Proof. Acccording to Theorem 2.1.4,

$$\mathbf{curl}: \boldsymbol{\mathcal{D}}(\boldsymbol{\Omega})/\boldsymbol{\mathcal{G}}(\Omega) \longrightarrow \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$$

is one to one and onto. Then, its adjoint

$$\mathbf{curl}: (\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega))' \longrightarrow \boldsymbol{\mathcal{D}}'(\boldsymbol{\Omega}) \perp \boldsymbol{\mathcal{G}}(\Omega) \tag{2.2.2}$$

is also one to one and onto, where $\boldsymbol{\mathcal{D}}'(\boldsymbol{\Omega}) \perp \boldsymbol{\mathcal{G}}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\boldsymbol{\Omega}),\ \langle \boldsymbol{v},\ \boldsymbol{\varphi} = 0,\ \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}(\Omega)\}$.

Let $\boldsymbol{L} \in (\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega))'$. As $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ is closed in $\boldsymbol{\mathcal{D}}(\Omega)$, we can extend $\boldsymbol{L}$ by $\widetilde{\boldsymbol{L}} \in \boldsymbol{\mathcal{D}}'(\Omega)$. Two expressions $\boldsymbol{g} \in \boldsymbol{\mathcal{D}}'(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{\mathcal{D}}'(\Omega)$ of $\widetilde{\boldsymbol{L}}$ coincide on $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ if and only if

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega),\ {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{g} - \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

Using again Theorem 2.1.4, we have

$$\begin{aligned}
\forall\, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega),\ {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{g} - \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} &= {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{g} - \boldsymbol{h}, \mathbf{curl}\,\boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} \\
&= {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \mathbf{curl}(\boldsymbol{g} - \boldsymbol{h}), \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)} \\
&= 0.
\end{aligned}$$

Which means that

$$\boldsymbol{g} - \boldsymbol{h} \in \mathbf{Ker}\,\mathbf{curl}, \quad \text{where} \quad \mathbf{curl} : \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \boldsymbol{\mathcal{D}}'(\Omega),$$

and consequently

$$(\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega))' = \boldsymbol{\mathcal{D}}'(\Omega)/\mathbf{Ker}\,\mathbf{curl}. \tag{2.2.3}$$

Let $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ satisfies (2.2.1). In other words, this means that $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega) \perp \boldsymbol{\mathcal{G}}(\Omega)$. From (2.2.2) and the characterization (2.2.3), there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$, such that

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f} \quad \text{in} \quad \Omega.$$

□

2.3 A weak rotational extension of De Rham's theorem

In this section, we will use Theorem 2.1.4 to show another surjectivity result of the **curl** operator. Then, we will use this result to prove a weak rotational extension of De Rham's theorem. First, we need the following lemma:

Lemma 2.3.1. *Let m be a nonnegative integer. Then, the space $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ is dense in $\boldsymbol{U}^{m,p}(\Omega)$.*

Proof. **Step 1**: we show that the linear mapping $\boldsymbol{\mathcal{R}} : \boldsymbol{\mathcal{V}}(\Omega) \longrightarrow \mathbb{R}^J$ defined by

$$(\boldsymbol{\mathcal{R}}(\boldsymbol{v}))_j = \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma, \quad 1 \leq j \leq J,$$

is onto, where J is the dimension of $\boldsymbol{K}_T(\Omega)$. For that purpos, we proceed by contradiction. We suppose that $\boldsymbol{\mathcal{R}}$ is not onto, which implies that there exists j_0 such that $1 \leq j_0 \leq J$ and a family of numbers $\{\lambda_j\}_{\substack{1 \leq j \leq J,\\ j \neq j_0}}$ such that for any $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\Omega)$, we have

$$\int_{\Sigma_{j_0}} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma = \sum_{\substack{j=1\\ j \neq j_0}}^{J} \lambda_j \int_{\Sigma_j} \boldsymbol{v} \cdot \boldsymbol{n}\, d\sigma.$$

Using the Green's formula of Lemma 3.10 of [3], then

$$\int_{\Sigma_{j_0}} \boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma - \sum_{\substack{j=1\\ j\neq j_0}}^{J} \lambda_j \int_{\Sigma_j} \boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma = \int_\Omega \boldsymbol{v}\cdot\left(\widetilde{\mathbf{grad}\,q_{j_0}^T} - \sum_{\substack{j=1\\ j\neq j_0}}^{J}\lambda_j \widetilde{\mathbf{grad}\,q_j^T}\right)dx = 0,$$

where the vector fields $\widetilde{\mathbf{grad}\,q_j^T}$ are the elements of the basis of $\boldsymbol{K}_T(\Omega)$ (see [3]). Then, the usual extension of De Rham's theorem (see [7]) implies that there exists $p \in H^1(\Omega)$, unique up to an additive constant, such that

$$\widetilde{\mathbf{grad}\,q_{j_0}^T} - \sum_{\substack{j=1\\ j\neq j_0}}^{J}\lambda_j \widetilde{\mathbf{grad}\,q_j^T} = \mathbf{grad}\,p.$$

Consequently, p is harmonic and $\dfrac{\partial p}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$. So, p is a constant and then the dimension of $\boldsymbol{K}_T(\Omega)$ is less then J, which is a contradiction. We have proved that for any $1 \le j \le J$ there exists $\boldsymbol{\varphi}_j \in \boldsymbol{\mathcal{V}}(\Omega)$ such that

$$\text{for} \quad \text{all } 1 \le k \le J, \int_{\Sigma_k} \boldsymbol{\varphi}_j\cdot\boldsymbol{n}\,d\sigma = \delta_{kj}. \tag{2.3.1}$$

Step 2: we show that $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ is dense in $\boldsymbol{U}^{m,p}(\Omega)$. Let $\boldsymbol{v} \in \boldsymbol{U}^{m,p}(\Omega)$, then there exists a sequence $(\boldsymbol{v}_k) \in \boldsymbol{\mathcal{V}}(\Omega)$ that converges to $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$. For any $1 \le j \le J$, let $\boldsymbol{\varphi}_j$ be the function in $\boldsymbol{\mathcal{V}}(\Omega)$ which satisfies (2.3.1). Now, setting

$$\boldsymbol{u}_k = \boldsymbol{v}_k - \sum_{j=1}^{J}\Big(\int_{\Sigma_j}\boldsymbol{v}_k\cdot\boldsymbol{n}\,d\sigma\Big)\,\boldsymbol{\varphi}_j,$$

the function $\boldsymbol{u}_k$ belongs to $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$. Also the sequence $(\boldsymbol{u}_k)$ converges to $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$, which is the required result. □

Theorem 2.3.2. *Let m be a nonnegative integer. For any $\boldsymbol{f} \in \boldsymbol{U}^{m,p}(\Omega)$, there exists $\boldsymbol{\psi} \in \boldsymbol{W}_0^{m+1,p}(\Omega)$ that satisfies*

$$\mathbf{curl}\,\boldsymbol{\psi} = \boldsymbol{f} \quad \textit{in} \quad \Omega,$$

and there exists a constant C such that

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{m+1,p}(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{W}^{m,p}(\Omega)}. \tag{2.3.2}$$

Proof. Let $\boldsymbol{f} \in \boldsymbol{U}^{m,p}(\Omega)$ and $(\boldsymbol{f}_n)$ a sequence in $\boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K_T}(\Omega)$, such that

$$\boldsymbol{f}_n \longrightarrow \boldsymbol{f} \quad \text{in} \quad \boldsymbol{W}^{m,p}(\Omega).$$

Theorem 2.1.4 shows that for any $n \in \mathbb{N}$, there exists a vector field $\boldsymbol{\psi}_n \in \boldsymbol{\mathcal{D}}(\Omega)$ such that

$$\boldsymbol{\psi}_n \in \boldsymbol{\mathcal{D}}(\Omega), \quad \mathbf{curl}\, \boldsymbol{\psi}_n = \boldsymbol{f}_n \quad \text{and} \quad \|\boldsymbol{\psi}_n\|_{\boldsymbol{W}^{m+1,p}(\Omega)} \leq C \|\boldsymbol{f}_n\|_{\boldsymbol{W}^{m,p}(\Omega)}.$$

Clearly $(\boldsymbol{\psi}_n)$ is a Cauchy sequence. Then, there exists an element $\boldsymbol{\psi} \in \boldsymbol{W}_0^{m+1,p}(\Omega)$ such that

$$\boldsymbol{\psi}_n \longrightarrow \boldsymbol{\psi} \quad \text{in} \quad \boldsymbol{W}^{m+1,p}(\Omega),$$

with $\boldsymbol{\psi}$ satisfies (2.3.2). □

Remark 2.3.3.

i) Theorem 2.3.2 was proved for Ω bounded and simply-connected open set of $\mathbb{R}^3$ with Lipschitz-continuous boundary, $m = 1$ and $p = 2$ by Ciarlet and Ciarlet, Jr (see the proof of Theorem 3.1 in [18]) and for m a nonnegative integer and $p = 2$ by Amrouche, Ciarlet and Ciarlet, Jr (see [5]).

ii) For m nonnegative integer and $p = 2$, as in [5], we can define a vector field $\boldsymbol{\psi_0} \in \boldsymbol{H}_0^{m+1}(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{\psi_0} = \boldsymbol{f}$ in Ω and $\mathrm{div} \boldsymbol{\Delta}^{m+1} \boldsymbol{\psi_0} = 0$ in Ω. For that, it is sufficient to choose $\boldsymbol{\psi_0} = \boldsymbol{\psi} - \mathbf{grad}\, p$, where p is the unique solution in $\boldsymbol{H}_0^{m+2}(\Omega)$ of $\Delta^{m+2} p = \mathrm{div} \boldsymbol{\Delta}^{m+1} \boldsymbol{\psi}$ and $\boldsymbol{\psi}$ given by Theorem 2.3.2.

iii) For Ω bounded and connected open set of $\mathbb{R}^3$ with boundary of class $\mathcal{C}^{m+2}$, Borchers and Sohr in [14] established the same result that Theorem 2.3.2 with $\mathrm{div}\, \boldsymbol{\Delta}^{m+1} \boldsymbol{\psi} = 0$. Moreover, for $m = 1$ and Ω of class $\mathcal{C}^{1,1}$, Amrouche, Bernardi, Dauge and Girault in [3] gave another proof of the result established by Borchers and Sohr. Furthermore, they proved that the vector field $\boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega)$ is unique, provided that

$$\langle \partial_n (\mathrm{div}\, \boldsymbol{\psi}), 1 \rangle_{\Gamma_i} = 0, \; 1 \leq i \leq I,$$

where Γ_i are the different connected components of $\partial\Omega$.

The following weak rotational extension of De Rham's theorem is a direct consequence of Theorem 2.3.2. We define the space $\boldsymbol{\mathcal{G}}^{\boldsymbol{m,p}}(\Omega)$ by

$$\boldsymbol{\mathcal{G}}^{\boldsymbol{m,p}}(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{W}_0^{m,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0} \quad \text{in} \quad \Omega\}.$$

Theorem 2.3.4. *Let m be an integer such that $m \geq 1$, $\boldsymbol{f} \in \boldsymbol{W}^{-m,p'}(\Omega)$ and satisfies*

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}^{\boldsymbol{m,p}}(\Omega), \quad {}_{\boldsymbol{W}^{-m,p'}(\Omega)}\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}_0^{m,p}(\Omega)} = 0. \tag{2.3.3}$$

Then, there exists $\boldsymbol{\Psi} \in \boldsymbol{W}^{-m+1,p'}(\Omega)$, such that

$$\mathbf{curl}\,\boldsymbol{\Psi} = \boldsymbol{f} \quad \textit{in} \quad \Omega.$$

Proof. According to Theorem 2.3.2, the operator

$$\mathbf{curl} : \boldsymbol{W}_0^{m,p}(\Omega)/\boldsymbol{\mathcal{G}}^{\boldsymbol{m,p}}(\Omega) \longrightarrow \boldsymbol{U}^{m-1,p}(\Omega),$$

is one to one and onto. Then, its adjoint

$$\mathbf{curl} : (\boldsymbol{U}^{m-1,p}(\Omega))' \longrightarrow \boldsymbol{W}^{-m,p'}(\Omega) \perp \boldsymbol{\mathcal{G}}^{\boldsymbol{m,p}}(\Omega), \tag{2.3.4}$$

is also one to one and onto. Proceeding as in the proof of Theorem 2.2.1 and using Theorem 2.3.2, it is easy to prove that

$$(\boldsymbol{U}^{m-1,p}(\Omega))' = \boldsymbol{W}^{-m+1,p'}(\Omega)/\mathbf{Ker\,curl}, \tag{2.3.5}$$

where

$$\mathbf{curl} :\ \boldsymbol{W}^{-m+1,p'}(\Omega) \longrightarrow \boldsymbol{W}^{-m,p'}(\Omega).$$

Let $\boldsymbol{f} \in \boldsymbol{W}^{-m,p'}(\Omega)$ satisfying (2.3.3). In other words, $\boldsymbol{f} \in \boldsymbol{W}^{-m,p'}(\Omega) \perp \boldsymbol{\mathcal{G}}^{\boldsymbol{m,p}}(\Omega)$. Since the operator (2.3.4) is an isomorphism, the characterization (2.3.5) implies that there exists $\boldsymbol{\Psi} \in \boldsymbol{W}^{-m+1,p'}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{\Psi} = \boldsymbol{f}$ in Ω. $\square$

2.4 A new proof of the general extension of Poincaré's Lemma

The classical Poincaré's lemma asserts that if Ω is a simply-connected open set, then for any $\boldsymbol{h} \in \boldsymbol{\mathcal{C}}^{\mathbf{1}}(\Omega)$ which satisfies $\mathbf{curl}\, \boldsymbol{h} = \mathbf{0}$ in Ω, there exists $\chi \in \mathcal{C}^2(\Omega)$ such that $\boldsymbol{h} = \mathbf{grad}\chi$. This lemma is also true in the general case where $\boldsymbol{h} \in \boldsymbol{L}^2(\Omega)$ and Ω is a bounded and simply-connected open set with a Lipschitz-continuous boundary (see Theorem 2.9 chapter 1 in [33]). A general extension when $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$ was proved by Ciarlet and Ciarlet, Jr (see [18]).

In this section, we study the case where $\boldsymbol{h}$ is a distribution. The first proof of this extension in the case where Ω is a simply-connected open, based on differential geometry tools, was given by S. Mardare [37] in 2008 (Schwartz proved this extension for $\Omega = \mathbb{R}^3$, see Section 3 of [47]). Here, we give a simpler proof, using the characterization of the dual space $\boldsymbol{\mathcal{V}}(\Omega)'$ given in the proof of Theorem 2.2.1.

Lemma 2.4.1. *Let* $\boldsymbol{h} \in \boldsymbol{\mathcal{D}}'(\Omega)$. *If*

$$\mathbf{curl}\, \boldsymbol{h} = \mathbf{0} \quad in \quad \Omega,$$

then, there exists $p \in \mathcal{D}'(\Omega)$ *such that*

$$\boldsymbol{h} = \mathbf{grad}\, \chi \quad in \quad \Omega.$$

Proof. Let $\boldsymbol{L} \in \boldsymbol{\mathcal{V}}(\Omega)'$. Since $\boldsymbol{\mathcal{V}}(\Omega)$ is closed in $\boldsymbol{\mathcal{D}}(\Omega)$, we can extend $\boldsymbol{L}$ by $\widetilde{\boldsymbol{L}} \in \boldsymbol{\mathcal{D}}'(\Omega)$. Two expressions $\boldsymbol{g} \in \boldsymbol{\mathcal{D}}'(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{\mathcal{D}}'(\Omega)$ of $\widetilde{\boldsymbol{L}}$ coincide on $\boldsymbol{\mathcal{V}}(\Omega)$ if and only if

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{\mathcal{V}}(\Omega), \;\; {}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \boldsymbol{g} - \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{\mathcal{D}}(\Omega)} = 0.$$

According to the usual De Rham's theorem, there exists $p \in \mathcal{D}'(\Omega)$ such that $\boldsymbol{g} - \boldsymbol{h} = \mathbf{grad}\, p$. This means that we can define $\boldsymbol{\mathcal{V}}(\Omega)'$ as follows:

$$\boldsymbol{\mathcal{V}}(\Omega)' = \boldsymbol{\mathcal{D}}'(\Omega)/\mathbf{Im}\,(\mathbf{grad}) \quad \text{where} \quad \mathbf{grad} : \mathcal{D}'(\Omega) \longrightarrow \boldsymbol{\mathcal{D}}'(\Omega). \tag{2.4.1}$$

It has already been shown that

$$\boldsymbol{\mathcal{V}}(\Omega)' = \boldsymbol{\mathcal{D}}'(\Omega)/\mathbf{ker}\,\mathbf{curl} \quad \text{where} \quad \mathbf{curl} : \boldsymbol{\mathcal{D}}'(\Omega) \longrightarrow \boldsymbol{\mathcal{D}}'(\Omega). \tag{2.4.2}$$

According to (2.4.1) and (2.4.2), we conclude that

$$\mathbf{Ker}\,(\mathbf{curl}) = \mathbf{Im}\,(\mathbf{grad}), \tag{2.4.3}$$

hence the required result. $\square$

Chapter 3

Beltrami's completeness for distributions symmetric matrix fields

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In Chapter 2, we have shown that the operator (2.1.2) is onto. Then, we have used this surjectivity result to prove a rotational extension of De Rham's theorem. In this chapter, we will use the same argument to prove some results for symmetric matrix fields, specially some extensions of the Beltrami completeness for data in $\mathbb{D}_s(\Omega)$ and for data in $\mathbb{D}'_s(\Omega)$.

In this chapter, Ω is a bounded and connected open set of $\mathbb{R}^3$ with Lipschitz-continuous boundary.

3.1 Beltrami's completeness for symmetric matrix fields in $\mathbb{D}_s(\Omega)$

Theorem 3.1.1. *Let m be a nonnegative integer. For any matrix $\boldsymbol{S} \in \mathbb{V}_s(\Omega)$ satisfies*

$$\int_\Omega \boldsymbol{S} : \boldsymbol{M}\, dx = 0 \quad \textit{for all} \quad \boldsymbol{M} \in \mathbb{K}_{T,s}(\Omega),$$

there exists $\boldsymbol{A} \in \mathbb{D}_s(\Omega)$ such that

$$\mathbf{Curl\,Curl}\,\boldsymbol{A} = \boldsymbol{S} \quad \textit{in} \quad \Omega.$$

Moreover, there exists a constant C depending only on p, m and Ω such that

$$\|\boldsymbol{A}\|_{\mathbb{W}^{m+2,p}(\Omega)} \le C\|\boldsymbol{S}\|_{\mathbb{W}^{m,p}(\Omega)}. \tag{3.1.1}$$

Proof. The proof follows the lines of the proof of Theorem 2.2 in [31]. Let $\boldsymbol{S} \in \mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega)$ i.e., $\boldsymbol{S} \in \mathbb{V}_s(\Omega)$ and satisfying $\int_\Omega \boldsymbol{S} : \boldsymbol{M}\, dx = 0$ for all $\boldsymbol{M} \in \mathbb{K}_{T,s}(\Omega)$. That means that for any $1 \le i \le 3$ and any $1 \le j \le J$ (see [19])

$$\mathbf{div}\,\boldsymbol{S} = \mathbf{0} \quad \text{in} \quad \Omega, \tag{3.1.2}$$

$$\int_{\Sigma_j} (\boldsymbol{S}\cdot\boldsymbol{n})\cdot\boldsymbol{e}^i\, d\sigma = 0, \tag{3.1.3}$$

$$\int_{\Sigma_j} (\boldsymbol{S}\cdot\boldsymbol{n})\cdot\boldsymbol{P}^i\, d\sigma = 0. \tag{3.1.4}$$

Observe that conditions (3.1.2), (3.1.3) are equivalent to: for each $1 \le i \le 3$, $\boldsymbol{S}^i \in \boldsymbol{\mathcal{V}}(\Omega) \perp \boldsymbol{K}_T(\Omega)$ where $\boldsymbol{S}^i$ is the i-th line of matrix $\boldsymbol{S}$. Then, Theorem 2.1.4 implies that there exists some vector field $\boldsymbol{W}^i$ in $\boldsymbol{\mathcal{D}}(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{W}^i = \boldsymbol{S}^i$, and satisfying the estimate

$$\|\boldsymbol{W}^i\|_{\boldsymbol{W}^{m+1,p}(\Omega)} \le C\|\boldsymbol{S}^i\|_{\boldsymbol{W}^{m,p}(\Omega)}.$$

We define $\boldsymbol{W}$ the matrix field whose lines are the vectors $\boldsymbol{W}^i$. So $\boldsymbol{W}$ satisfies $\mathbf{curl}\,\boldsymbol{W} = \boldsymbol{S}^T$ in Ω and

$$\|\boldsymbol{W}\|_{\mathbb{W}^{m+1,p}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{W}^{m,p}(\Omega)}.$$

Now setting $\boldsymbol{B} = \boldsymbol{W}^T - \mathrm{tr}(\boldsymbol{W})\boldsymbol{I}$. The symmetry of $\boldsymbol{S}$ implies that

$$\mathbf{Div}\,\boldsymbol{B} = \mathbf{0} \quad \text{in} \quad \Omega. \tag{3.1.5}$$

Indeed, for $i = 1$ for example, we have

$$\begin{aligned} \mathbf{div}\,\boldsymbol{B}^1 &= \partial_2 W_{21} - \partial_1 W_{22} + \partial_3 W_{31} - \partial_1 W_{33} \\ &= 0. \end{aligned}$$

Moreover,

$$\int_{\Sigma_j} ((\mathbf{Curl}\,\boldsymbol{W})^T\boldsymbol{n})\cdot\boldsymbol{P}^i\,d\sigma = \int_{\Sigma_j} ((\mathbf{Curl}(\boldsymbol{P}\boldsymbol{W}))^T\boldsymbol{n})\cdot\boldsymbol{e}^i\,d\sigma + \int_{\Sigma_j} (\boldsymbol{W}^T\boldsymbol{n})\cdot\boldsymbol{e}^i\,d\sigma - \int_{\Sigma_j} (\mathrm{tr}(\boldsymbol{W})\mathrm{I}\,\boldsymbol{n})\cdot\boldsymbol{e}^i. \tag{3.1.6}$$

Because $\boldsymbol{P}\boldsymbol{W} \in \boldsymbol{D}(\Omega)$, we get

$$\int_{\Sigma_j} (\mathbf{Curl}(\boldsymbol{P}\boldsymbol{W}))^T\boldsymbol{n}) \cdot \boldsymbol{e}^i\,d\sigma = 0. \tag{3.1.7}$$

Hence (3.1.4), (3.1.6) and (3.1.7) imply that

$$\int_{\Sigma_j} (\boldsymbol{B}\boldsymbol{n}) \cdot \boldsymbol{e}^i\,d\sigma = 0. \tag{3.1.8}$$

By using (3.1.5), (3.1.8) and applying again Theorem 2.1.4, there exists a matrix field $\boldsymbol{D}$ in $\mathbb{D}(\Omega)$ such that

$$\mathbf{curl}\,\boldsymbol{D} = \boldsymbol{B}^T = \boldsymbol{W} - \mathrm{tr}(\boldsymbol{W})\mathrm{I}, \tag{3.1.9}$$

with

$$\|\boldsymbol{D}\|_{\mathbb{W}^{m+2,p}(\Omega)} \leq C\|\boldsymbol{B}\|_{\mathbb{W}^{m+1,p}(\Omega} \leq C\|\boldsymbol{S}\|_{\mathbb{W}^{m,p}(\Omega)}.$$

Therefore

$$\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{D} = \boldsymbol{S}^T - \mathbf{curl}(\mathrm{tr}(\boldsymbol{W})\boldsymbol{I}). \tag{3.1.10}$$

We also have

$$\begin{aligned} \textbf{Curl}\,\textbf{Curl}\,\boldsymbol{D}^T &= (\textbf{Curl}\,\textbf{Curl}\,\boldsymbol{D})^T \\ &= \boldsymbol{S} + \textbf{curl}(\mathrm{tr}(\boldsymbol{W})\boldsymbol{I}). \end{aligned} \tag{3.1.11}$$

Define $\boldsymbol{A} = \dfrac{\boldsymbol{D}+\boldsymbol{D}^T}{2}$, then (3.1.10) and (3.1.11) imply

$$\textbf{Curl}\,\textbf{Curl}\,\boldsymbol{A} = \frac{\boldsymbol{S}+\boldsymbol{S}^T}{2} = \boldsymbol{S}.$$

Which is the required result. □

P.G. Ciarlet et al in [19] stated the range of the operator $\textbf{Curl}\,\textbf{Curl} : \mathbb{H}^2_{0,s}(\Omega) \longrightarrow \mathbb{L}^2_s(\Omega)$ is the space $\mathbb{U}^{0,2}_s(\Omega)$. In the following, We will use the Beltrami's completeness, which had been proved in Theorem 3.1.1 to show that the operator $\textbf{Curl}\,\textbf{Curl} : \mathbb{W}^{m+2,p}_{0,s}(\Omega) \longrightarrow \mathbb{U}^{m,p}_s(\Omega)$ is onto, where m is a nonnegative integer. Using the same argument of the proof of Lemma 2.3.1, the following result holds:

Lemma 3.1.2. *Let m be a nonnegative integer. Then the space $\mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega)$ is dense in $\mathbb{U}^{m,p}_s(\Omega)$.*

Theorem 3.1.3. *Let m be a nonnegative integer. For any matrix $\boldsymbol{S}$ in $\mathbb{U}^{m,p}_s(\Omega)$, there exists $\boldsymbol{A} \in \mathbb{W}^{m+2,p}_{0,s}(\Omega)$ such that*

$$\textbf{Curl}\,\textbf{Curl}\,\boldsymbol{A} = \boldsymbol{S} \quad \textit{in} \quad \Omega \quad \textit{and} \quad \|\boldsymbol{A}\|_{\mathbb{W}^{m+2,p}_s(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{W}^{m,p}_s(\Omega)}.$$

Proof. Let $\boldsymbol{A} \in \mathbb{U}^{m,p}_s(\Omega)$. Since $\mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega)$ is dense in $\mathbb{U}^{m,p}_s(\Omega)$, there exists a sequence $(\boldsymbol{S}_k)$ of $\mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega)$ such that

$$\boldsymbol{S}_k \longrightarrow \boldsymbol{S} \quad \text{in} \quad \mathbb{W}^{m,p}_s(\Omega) \quad \text{when} \quad k \longrightarrow \infty.$$

From Lemma 3.1.1, for any $k \in \mathbb{N}$, there exists $\boldsymbol{A}_k \in \mathbb{D}_s(\Omega)$ such that

$$\textbf{Curl}\,\textbf{Curl}\,\boldsymbol{A}_k = \boldsymbol{S}_k \quad \text{with} \quad \|\boldsymbol{A}_k\|_{\mathbb{W}^{m+2,p}_s(\Omega)} \leq C\|\boldsymbol{S}_k\|_{\mathbb{W}^{m,p}_s(\Omega)}.$$

Clearly $(\boldsymbol{A}_k)$ is a Cauchy sequence and there exists $\boldsymbol{A} \in \mathbb{W}^{m+2,p}_{0,s}(\Omega)$ such that

$$\boldsymbol{A}_k \longrightarrow \boldsymbol{A} \quad \text{in} \quad \mathbb{W}^{m+2,p}_s(\Omega).$$

with

$$\mathbf{Curl\,Curl}\,\boldsymbol{A} = \boldsymbol{S} \quad \text{in} \quad \Omega \quad \text{and} \quad \|\boldsymbol{A}\|_{\mathbb{W}^{m+2,p}_s(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{W}^{m,p}_s(\Omega)}.$$

□

3.2 Beltrami-s completeness for symmetric matrix fields in $\mathbb{D}'_s(\Omega)$

In Chapter 2 we have used the surjectivity of the operator (2.1.2) to present te rotational version of De Rham's theorem. Here, we will use the extension of Beltrami's completeness has been stated in Theorem 3.1.1 to present the symmetric analogous of Theorem 2.3.4 which can be considered as an extension of Beltrami's completeness in $\mathbb{D}'_s(\Omega)$.

Theorem 3.2.1. *Let $\boldsymbol{E} \in \mathbb{D}'_s(\Omega)$ satisfies*

$$_{\mathbb{D}'(\Omega)}\langle \boldsymbol{S}, \boldsymbol{E}\rangle_{\mathbb{D}(\Omega)} = 0 \quad \textit{for} \quad \textit{all} \quad \boldsymbol{E} \in \mathbb{G}_s(\Omega). \tag{3.2.1}$$

Then there exists $\boldsymbol{A} \in \mathbb{D}'_s(\Omega)$ such that

$$\mathbf{Curl\,Curl}\,\boldsymbol{A} = \boldsymbol{S} \quad \textit{in} \quad \Omega.$$

Remark 3.2.2. The converse is obvious, because for any $\boldsymbol{S} \in \mathbb{D}'_s(\Omega)$ and any $\boldsymbol{E} \in \mathbb{G}_s(\Omega)$, we have

$$_{\mathbb{D}'(\Omega)}\langle \mathbf{Curl\,Curl}\,\boldsymbol{S}, \boldsymbol{E}\rangle_{\mathbb{D}(\Omega)} = \,_{\boldsymbol{D}'(\Omega)}\langle \boldsymbol{S}, \mathbf{Curl\,Curl}\,\boldsymbol{E}\rangle_{\mathbb{D}(\Omega)} = 0.$$

Proof. According to Theorem 3.1.1,

$$\mathbf{Curl\,Curl} \,:\, \mathbb{D}_s(\Omega)/\mathbb{G}_s(\Omega) \longrightarrow \mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega) \tag{3.2.2}$$

is one to one and onto. Then its adjoint

$$\mathbf{Curl\,Curl}\ :\ (\mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega))' \longrightarrow \mathbb{D}'_s(\Omega) \perp \mathbb{G}_s(\Omega)$$

is one to one and onto.

Let $\boldsymbol{L} \in (\mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega))'$ and $\widetilde{\boldsymbol{L}}$ any extension of $\boldsymbol{L}$ in $\mathbb{D}'_s(\Omega)$. Two expressions $\boldsymbol{S}$ and $\boldsymbol{A}$ of $\widetilde{\boldsymbol{L}}$ coincide on $\mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega)$ if and only if,

$$\forall \boldsymbol{E} \in \mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega),\ {}_{\mathbb{D}'(\Omega)}\langle \boldsymbol{S} - \boldsymbol{A}, \boldsymbol{E}\rangle_{\mathbb{D}(\Omega)} = 0.$$

Using again Lemma 3.1.1, we get

$$\forall \boldsymbol{B} \in \mathbb{D}_s(\Omega),\ {}_{\mathbb{D}'(\Omega)}\langle \boldsymbol{S} - \boldsymbol{A}, \mathbf{Curl\,Curl}\,\boldsymbol{B}\rangle_{\mathbb{D}(\Omega)} \quad = \quad {}_{\mathbb{D}'(\Omega)}\langle \mathbf{Curl\,Curl}\,(\boldsymbol{S} - \boldsymbol{A}), \boldsymbol{B}\rangle_{\mathbb{D}(\Omega)} = 0,$$

which means that $\boldsymbol{S} - \boldsymbol{A} \in \mathbf{Ker\,Curl\,Curl}$, where

$$\mathbf{Curl\,Curl}\ :\ \mathbb{D}'_s(\Omega) \longrightarrow \mathbb{D}'_s(\Omega).$$

Consequently,

$$(\mathbb{V}_s(\Omega) \perp \mathbb{K}_{T,s}(\Omega))' = \mathbb{D}'_s(\Omega)/\mathbf{Ker\,Curl\,Curl}. \tag{3.2.3}$$

Let $\boldsymbol{S} \in \mathbb{D}'_s(\Omega)$ satisfies (3.2.1). In other words, that means that $\boldsymbol{S} \in \mathbb{D}'_s(\Omega) \perp \mathbb{G}_s(\Omega)$, then the operator is an isomorphism, d the characterization (3.2.3) implies that there exists $\boldsymbol{A} \in \mathbb{D}'_s(\Omega)$ such that $\mathbf{Curl\,Curl}\,\boldsymbol{A} = \boldsymbol{S}$ in Ω. □

3.3 The general extension of Saint-Venant's theorem

Podio-Guidugli in [45] have used a Beltrami's completeness to show the equivalence between the sufficient conditions of Donati's and Saint-Venant's theorems: Let Ω be a smooth bounded and simply-connected open set of $\mathbb{R}^3$, then any symmetric matrix field $\boldsymbol{E} = (E_{ij})$ with $E_{ij} \in \mathcal{C}^N(\Omega)$ $(N \geq 2)$ satisfies

$$\mathbf{Curl\,Curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } \Omega,$$

if and only if

$$\int_\Omega \boldsymbol{E} : \boldsymbol{M}\, dx = 0 \quad \text{for any } \boldsymbol{M} \in \mathbb{V}_s(\Omega).$$

Later, Geymonat and Krasucki in [30] have proved the above equivalence when $\boldsymbol{E} \in \mathbb{L}^2_s(\Omega)$ and they have used it together with Ting's theorem to conclude an extension of Saint-Venant's theorem in $\mathbb{L}^2_s(\Omega)$. In the following, we will use the same idea to present an extension of Saint-Venant's theorem in $\mathbb{D}'_s(\Omega)$.

Theorem 3.3.1. *Let $\boldsymbol{E} \in \mathbb{D}'_s(\Omega)$ satisfies*

$$\mathbf{Curl\,Curl}\,\boldsymbol{E} = \mathbf{0} \quad \textit{in} \quad \Omega. \tag{3.3.1}$$

Then there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that

$$\boldsymbol{\nabla_s v} = \boldsymbol{E} \quad \textit{in} \quad \Omega.$$

Proof. Let $\boldsymbol{E}$ be a symmetric matrix field in $\mathbb{D}'_s(\Omega)$ such that $\mathbf{Curl\,Curl}\,\boldsymbol{E} = \mathbf{0}$ in Ω. We have already shown that for any symmetric matrix field $\boldsymbol{A}$ in $\mathbb{V}_s(\Omega)$, there exists $\boldsymbol{B} \in \mathbb{D}_s(\Omega)$ such that $\mathbf{Curl\,Curl}\,\boldsymbol{B} = \boldsymbol{A}$. Then, we have

$$_{\mathbb{D}'(\Omega)}\langle \boldsymbol{E},\, \boldsymbol{A}\rangle_{\mathbb{D}(\Omega)} = \,_{\mathbb{D}'(\Omega)}\langle \boldsymbol{E},\, \mathbf{Curl\,Curl}\,\boldsymbol{B}\rangle_{\mathbb{D}(\Omega)} = \,_{\mathbb{D}'(\Omega)}\langle \mathbf{Curl\,Curl}\,\boldsymbol{E},\, \boldsymbol{B}\rangle_{\mathbb{D}(\Omega)} = 0.$$

Thus, Theorem 1.0.2 (Moreau's theorem) implies that there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that $\boldsymbol{\nabla_s v} = \boldsymbol{E}$ in Ω, which is the required result. □

Chapter 4

Beltrami's completeness and representation for $\mathbb{L}^p_s$-symmetric matrix fields

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Gurtin [35] has shown the Beltrami's completeness for smooth matrix fields. He proved that for any self-equilibrated matrix field $\boldsymbol{S} = (S_{ij}) \in \mathcal{C}^1(\Omega)$, there exists symmetric matrix field $\boldsymbol{A} = (A_{ij}) \in \mathcal{C}^3(\Omega)$ such that $\operatorname{Curl}\operatorname{Curl}\boldsymbol{A} = \boldsymbol{S}$ in Ω when Ω is smooth. In 2006, Geymonat and Krasucki [31] have shown a new extension of Beltrami's completeness for matrix fields in $\mathbb{L}^2_s(\Omega)$ when Ω is only Lipschitz. In this chapter, we will show two extensions of Beltrami's completeness for matrix fields in $\mathbb{L}^p_s(\Omega)$. Then, , we will present some extensions of Beltrami's representation, also for matrix fields in $\mathbb{L}^p_s(\Omega)$. .

In this chapter, Ω is a bounded and connected open set of $\mathbb{R}^3$.

4.1 Tangential Beltrami's completeness

We know that any vector field $\boldsymbol{v}$ in $\boldsymbol{H}^p(\mathrm{div},\,\Omega)$ has a normal trace $\boldsymbol{v}\cdot\boldsymbol{n}$ in $W^{-\frac{1}{p},p}(\Gamma)$ (see [9]). Using the same arguments of proofs of Theorem 2.4 and Theorem 2.5 of [33], then the following analogous results for matrix fields in $\mathbb{H}^p_s(\mathbf{Div},\,\Omega)$ hold:

Proposition 4.1.1. *Assume that Ω is Lipschitz.*
i) The space $\mathbb{D}_s(\overline{\Omega})$ of restriction to Ω of functions of $\mathbb{D}_s(\mathbb{R}^3)$ is dense in $\mathbb{H}^p_s(\mathbf{Div},\Omega)$.
ii) The mapping $\boldsymbol{S}\longrightarrow\boldsymbol{Sn}$ defined on $\mathbb{D}_s(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted by the same way, from $\mathbb{H}^p_s(\mathbf{Div},\,\Omega)$ into $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ and the following Green's formula holds:

$$\forall\boldsymbol{v}\in\boldsymbol{W}^{1,p'}(\Omega),\quad \langle\boldsymbol{Sn},\boldsymbol{v}\rangle_\Gamma=\int_\Omega\boldsymbol{S}:\boldsymbol{\nabla_s v}\,dx+\int_\Omega\mathbf{Div}\,\boldsymbol{S}\cdot\boldsymbol{v}\,dx. \tag{4.1.1}$$

We denote by $\mathbb{H}^p_{0,s}(\mathbf{Div},\,\Omega)$ the closure of $\mathbb{D}_s(\Omega)$ in $\mathbb{H}^p_s(\mathbf{Div},\,\Omega)$ and $\mathbb{H}^p_{0,s}(\mathbf{Curl\,Curl},\,\Omega)$ the closure of $\mathbb{D}_s(\Omega)$ in $\mathbb{H}^p_s(\mathbf{Curl\,Curl},\,\Omega)$. Here, we give characterizations of the above spaces:

Proposition 4.1.2. *Assume that Ω is Lipschitz.*
i) If $\boldsymbol{S}$ belongs to $\mathbb{H}^p_s(\mathbf{Div},\,\Omega)$ and satisfies

$$\textit{for}\quad\textit{all}\ \boldsymbol{v}\in\boldsymbol{\mathcal{D}}(\overline{\Omega}),\ \int_\Omega\boldsymbol{S}:\boldsymbol{\nabla}_s\boldsymbol{v}\,dx+\int_\Omega\mathbf{Div}\,\boldsymbol{S}\cdot\boldsymbol{v}\ dx=0, \tag{4.1.2}$$

then $\boldsymbol{S}\in\mathbb{H}^p_{0,s}(\mathbf{Div},\,\Omega)$.
ii) If $\boldsymbol{S}$ belongs to $\mathbb{H}^p_s(\mathbf{Curl\,Curl},\,\Omega)$ and satisfies

$$\textit{for}\quad\textit{all}\ \boldsymbol{\Psi}\in\mathbb{D}(\overline{\Omega}),\ \int_\Omega\boldsymbol{S}:\mathbf{Curl\,Curl}\,\boldsymbol{\psi}\,dx-\int_\Omega\boldsymbol{\Psi}:\mathbf{Curl\,Curl}\,\boldsymbol{S}\,dx=0, \tag{4.1.3}$$

then $\boldsymbol{S}\in\mathbb{H}^p_{0,s}(\mathbf{Curl\,Curl},\,\Omega)$.

Proof. **i)** Let $\boldsymbol{S}$ be an element of $\mathbb{H}^p_s(\mathbf{Div},\,\Omega)$ and satisfying the relation (4.1.2). We denote $\widetilde{\boldsymbol{S}}$ the extension of $\boldsymbol{S}$ by zero outside Ω. The fact that $\boldsymbol{S}$ satisfies (4.1.2) implies that $\widetilde{\boldsymbol{S}}$ belongs to $\mathbb{H}^p_s(\mathbf{Div},\,\mathbb{R}^3)$, *i.e.* $\widetilde{\boldsymbol{S}} \in \mathbb{L}^p_s(\mathbb{R}^3)$ and $\mathbf{Div}\,\widetilde{\boldsymbol{S}} \in \boldsymbol{L}^p(\mathbb{R}^3)$. Indeed, for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$, we have

$$\begin{aligned} |_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^3)}\langle \mathbf{Div}\,\widetilde{\boldsymbol{S}},\,\boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}(\mathbb{R}^3)}| &= |\int_{\mathbb{R}^3} \widetilde{\boldsymbol{S}} : \boldsymbol{\nabla}_s\boldsymbol{\varphi}\,dx| = |\int_{\Omega} \boldsymbol{S} : \boldsymbol{\nabla}_s\boldsymbol{\varphi}\,dx| \\ &= |\int_{\Omega} \mathbf{Div}\,\boldsymbol{S}\cdot\boldsymbol{\varphi}\,dx| \\ &\leq \|\mathbf{Div}\,\boldsymbol{S}\|_{\boldsymbol{L}^p(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'}(\mathbb{R}^3)} \end{aligned}$$

Step 1. We suppose that Ω is starlike with respect to an open ball centered at the origin. We make the change of variable

$$\widetilde{\boldsymbol{S}}_\theta(x) = \widetilde{\boldsymbol{S}}(\frac{x}{\theta}), \quad \theta \in\,]0,1[.$$

The choice of $\theta \in\,]0,1[$ implies that $\widetilde{\boldsymbol{S}}_\theta$ has a compact support in Ω. It is clear that $\widetilde{\boldsymbol{S}}_\theta$ belongs to $\mathbb{H}^p_s(\mathbf{Div},\,\mathbb{R}^3)$ and

$$\lim_{\theta\longrightarrow 1} \widetilde{\boldsymbol{S}}_\theta = \widetilde{\boldsymbol{S}} \quad \text{in} \quad \mathbb{H}^p_s(\mathbf{Div},\,\mathbb{R}^3).$$

For $\varepsilon > 0$, let ρ_ε be a mollifiers that vanishes for $|x| > \varepsilon$. We define the matrix field $\rho_\varepsilon * \widetilde{\boldsymbol{S}}_\theta$ by $(\rho_\varepsilon * \widetilde{\boldsymbol{S}}_\theta)_{ij} = \rho_\varepsilon * \widetilde{(S_\theta)}_{ij}$. The choice of ε sufficiently small implies that $\rho_\varepsilon * \widetilde{\boldsymbol{S}}_\theta$ has a compact support in Ω, then $(\rho_\varepsilon * \widetilde{\boldsymbol{S}}_\theta)_{|\Omega}$ belongs to $\mathbb{D}_s(\Omega)$ and

$$\lim_{\varepsilon\longrightarrow 0}\lim_{\theta\longrightarrow 1}(\rho_\varepsilon * \widetilde{\boldsymbol{S}}_\theta)_{|\Omega} = \boldsymbol{S} \quad \text{in} \quad \mathbb{H}^p_s(\mathbf{Div},\,\Omega).$$

Step 2. We suppose that Ω is Lipschitz but not necessarily starlike with respect to an open ball. We denote $\{\Omega_i\}_{i=1}^{I_0}$ the finite set of starlike open sets that recover Ω and let $(\alpha_i)_i$ be a partition of unity subordinate to $\{\Omega_i\}_{i=1}^{I_0}$. We know that for any $1 \leq i \leq I_0$, there exists a sequence $(\boldsymbol{A}^i_k)_k$ of $\mathbb{D}_s(\Omega_i)$ that converges to $\alpha_i\boldsymbol{S}$ in $\mathbb{H}^p_s(\mathbf{Div},\,\Omega_i)$. For any $k \in \mathbb{N}$, we set $\boldsymbol{A}_k = \sum_{i=1}^{I_0} \boldsymbol{A}^i_k$. Observe that this sequence belongs to $\mathbb{D}_s(\Omega)$ and it converges to $\boldsymbol{S}$ in $\mathbb{H}^p_s(\mathbf{Div},\,\Omega)$, which ends the proof of Point i).

ii) We use the same argument of the proof of Point i). □

Remark 4.1.3. Due to Proposition 4.1.2 and Proposition 4.1.1, the space $\mathbb{H}^p_{0,s}(\mathbf{Div},\,\Omega)$ can be defined by

$$\mathbb{H}^p_{0,s}(\mathbf{Div},\,\Omega) = \{\boldsymbol{S} \in \mathbb{H}^p_s(\mathbf{Div},\,\Omega),\ \boldsymbol{S}\boldsymbol{n} = \boldsymbol{0} \ \text{ on } \Gamma\}.$$

The following lemma was proved in [4] for $p = 2$. The proof below is very close to that given in the Hilbertian case.

Lemma 4.1.4. *Assume that* Ω *is of class* $\mathcal{C}^{1,1}$ *and let* $\boldsymbol{S} \in \mathbb{H}^p_{0,s}(\mathbf{Div}, \Omega)$. *Then, the restriction* $\boldsymbol{Sn}$ *to any* Σ_j *belongs to* $[\boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j)]'$ *and the following Green's formula holds: for all* $\boldsymbol{v} \in \boldsymbol{W}^{1,p'}(\Omega^\circ)$,

$$\sum_{j=1}^{J} \langle \boldsymbol{Sn},\, [\boldsymbol{v}]_j \rangle_{\Sigma_j} = \int_{\Omega^\circ} \boldsymbol{S} : \boldsymbol{\nabla}_s \boldsymbol{v}\, dx + \int_{\Omega^\circ} \boldsymbol{v} \cdot \mathbf{Div}\, \boldsymbol{S}\, dx, \tag{4.1.4}$$

where

$$\boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j) = \{\boldsymbol{\mu} \in \boldsymbol{W}^{\frac{1}{p},p'}(\Sigma_j),\, \widetilde{\boldsymbol{\mu}} \in \boldsymbol{W}^{\frac{1}{p},p'}(\mathcal{M}_j)\}$$

and $\widetilde{\boldsymbol{\mu}}$ *is the extension of* $\boldsymbol{\mu}$ *by zero outside of* Σ_j.

Proof. Let $1 \leq j \leq J$, we extend the cut Σ_j by the cut Σ'_j, which allows us to divide Ω on two parts Ω_j and Ω'_j such that $\Omega = \Omega_j \cup \Omega'_j \cup \Sigma_j \cup \Sigma'_j$. Let $\boldsymbol{\mu} \in \boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j)$, we denote $\boldsymbol{\psi}_j$ the solution in $\boldsymbol{W}^{1,p'}(\Omega_j)$ of the problem

$$\mathbf{Div}\, \boldsymbol{\nabla}_s \boldsymbol{\psi}_j = \mathbf{0} \text{ in } \Omega_j, \quad \boldsymbol{\psi}_j = \mathbf{0} \text{ on } \partial\Omega_j \backslash \Sigma_j \quad \text{and} \quad \boldsymbol{\psi}_j = \frac{\boldsymbol{\mu}}{2} \text{ on } \Sigma_j, \tag{4.1.5}$$

and $\boldsymbol{\psi}'_j$ the solution in $\boldsymbol{W}^{1,p'}(\Omega'_j)$ of the problem

$$\mathbf{Div}\, \boldsymbol{\nabla}_s \boldsymbol{\psi}'_j = \mathbf{0} \text{ in } \Omega'_j, \quad \boldsymbol{\psi}'_j = \mathbf{0} \text{ on } \partial\Omega'_j \backslash \Sigma_j \quad \text{and} \quad \boldsymbol{\psi}'_j = -\frac{\boldsymbol{\mu}}{2} \text{ on } \Sigma_j. \tag{4.1.6}$$

We know that there exists a constant C_j depending only on p and Ω_j such that

$$\|\boldsymbol{\psi}_j\|_{\boldsymbol{W}^{1,p'}(\Omega_j)} \leq C_j \|\boldsymbol{\mu}\|_{\boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j)},$$

and there exists a constant C'_j depending only on p and Ω'_j such that

$$\|\boldsymbol{\psi}_j\|_{\boldsymbol{W}^{1,p'}(\Omega'_j)} \leq C_j \|\boldsymbol{\mu}\|_{\boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j)}.$$

We define the vector field $\boldsymbol{w}_j$ by

$$\boldsymbol{w}_j = \begin{cases} \boldsymbol{\psi}_j & \text{in} \quad \Omega_j, \\ \boldsymbol{\psi}'_j & \text{in} \quad \Omega'_j, \\ \mathbf{0} & \text{on} \quad \Sigma'_j. \end{cases}$$

Observe that $\boldsymbol{w}_j$ satisfies

$$\forall 1 \leq k \leq J, \qquad [\boldsymbol{w}_j]_{\Sigma_k} = \delta_{jk}\,\boldsymbol{\mu}, \quad \boldsymbol{w}_j \in \boldsymbol{W}^{1,p'}(\mathring{\Omega}_j),$$

where $\mathring{\Omega}_j = \Omega \backslash \Sigma_j$ and there exists a constant C depends only on p and Ω such that

$$\|\boldsymbol{w}_j\|_{\boldsymbol{W}^{1,p'}(\mathring{\Omega}_j)} \leq C_j \|\boldsymbol{\mu}\|_{\boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j)}.$$

Next, setting $\boldsymbol{\omega}_j = \boldsymbol{w}_j|_{\Omega^\circ}$, it satisfies

$$\boldsymbol{\omega}_j \in \boldsymbol{W}^{1,p'}(\Omega^\circ), \quad [\boldsymbol{\omega}]_{\Sigma_k} = \delta_{jk}\,\boldsymbol{\mu}, \quad 1 \leq k \leq J, \quad \boldsymbol{\omega}_j = \boldsymbol{0} \quad \text{on} \quad \Gamma$$

$$\text{and} \quad \|\boldsymbol{\omega}_j\|_{\boldsymbol{W}^{1,p'}(\Omega^\circ)} \leq C \|\boldsymbol{\mu}\|_{\boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j)}.$$

Now, let $\boldsymbol{A} \in \mathbb{D}_s(\Omega)$, the Green's formula gives

$$\langle \boldsymbol{A}\boldsymbol{n},\, \boldsymbol{\mu} \rangle_{\Sigma_j} = \int_{\Omega^\circ} \boldsymbol{A} : \boldsymbol{\nabla}_s \boldsymbol{\omega}_j \, dx + \int_{\Omega^\circ} \boldsymbol{\omega}_j \cdot \mathbf{Div}\, \boldsymbol{A} \, dx. \tag{4.1.7}$$

Moreover, we have

$$|\langle \boldsymbol{A}\boldsymbol{n},\, \boldsymbol{\mu} \rangle_{\Sigma_j}| \leq C \|\boldsymbol{A}\|_{\mathbb{H}^p_s(\mathbf{Div},\,\Omega)} \|\boldsymbol{\mu}\|_{\boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j)}.$$

Then, the linear mapping

$$\mathbb{D}_s(\Omega) \longrightarrow [\boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j)]'$$
$$\boldsymbol{A} \longrightarrow \boldsymbol{A}\boldsymbol{n}|_{\Sigma_j}$$

is continuous in $\mathbb{D}_s(\Omega)$ equipped with the norm of $\mathbb{H}^p_s(\mathbf{Div},\,\Omega)$. As $\mathbb{D}_s(\Omega)$ is dense in $\mathbb{H}^p_{0,s}(\mathbf{Div},\,\Omega)$, it can be extended to an unique linear and continuous mapping from $\mathbb{H}^p_{0,s}(\mathbf{Div},\,\Omega)$ into $[\boldsymbol{W}_{00}^{\frac{1}{p},p'}(\Sigma_j)]'$ and by using adequate partition of unity, the Green's formula (4.1.4) follows from (4.1.7). □

Remark 4.1.5. In the case of $p = 2$ and Ω is only Lipschitz, the elliptic problems (4.1.5) and (4.1.6) have solutions in $\boldsymbol{H}^1(\Omega)$ and then Lemma 4.1.4 still true.

Notation 4.1.1. *For any vector $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega^\circ)$, we denote by $[\boldsymbol{v}]_{\Sigma_j}$ the jump of $\boldsymbol{v}$ throught Σ_j.*

We define the Kernel space $\mathbb{K}^p_{T,s}(\Omega)$ by

$$\mathbb{K}^p_{T,s}(\Omega) = \{\boldsymbol{S} \in \mathbb{X}^p_{T,s}(\Omega),\, \mathbf{Div}\, \boldsymbol{S} = \mathbf{0} \quad \text{and} \quad \mathbf{Curl\, Curl}\, \boldsymbol{S} = \mathbf{0} \text{ in } \Omega\}.$$

The regularity of the elements of $\mathbb{K}^p_{T,s}(\Omega)$ depends on the regularity of the domain Ω as follows:

Lemma 4.1.6. *Let m be a positive integer. Assume that Ω is of class $\mathcal{C}^{m,1}$. Then, the space $\mathbb{K}^p_{T,s}(\Omega)$ embedded in the space $\mathbb{W}^{m,p}(\Omega)$.*

Proof. To simplify the proof, we consider $J = 1$ and $m = 1$. Let Σ and Σ' be two disjoint cuts. We define $\Omega_\Sigma = \Omega \setminus \Sigma$ and $\Omega_{\Sigma'} = \Omega \setminus \Sigma'$ which are simply-connected open sets and let $\boldsymbol{S} \in \mathbb{K}^p_{T,s}(\Omega)$. As $\mathbf{Curl\, Curl}\, \boldsymbol{S}|_{\Omega_\Sigma} = \mathbf{0}$ and $\mathbf{Curl\, Curl}\, \boldsymbol{S}|_{\Omega_{\Sigma'}} = \mathbf{0}$, then Theorem 3.3.1 together with Theorem 3.1 of [6] imply that there exist $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega_\Sigma)$ and $\boldsymbol{v}' \in \boldsymbol{W}^{1,p}(\Omega_{\Sigma'})$ such that

$$\boldsymbol{\nabla}_s \boldsymbol{v} = \boldsymbol{S} \quad \text{in} \quad \Omega_\Sigma \quad \text{and} \quad \boldsymbol{\nabla}_s \boldsymbol{v}' = \boldsymbol{S} \quad \text{in} \quad \Omega_{\Sigma'}.$$

As $\mathbf{Div}\, \boldsymbol{\nabla}_s \boldsymbol{v} = \mathbf{0}$ in Ω_Σ and $\mathbf{Div}\, \boldsymbol{\nabla}_s \boldsymbol{v}' = \mathbf{0}$ in $\Omega_{\Sigma'}$, then

$$\boldsymbol{v} \in \mathbb{W}^{2,p}_{loc}(\Omega_\Sigma) \quad \text{and} \quad \boldsymbol{v}' \in \mathbb{W}^{2,p}_{loc}(\Omega_{\Sigma'}). \tag{4.1.8}$$

The condition $\boldsymbol{Sn} = \mathbf{0}$ on Γ and the regularity $\mathcal{C}^{1,1}$ of Γ together with (4.1.8) imply that there exist two open neighborhoods $\mathcal{O}$ and $\mathcal{O}'$ of Σ and Σ' respectively such that

$$\mathcal{O} \cap \Sigma' = \emptyset \quad \text{and} \quad \mathcal{O}' \cap \Sigma = \emptyset. \tag{4.1.9}$$

Since $\boldsymbol{v}' \in \mathbb{W}^{2,p}(\mathcal{O})$ and $\boldsymbol{\nabla}_s \boldsymbol{v}' = \boldsymbol{\nabla}_s \boldsymbol{v}$ in $\mathcal{O} \setminus \Sigma$, we deduce that the jump on Σ of the traces of the matrix $\boldsymbol{\nabla}_s \boldsymbol{v}$ is equal to zero. As $\boldsymbol{\nabla}_s \boldsymbol{v} \in \mathbb{W}^{1,p}(\mathcal{O} \setminus \Sigma)$, we get $\boldsymbol{\nabla}_s \boldsymbol{v} \in \mathbb{W}^{1,p}(\mathcal{O})$ and then $\boldsymbol{v} \in \mathbb{W}^{2,p}(\Omega)$, which implies that $\boldsymbol{S} \in \mathbb{W}^{1,p}(\Omega)$. □

In the following, we will show that $\mathbb{K}^p_{T,s}(\Omega)$ is independent of p. In other words, if p and q are two real numbers such that $1 < p < \infty$ and $1 < q < \infty$, then $\mathbb{K}^p_{T,s}(\Omega) = \mathbb{K}^q_{T,s}(\Omega)$.

Proposition 4.1.7. *Assume that Ω is of class $\mathcal{C}^{1,1}$. Then for all $p \in\,]1, \infty[$, we have*

$$\mathbb{K}^p_{T,s}(\Omega) = \mathbb{K}^2_{T,s}(\Omega),$$

which means in particular that each vector field of $\mathbb{K}^2_{T,s}(\Omega)$ belongs to $\mathbb{W}^{1,p}(\Omega)$ for any $p > 1$.

Proof. **Step 1. We show that, for any $1 < p < \infty$, $\mathbb{K}^2_{T,s}(\Omega) \subset \mathbb{K}^p_{T,s}(\Omega)$.**
Lemma 4.1.6 implies that $\mathbb{K}^2_{T,s}(\Omega)$ embedded in $\mathbb{H}^1(\Omega)$, so the Sobolev embedding implies that $\mathbb{K}^2_{T,s}(\Omega) \subset \mathbb{K}^6_{T,s}(\Omega)$. Again, $\mathbb{K}^6_{T,s}(\Omega)$ embedded in $\mathbb{W}^{1,6}(\Omega)$, which embedded in $\mathbb{L}^\infty(\Omega)$. So $\mathbb{K}^2_{T,s}(\Omega) \subset \mathbb{L}^\infty(\Omega)$ and consequently the required inclusion.
Step 2. We show that, for any $1 < p < \infty$, $\mathbb{K}^p_{T,s}(\Omega) \subset \mathbb{K}^2_{T,s}(\Omega)$.
Using again Lemma 4.1.6, we know that $\mathbb{K}^p_{T,s}(\Omega)$ embedded in $\mathbb{W}^{1,p}(\Omega)$. Let $p > \frac{6}{5}$. As $\mathbb{W}^{1,p}(\Omega) \hookrightarrow \mathbb{L}^2(\Omega)$, then we have $\mathbb{K}^p_{T,s}(\Omega) \subset \mathbb{K}^2_{T,s}(\Omega)$. Now, if $1 < p \leq \frac{6}{5}$, we have $\mathbb{W}^{1,p}(\Omega) \hookrightarrow \mathbb{L}^{\frac{3}{2}}(\Omega)$. So $\mathbb{K}^p_{T,s}(\Omega) \subset \mathbb{K}^{\frac{3}{2}}_{T,s}(\Omega) \subset \mathbb{K}^2_{T,s}(\Omega)$. □

Remark 4.1.8. Because the above identity, we will use the notation $\mathbb{K}_{T,s}(\Omega)$ instead of $\mathbb{K}^p_{T,s}(\Omega)$ in the rest of the paper.

Notation 4.1.2. *For any vector field $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega^\circ)$, $\boldsymbol{\nabla}_s \boldsymbol{v}$ belongs to $\mathbb{L}^2_s(\Omega^\circ)$ and it can be extended to $\mathbb{L}^2_s(\Omega)$, we denote it $\widetilde{\boldsymbol{\nabla}_s \boldsymbol{v}}$.*

P.G Ciarlet et al [19] have shown that the space $\mathbb{K}_{T,s}(\Omega)$ is of finite dimension and its dimension is equal to 6J. Furthermore, they have characterized the basis of $\mathbb{K}_{T,s}(\Omega)$. They have shown that $\mathbb{K}_{T,s}(\Omega)$ is spanned by the matrix fields $\widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}_i^j}$ and $\widetilde{\boldsymbol{\nabla}_s \boldsymbol{r}_i^j}$, $1 \leq i \leq 3$, $1 \leq j \leq J$, where $\boldsymbol{u}_i^j$ and $\boldsymbol{r}_i^j$ are the solutions belonging to the space

$$\boldsymbol{\mathcal{V}}^{\boldsymbol{\Sigma}}_{\boldsymbol{D}} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega^\circ),\ [\boldsymbol{v}]_{\Sigma_j} = \sum_{i=1}^{3} (a_i^j(\boldsymbol{v})\boldsymbol{e}^i + b_i^j(\boldsymbol{v})\boldsymbol{P}^i),\ 1 \leq j \leq J\}$$

of the variational problems

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}^{\boldsymbol{\Sigma}}_{\boldsymbol{D}}, \quad \int_{\Omega^\circ} \boldsymbol{\nabla}_s \boldsymbol{u}_i^j : \boldsymbol{\nabla}_s \boldsymbol{v}\, dx = a_i^j(\boldsymbol{v}), \tag{4.1.10}$$

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}^{\boldsymbol{\Sigma}}_{\boldsymbol{D}}, \quad \int_{\Omega^\circ} \boldsymbol{\nabla}_s \boldsymbol{r}_i^j : \boldsymbol{\nabla}_s \boldsymbol{v}\, dx = b_i^j(\boldsymbol{v}). \tag{4.1.11}$$

In the following, we will show more properties of the vector fields $\boldsymbol{u}_i^j$ and $\boldsymbol{r}_i^j$.

Theorem 4.1.9. *Assume that Ω is Lipschitz. For $1 \leq i \leq 3$ and $1 \leq j \leq J$, the vector field*

$\boldsymbol{u}_i^j$ is the unique solution in $\boldsymbol{H}^1(\Omega^\circ)$, up to an additive rigid displacement, of the problem

$$\begin{cases} \mathbf{Div}\, \boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j = \mathbf{0} \quad \textit{in} \quad \Omega^\circ, \\ (\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j)\, \boldsymbol{n} = \mathbf{0} \quad \textit{on} \quad \Gamma, \\ \left[\boldsymbol{u}_j^i\right]_k = \boldsymbol{rig} \quad \textit{and} \quad \left[(\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j)\, \boldsymbol{n}\right]_k = \mathbf{0},\, 1 \le k \le J, \\ \langle (\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j)\, \boldsymbol{n},\, \boldsymbol{e}^\ell \rangle_{\Sigma_k} = \delta_{i\ell}\delta_{jk}, \quad 1 \le \ell \le 3 \quad \textit{and} \quad 1 \le k \le J, \\ \langle (\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j)\, \boldsymbol{n},\, \boldsymbol{P}^\ell \rangle_{\Sigma_k} = 0, \quad 1 \le \ell \le 3 \quad \textit{and} \quad 1 \le k \le J, \end{cases} \tag{4.1.12}$$

and $\boldsymbol{r}_i^j$ is the solution in $\boldsymbol{H}^1(\Omega^\circ)$, up to an additive rigid displacement, of the problem

$$\begin{cases} \mathbf{Div}\, \boldsymbol{\nabla}_s\, \boldsymbol{r}_i^j = \mathbf{0} \quad \textit{in} \quad \Omega^\circ, \\ (\boldsymbol{\nabla}_s\, \boldsymbol{r}_i^j)\, \boldsymbol{n} = \mathbf{0} \quad \textit{on} \quad \Gamma, \\ \left[\boldsymbol{r}_j^i\right]_k = \boldsymbol{rig} \quad \textit{and} \quad \left[(\boldsymbol{\nabla}_s\, \boldsymbol{r}_i^j)\, \boldsymbol{n}\right]_k = \mathbf{0},\, 1 \le k \le J, \\ \langle (\boldsymbol{\nabla}_s\, \boldsymbol{r}_i^j)\, \boldsymbol{n},\, \boldsymbol{e}^\ell \rangle_{\Sigma_k} = 0, \quad 1 \le \ell \le 3 \quad \textit{and} \quad 1 \le k \le J, \\ \langle (\boldsymbol{\nabla}_s\, \boldsymbol{r}_i^j)\, \boldsymbol{n},\, \boldsymbol{P}^\ell \rangle_{\Sigma_k} = \delta_{i\ell}\delta_{jk}, \quad 1 \le \ell \le 3 \quad \textit{and} \quad 1 \le k \le J, \end{cases} \tag{4.1.13}$$

where the notation $\boldsymbol{rig}$ means "rigid displacement".

Proof. We follow the same steps as in the proof of Proposition 3.14 of [3]. Let $1 \le i \le 3$ and $1 \le j \le J$, we will show that the solution $\boldsymbol{u}_i^j$ of (4.1.10) solves the problem (4.1.12). Note that it suffices to use the same argument to show that the solution $\boldsymbol{r}_i^j$ of (4.1.11) solves the problem (4.1.13).

Let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega)$, using the variational formulation (4.1.10), we obtain

$$\langle \mathbf{Div}\, (\widetilde{\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j}),\, \boldsymbol{v} \rangle_\Omega = -\int_\Omega \widetilde{\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j} : \boldsymbol{\nabla}_s\, \boldsymbol{v}\, dx = -\int_{\Omega^\circ} \boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j : \boldsymbol{\nabla}_s\, \boldsymbol{v}\, dx = 0.$$

Then $\widetilde{\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j}$ belongs to $\mathbb{H}_s^p(\mathbf{Div},\, \Omega)$ and $\mathbf{Div}\, (\widetilde{\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j}) = \mathbf{0}$ in Ω. Using Green's formula with $\boldsymbol{v}$ in $\boldsymbol{H}_0^1(\Omega)$, we conclude that the jump of $(\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j)\, \boldsymbol{n}$ across any cut Σ_k, $1 \le k \le J$ is zero. Also, by applying (4.1.10) with $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, we obtain

$$0 = \int_{\Omega^\circ} (\mathbf{Div}\, \boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j) \cdot \boldsymbol{v}\, dx = -\int_{\Omega^\circ} \boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j : \boldsymbol{\nabla}_s\, \boldsymbol{v}\, dx + \langle (\boldsymbol{\nabla}_s\, \boldsymbol{u}_i^j)\, \boldsymbol{n},\, \boldsymbol{v} \rangle_\Gamma,$$

then $(\nabla_s \boldsymbol{u}_i^j)\,\boldsymbol{n} = \boldsymbol{0}$ on Γ, which implies that $\widetilde{\nabla_s \boldsymbol{u}_i^j}$ belongs to $\mathbb{H}^p_{0,s}(\mathbf{Div},\,\Omega)$. From Lemma 4.1.4 we deduce that the restriction of $(\nabla_s \boldsymbol{u}_i^j)\,\boldsymbol{n}$ to any cut Σ_k belongs to $[\boldsymbol{H}^{1/2}_{00}(\Sigma_j)]'$. Finally, to show the two last equalities of (4.1.12), we choose $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}^{\boldsymbol{\Sigma}}_{\boldsymbol{D}}$. Applying Green's formula (4.1.4), we obtain

$$\sum_k \langle (\nabla_s \boldsymbol{u}_i^j)\,\boldsymbol{n},\, [\boldsymbol{v}]_k \rangle_{\Sigma_k} = \int_{\Omega^\circ} \nabla_s \boldsymbol{u}_i^j : \nabla_s \boldsymbol{s}_i^j \, dx = a_i^j(\boldsymbol{v}).$$

In particular, if for any k the jump $[\boldsymbol{v}]_k$ is constant, we have

$$\sum_k a_\ell^k(\boldsymbol{v}) \langle (\nabla_s \boldsymbol{u}_i^j)\,\boldsymbol{n},\, \boldsymbol{e}^\ell \rangle_{\Sigma_k} = a_i^j(\boldsymbol{v})$$

and then for any $1 \le \ell \le 3$ and any $1 \le k \le J$,

$$\langle (\nabla_s \boldsymbol{u}_i^j)\,\boldsymbol{n},\, \boldsymbol{e}^\ell \rangle_{\Sigma_k} = \delta_{i\ell}\delta_{jk}.$$

To finish, we deduce the last relation in (4.1.12) by choosing the jump $[\boldsymbol{v}]_k = \sum_{i=1}^{3} b_i^k(\boldsymbol{v})\boldsymbol{P}^i$, for any $1 \le k \le J$. $\square$

Now, we introduce our first extension of Beltrami's completeness for matrix fields in $\mathbb{L}^p_s(\Omega)$. Note that the case $p = 2$ has been shown by Geymonat and Krasucki in [31] and in [32].

Theorem 4.1.10. *Assume that Ω is of class $\mathcal{C}^{1,1}$. A matrix $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$ satisfies*

$$\mathbf{Div}\,\boldsymbol{S} = \boldsymbol{0} \quad in \quad \Omega, \tag{4.1.14}$$

$$\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{e}^i \rangle_{\Gamma_k} = 0, \quad 1 \le i \le 3 \quad \text{and} \quad 0 \le k \le I, \tag{4.1.15}$$

$$\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{P}^i \rangle_{\Gamma_k} = 0,\; 1 \le i \le 3 \quad \text{and} \quad 0 \le k \le I, \tag{4.1.16}$$

if and only if there exists a matrix $\boldsymbol{A} \in \mathbb{W}^{2,p}_s(\Omega)$ such that

$$\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} = \boldsymbol{S} \quad in \quad \Omega. \tag{4.1.17}$$

Moreover, there exists a positive constant C which depends only on p and Ω such that

$$\|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} \le C \|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \tag{4.1.18}$$

To show Theorem 4.1.10, we need the following vector potential theorem which has been shown by Amrouche et al in [9]:

Theorem 4.1.11. *i) Assume that Ω is of class $\mathcal{C}^{1,1}$. A vector field $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ satisfies*

$$\operatorname{div} \boldsymbol{v} = 0 \quad in \quad \Omega, \tag{4.1.19}$$

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_k} = 0, \quad 0 \leq k \leq I, \tag{4.1.20}$$

if and only if there exists a vector field $\boldsymbol{\psi} \in \boldsymbol{W}^{1,p}(\Omega)$ such that

$$\operatorname{div} \boldsymbol{\psi} = 0 \quad and \quad \mathbf{curl}\, \boldsymbol{\psi} = \boldsymbol{v} \quad in \quad \Omega,$$

and there exists a positive constant C_1 which depends only on p and Ω such that

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C_1 \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.1.21}$$

ii) If $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$, then $\boldsymbol{\psi} \in \boldsymbol{W}^{2,p}(\Omega)$. Furthermore, there exists a positive constant C_2 which depends only on p and Ω such that

$$\|\boldsymbol{\psi}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C_2 \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}. \tag{4.1.22}$$

Proof of Theorem 4.1.10. We follow the steps of the proof of Theorem 2.2 in [31]. Let $\boldsymbol{A} \in \mathbb{W}^{2,p}_s(\Omega)$ and $\boldsymbol{S} = \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}$. We know that $\mathbf{Div}\,\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} = \mathbf{0}$, then (4.1.14) holds. Now, we show that $\boldsymbol{S}$ satisfies (4.1.15) and (4.1.16). Let $\chi_k \in \mathcal{C}^\infty(\overline{\Omega})$, such that χ_k equals to 1 in the neighbourhood of Γ_k and equals to 0 in the neighbourhood of $\Gamma_{k'}$ if $0 \leq k' \leq I$ and $k \neq k'$. Then, using Proposition 4.1.1, we get

$$\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{e}^i \rangle_{\Gamma_k} = \langle \mathbf{Curl}\,\mathbf{Curl}\,(\chi_k \boldsymbol{A})\boldsymbol{n}, \boldsymbol{e}^i \rangle_{\Gamma} = \int_\Omega \mathbf{Div}\,(\mathbf{Curl}\,\mathbf{Curl}\,(\chi_k \boldsymbol{A}))\boldsymbol{e}^i \, dx = 0,$$

$$\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{P}^i \rangle_{\Gamma_k} = \langle \mathbf{Curl}\,\mathbf{Curl}\,(\chi_k \boldsymbol{A})\boldsymbol{n}, \boldsymbol{P}^i \rangle_{\Gamma} = \int_\Omega \mathbf{Div}\,(\mathbf{Curl}\,\mathbf{Curl}\,(\chi_k \boldsymbol{A}))\boldsymbol{P}^i \, dx = 0.$$

Conversely, let $\boldsymbol{S} \in \mathbb{H}^p_s(\mathbf{Div}\,\Omega)$ and satisfies the conditions (4.1.14)-(4.1.16). For $1 \leq i \leq 3$, we set $\boldsymbol{S}^i$ the i^{th} line of $\boldsymbol{S}$. The conditions (4.1.14) and (4.1.15) imply that $\boldsymbol{S}^i$ satisfies the compatibility conditions (4.1.19) and (4.1.20) of Theorem 4.1.11. Then, there exists a vector field $\boldsymbol{B}^i \in \boldsymbol{W}^{1,p}(\Omega)$ such that $\operatorname{div} \boldsymbol{B}^i = 0$ and $\mathbf{curl}\, \boldsymbol{B}^i = \boldsymbol{S}^i$ in Ω. We set $\boldsymbol{B}$ the matrix whose lines are the vector $\boldsymbol{B}^i$. Then the matrix $\boldsymbol{B}$ satisfies $\mathbf{Curl}\, \boldsymbol{B} = \boldsymbol{S}^T = \boldsymbol{S}$ and by applying the estimate (4.1.21) on the vector lines of $\boldsymbol{B}$, we obtain

$$\|\boldsymbol{B}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C_1 \|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \tag{4.1.23}$$

Now, we define the matrix $\boldsymbol{C} = \boldsymbol{B}^T - \operatorname{tr}(\boldsymbol{B})\, \boldsymbol{I}$. Since $\boldsymbol{S}$ is symmetric and $\mathbf{Curl}\, \boldsymbol{B} = \boldsymbol{S}$, then $\mathbf{Div}\, \boldsymbol{C} = \mathbf{0}$ in Ω, which implies that the vector lines $(\boldsymbol{C}^i)$ of $\boldsymbol{C}$ satisfy the compatibility condition (4.1.19). Moreover, the identity

$$\langle (\mathbf{Curl}\, \boldsymbol{B})^T \boldsymbol{n}, \boldsymbol{P}^i \rangle_{\Gamma_k} = \langle (\mathbf{Curl}(\boldsymbol{P}\, \boldsymbol{B})^t \boldsymbol{n}, \boldsymbol{e}^i \rangle_{\Gamma_k} + \langle \boldsymbol{B}^T \boldsymbol{n}, \boldsymbol{e}^i \rangle_{\Gamma_k} - \langle (\operatorname{tr}(\boldsymbol{B})\, \boldsymbol{I}) \boldsymbol{n}, \boldsymbol{e}^i \rangle_{\Gamma_k}$$

together with the condition (4.1.16) imply that for any $1 \leq i \leq 3$, the vector $\boldsymbol{C}^i$ satisfies the compatibility condition (4.1.20). Let us apply once again Theorem 4.1.11 on the vector $\boldsymbol{C}^i$, then there exists $\boldsymbol{D}^i \in \boldsymbol{W}^{2,p}(\Omega)$ such that $\mathbf{Curl}\, \boldsymbol{D}^i = \boldsymbol{C}^i$ in Ω. The matrix $\boldsymbol{D}$, whose lines are the vectors $(\boldsymbol{D}^i)$, satisfies

$$\mathbf{Curl}\, \boldsymbol{D} = \boldsymbol{C}^T = \boldsymbol{B} - \operatorname{tr}(\boldsymbol{B})\boldsymbol{I}.$$

So

$$\mathbf{Curl}\, \mathbf{Curl}\, \boldsymbol{D} = \boldsymbol{S} - \mathbf{Curl}(\operatorname{tr}(\boldsymbol{B})\boldsymbol{I}). \tag{4.1.24}$$

We have also

$$\mathbf{Curl}\, \mathbf{Curl}\, \boldsymbol{D}^T = \boldsymbol{S} + \mathbf{Curl}(\operatorname{tr}(\boldsymbol{B})\boldsymbol{I}). \tag{4.1.25}$$

Setting $\boldsymbol{A} = \dfrac{\boldsymbol{D} + \boldsymbol{D}^T}{2}$, then (4.1.24) and (4.1.25) imply that

$$\mathbf{Curl}\, \mathbf{Curl}\, \boldsymbol{A} = \boldsymbol{S} \quad \text{in} \quad \Omega.$$

By applying the estimate (4.1.22) on the vector lines of $\boldsymbol{D}$, we obtain

$$\|\boldsymbol{D}\|_{\mathbb{W}^{2,p}(\Omega)} \leq 2C_2 \|\boldsymbol{B}\|_{\mathbb{W}^{1,p}(\Omega)}. \tag{4.1.26}$$

Then (4.1.23) and (4.1.26) imply that the estimate (4.1.18) holds. □

Remark 4.1.12. If $p = 2$, Theorem 4.1.11 still true when Ω is only Lipschitz, then Theorem 4.1.10 still also true in this case.

Now, we show the continuous embedding of $\mathbb{X}^p_{T,s}(\Omega)$ in $\mathbb{W}^{1,p}(\Omega)$ if the domain Ω is of class $\mathcal{C}^{1,1}$ and the continuous embedding of $\mathbb{Y}^p_{T,s}(\Omega)$ in $\mathbb{W}^{2,p}(\Omega)$ if the domain Ω is of class $\mathcal{C}^{2,1}$.

Theorem 4.1.13. *i) Assume that Ω is of class $\mathcal{C}^{1,1}$. Then, the space $\mathbb{X}^p_{T,s}(\Omega)$ is continuously embedded in $\mathbb{W}^{1,p}(\Omega)$.*
ii) Moreover, if Ω is of class $\mathcal{C}^{2,1}$. Then, the space $\mathbb{Y}^p_{T,s}(\Omega)$ is continuously embedded in $\mathbb{W}^{2,p}(\Omega)$.

Proof. **i)** Assume that Ω is of class $\mathcal{C}^{1,1}$. Let $\boldsymbol{S} \in \mathbb{X}^p_{T,s}(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ the solution of the problem

$$\begin{cases} \mathbf{Div}(\boldsymbol{\nabla}_s \boldsymbol{v}) &= \mathbf{Div}\, \boldsymbol{S} \quad \text{in} \quad \Omega, \\ (\boldsymbol{\nabla}_s \boldsymbol{v})\, \boldsymbol{n} &= \mathbf{0} \quad \text{on} \quad \Gamma, \end{cases}$$

with the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\|\mathbf{Div}\, \boldsymbol{S}\|_{\boldsymbol{L}^p(\Omega)}, \tag{4.1.27}$$

where C is a positive constant which depends only on p and Ω. Setting $\boldsymbol{A} = \boldsymbol{S} - \boldsymbol{\nabla}_s \boldsymbol{v}$, we have $\boldsymbol{A} \in \mathbb{L}^p_s(\Omega)$, $\mathbf{Div}\, \boldsymbol{A} = \mathbf{0}$ in Ω, $\mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{A} \in \mathbb{L}^p_s(\Omega)$ and $\boldsymbol{A}\boldsymbol{n} = \mathbf{0}$ on Γ. The matrix $\boldsymbol{B} = \mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{A}$ belongs to $\mathbb{H}^p_s(\mathbf{Div}, \Omega)$ and satisfies the compatibility conditions (4.1.14)-(4.1.16). Then, Theorem 4.1.10 implies that there exists $\boldsymbol{D} \in \mathbb{W}^{2,p}_s(\Omega)$ such that $\mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{D} = \boldsymbol{B}$ in Ω and satisfies the estimate

$$\|\boldsymbol{D}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C\|\boldsymbol{B}\|_{\mathbb{L}^p(\Omega)}. \tag{4.1.28}$$

Now, let $\boldsymbol{u}$ be the solution of the problem

$$\begin{cases} \mathbf{Div}(\boldsymbol{\nabla}_s \boldsymbol{u}) &= \mathbf{Div}\, \boldsymbol{D} \quad \text{in} \quad \Omega, \\ (\boldsymbol{\nabla}_s \boldsymbol{v})\, \boldsymbol{n} &= \boldsymbol{D}\boldsymbol{n} \quad \text{on} \quad \Gamma. \end{cases}$$

Since Ω is of class $\mathcal{C}^{1,1}$, $\mathbf{Div}\, \boldsymbol{D} \in \boldsymbol{L}^p(\Omega)$ and $\boldsymbol{D}\boldsymbol{n} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, then $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{2,p}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\left(\|\mathbf{Div}\, \boldsymbol{D}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{D}\boldsymbol{n}\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\right) \leq C\|\mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{A}\|_{\mathbb{L}^p(\Omega)}. \tag{4.1.29}$$

Next, the symmetric matrix $\boldsymbol{E} = \boldsymbol{D} - \boldsymbol{\nabla}_s \boldsymbol{u}$ belongs to $\mathbb{W}^{1,p}(\Omega)$ and satisfies $\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E} = \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}$, $\mathbf{Div}\,\boldsymbol{E} = \mathbf{0}$ in Ω and $\boldsymbol{E}\boldsymbol{n} = \mathbf{0}$ on Γ. Moreover, from (4.1.28) and (4.1.29), we get

$$\|\boldsymbol{E}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}\|_{\mathbb{L}^p(\Omega)}. \tag{4.1.30}$$

Finally, the matrix $\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{E}$ belongs to $\mathbb{L}^p_s(\Omega)$ and satisfies

$$\mathbf{Div}\,\boldsymbol{F} = \mathbf{0}, \quad \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{F} = \mathbf{0} \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{F}\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma.$$

Observe that $\boldsymbol{F} \in \mathbb{K}_{T,s}(\Omega)$. Since Ω is of class $\mathcal{C}^{1,1}$, Lemma 4.1.6 implies that $\boldsymbol{F}$ belongs to $\mathbb{W}^{1,p}(\Omega)$ and since $\mathbb{K}_{T,s}(\Omega)$ is of finite dimension, there exists a constant C depending only on Ω such that

$$\|\boldsymbol{F}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{F}\|_{\mathbb{L}^p(\Omega)}.$$

Then, we obtain the estimate

$$\|\boldsymbol{F}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{X}^p_s(\Omega)}. \tag{4.1.31}$$

Knowing that $\boldsymbol{S} = \boldsymbol{E} + \boldsymbol{F} + \boldsymbol{\nabla}_s \boldsymbol{v}$, then $\boldsymbol{S}$ belongs to $\mathbb{W}^{1,p}(\Omega)$. Furthermore, the estimates (4.1.27), (4.1.30) and (4.1.31) imply that there exists a constant C_1 depending only on p and Ω such that

$$\|\boldsymbol{S}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C_1\|\boldsymbol{S}\|_{\mathbb{X}^p_s(\Omega)}.$$

ii) The proof of the continuously embedding of $\mathbb{Y}^p_{T,s}(\Omega)$ in $\mathbb{W}^{2,p}_s(\Omega)$ is similar to the previous one. Let $\boldsymbol{S} \in \mathbb{Y}^p_{T,s}(\Omega)$. We define $\boldsymbol{v}, \boldsymbol{A}, \boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{u}$, $\boldsymbol{E}$ and $\boldsymbol{F}$ like the proof of Point i). The fact that Ω is of class $\mathcal{C}^{2,1}$ implies that $\boldsymbol{v} \in \boldsymbol{W}^{3,p}(\Omega)$. Moreover, since $\mathbf{Div}\,\boldsymbol{D} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{D}\boldsymbol{n} \in \boldsymbol{W}^{2-\frac{1}{p},p}(\Gamma)$, then $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{3,p}(\Omega)$. So $\boldsymbol{E} \in \mathbb{W}^{2,p}(\Omega)$ and Lemma 4.1.6 imply that $\boldsymbol{F} \in \mathbb{W}^{2,p}(\Omega)$. Consequently, $\boldsymbol{S} \in \mathbb{W}^{2,p}(\Omega)$. Furthermore, there exists a positive constant C_2 which depends only on Ω and p such that

$$\|\boldsymbol{S}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C_2\|\boldsymbol{S}\|_{\mathbb{Y}^p_s(\Omega)}.$$

□

Using Theorem 4.1.13, and the fact that the embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is compact, then the following result holds true.

Lemma 4.1.14. *Assume that Ω is of class $\mathcal{C}^{1,1}$. We affirm that the embedding of $\mathbb{X}^p_{T,s}(\Omega)$ in $\mathbb{L}^p_s(\Omega)$ is compact.*

Lemma 4.1.14, together with Peetre-Tartar theorem, allow us to prove the following corollary.

Corollary 4.1.15. *Assume that Ω is of class $\mathcal{C}^{1,1}$. On the space $\mathbb{X}^p_{T,s}(\Omega)$, the semi-norm*

$$\boldsymbol{S} \mapsto \|\mathbf{Div}\,\boldsymbol{S}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \sum_{i=1}^{3}\sum_{j=1}^{J}\left(|\langle \boldsymbol{Sn}, \boldsymbol{e}^i\rangle_{\Sigma_j}| + |\langle \boldsymbol{Sn}, \boldsymbol{P}^i\rangle_{\Sigma_j}|\right) \tag{4.1.32}$$

is a norm equivalent to the norm $\|\cdot\|_{\mathbb{W}^{1,p}(\Omega)}$. In particular, we have the following Friedrich's inequality type for every matrix $\boldsymbol{S} \in \mathbb{X}^p_{T,s}(\Omega)$:

$$\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} \leq C(\|\mathbf{Div}\,\boldsymbol{S}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \sum_{i=1}^{3}\sum_{j=1}^{J}(|\langle \boldsymbol{Sn}, \boldsymbol{e}^i\rangle_{\Sigma_j}| + |\langle \boldsymbol{Sn}, \boldsymbol{P}^i\rangle_{\Sigma_j}|)). \tag{4.1.33}$$

Moreover, if Ω is of class $\mathcal{C}^{2,1}$, then the semi-norm

$$\boldsymbol{S} \mapsto \|\mathbf{Div}\,\boldsymbol{S}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \sum_{i=1}^{3}\sum_{j=1}^{J}\left(|\langle \boldsymbol{Sn}, \boldsymbol{e}^i\rangle_{\Sigma_j}| + |\langle \boldsymbol{Sn}, \boldsymbol{P}^i\rangle_{\Sigma_j}|\right) \tag{4.1.34}$$

is a norm equivalent on $\mathbb{Y}^p_{T,s}(\Omega)$ to the norm $\|\cdot\|_{\mathbb{W}^{2,p}(\Omega)}$.

Now, we introduce the second extension of Beltrami's completeness, with tangential boundary conditions.

Theorem 4.1.16. *Assume that Ω is of class $\mathcal{C}^{1,1}$. A matrix $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$ satisfies (4.1.14)-(4.1.16), if and only if there exists a matrix $\boldsymbol{A} \in \mathbb{X}^p_s(\Omega)$ such that*

$$\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} = \boldsymbol{S} \quad \textit{and} \quad \mathbf{Div}\,\boldsymbol{A} = \mathbf{0} \quad \textit{in} \quad \Omega, \tag{4.1.35}$$

$$\boldsymbol{An} = \mathbf{0} \quad \textit{on} \quad \Gamma, \tag{4.1.36}$$

$$\langle \boldsymbol{An}, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \boldsymbol{An}, \boldsymbol{P}^i\rangle_{\Sigma_j} = 0, \quad 1 \leq i \leq 3, \quad 1 \leq j \leq J. \tag{4.1.37}$$

Moreover, this matrix $\boldsymbol{A}$ is unique and we have the estimate

$$\|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C_1\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \tag{4.1.38}$$

If Ω is of class $\mathcal{C}^{2,1}$, then $\boldsymbol{A} \in \mathbb{W}^{2,p}_s(\Omega)$ and we have the estimate

$$\|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C_2 \|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \tag{4.1.39}$$

Proof. **i)** Assume that $\boldsymbol{A}$ belongs to $\mathbb{X}^p_s(\Omega)$ and satisfies (4.1.35), from the proof of Theorem 4.1.10, we know that $\mathbf{Curl\,Curl}\,\boldsymbol{A}$ satisfies (4.1.14)-(4.1.16).

ii) Conversely, let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$ and satisfies conditions (4.1.14)-(4.1.16). Let us consider the matrix $\boldsymbol{A}_0$ given by Theorem 4.1.10, and the solution $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ of the following problem:

$$\begin{cases} -\mathbf{Div}(\boldsymbol{\nabla}_s \boldsymbol{v}) = \mathbf{Div}\,\boldsymbol{A}_0 & \text{in} \quad \Omega, \\ (\boldsymbol{\nabla}_s \boldsymbol{v})\,\boldsymbol{n} = \boldsymbol{A}_0 \boldsymbol{n} & \text{on} \quad \Gamma, \end{cases}$$

(note that $\boldsymbol{v} \in \boldsymbol{W}^{3,p}(\Omega)$ when Ω is of class $\mathcal{C}^{2,1}$).

We set

$$\boldsymbol{A} = \boldsymbol{A}_0 + \boldsymbol{\nabla}_s \boldsymbol{v} - \sum_{i=1}^{3} \sum_{j=1}^{J} \left(\langle (\boldsymbol{A}_0 + \boldsymbol{\nabla}_s \boldsymbol{v})\boldsymbol{n}, \boldsymbol{e}^i \rangle_{\Sigma_j} \widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}^j_i} + \langle (\boldsymbol{A}_0 + \boldsymbol{\nabla}_s \boldsymbol{v})\boldsymbol{n}, \boldsymbol{P}^i \rangle_{\Sigma_j} \widetilde{\boldsymbol{\nabla}_s \boldsymbol{r}^j_i} \right).$$

Since Ω is of class $\mathcal{C}^{1,1}$, then for all $1 \leq i \leq 3$ and $1 \leq j \leq J$, the matrix $\widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}^j_i}$ and $\widetilde{\boldsymbol{\nabla}_s \boldsymbol{r}^j_i}$ are in $\mathbb{W}^{1,p}(\Omega)$ (respectively in $\mathbb{W}^{2,p}(\Omega)$ if Ω is of class $\mathcal{C}^{2,1}$); then the matrix $\boldsymbol{A} \in \mathbb{W}^{1,p}(\Omega)$ (respectively $\boldsymbol{A} \in \mathbb{W}^{2,p}(\Omega)$ if Ω is of class $\mathcal{C}^{2,1}$) and satisfies the conditions (4.1.35)-(4.1.37)). Finally, Corollary 4.1.15 implies that estimates (4.1.38) and (4.1.39) are true, and the uniqueness of $\boldsymbol{A}$ is due to the characterization of the kernel space $\mathbb{K}_{T,s}(\Omega)$. □

4.2 Normal Beltrami's completeness

It is well known that if Ω is a Lipschitz domain, then any vector field $\boldsymbol{v}$ of $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ has a tangential trace $\boldsymbol{v} \times \boldsymbol{n}$ in $\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$. Amrouche et al [3] used this characterization of $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$ to show potential vector theorems in both cases of Hilbert spaces case (see Theorem 3.17 of [3]), and in the Banach spaces case (see Theorem 4.3 of [9]).

In this section, we present some analogous results for symmetric matrix fields. We show that if the domain Ω is of class $\mathcal{C}^{1,1}$, then any matrix field $\boldsymbol{S}$ of $\mathbb{H}^p_s(\mathbf{Curl\,Curl}, \Omega)$ has a tangential trace $\boldsymbol{S}\times\boldsymbol{n}$ in $\mathbb{W}^{-\frac{1}{p},p}(\Gamma)$; and the matrix field $\mathbf{Curl}\,\boldsymbol{S}$ (which is not even in $\mathbb{L}^p(\Omega)$) has a tangential trace $\mathbf{Curl}\,\boldsymbol{S}\times\boldsymbol{n}$ in $\mathbb{W}^{-1-\frac{1}{p},p}(\Gamma)$. After that, we focus our attention to show an extension of Beltrami's completeness with normal boundary conditions.

Proposition 4.2.1. *i) The space $\mathbb{D}_s(\overline{\Omega})$ is dense in $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\Omega)$.*
ii) If Ω is of class $\mathcal{C}^{1,1}$, then:
a) The linear mapping $\boldsymbol{S}\longrightarrow\boldsymbol{S}\times\boldsymbol{n}$ defined on $\mathbb{D}_s(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping, still denoted in the same way, from $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$ into $\mathbb{W}^{-\frac{1}{p},p}(\Gamma)$.
b) The linear mapping $\boldsymbol{S}\mapsto\mathbf{Curl}\,\boldsymbol{S}\times\boldsymbol{n}_{|\Gamma}$ can be extended by continuity to a linear and continuous mapping from $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$ into $\mathbb{W}^{-1-\frac{1}{p},p}(\Gamma)$ and the following Green's formula holds true: for all $\boldsymbol{E}\in\mathbb{W}^{2,p'}_s(\Omega)$,

$$\langle\boldsymbol{S}\times\boldsymbol{n},\mathbf{Curl}\,\boldsymbol{E}\rangle_\Gamma+\langle\mathbf{Curl}\,\boldsymbol{S}\times\boldsymbol{n},\,\boldsymbol{E}\rangle_\Gamma=\int_\Omega\boldsymbol{S}:\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E}\,dx-\int_\Omega\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S}:\boldsymbol{E}\,dx. \tag{4.2.1}$$

Proof. **i)** Let $\boldsymbol{\ell}\in(\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega))'$ such that $\langle\boldsymbol{\ell},\boldsymbol{A}\rangle=0$ for all $\boldsymbol{A}\in\mathbb{D}_s(\overline{\Omega})$. We associate to $\boldsymbol{\ell}$ the matrix $\boldsymbol{L}$ in $\mathbb{H}^{p'}_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$ such that:

$$\text{for all }\boldsymbol{A}\in\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega),\ \langle\boldsymbol{\ell},\,\boldsymbol{A}\rangle=\int_\Omega\boldsymbol{L}:\boldsymbol{A}\,dx+\int_\Omega\boldsymbol{S}:\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}\,dx$$

where

$$\boldsymbol{S}=\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{L}.$$

Now, we assume that $\boldsymbol{\ell}$ vanishes on $\mathbb{D}_s(\overline{\Omega})$. We set $\widetilde{L_{ij}}$ (resp $\widetilde{S_{ij}}$) the extension of L_{ij} (resp S_{ij}) by zero outside Ω and let $\boldsymbol{A}\in\mathbb{D}_s(\mathbb{R}^3)$, so we have

$$\int_{\mathbb{R}^3}\widetilde{\boldsymbol{L}}:\boldsymbol{A}\,dx+\int_{\mathbb{R}^3}\widetilde{\boldsymbol{S}}:\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}\,dx=0.$$

Thus

$$-\widetilde{\boldsymbol{L}}=\mathbf{Curl}\,\mathbf{Curl}\,\widetilde{\boldsymbol{S}}\quad\text{in}\quad\mathbb{R}^3.$$

Consequently, Proposition 4.1.2 implies that the matrix field $\boldsymbol{S}$ belongs to $\mathbb{H}^p_{0,s}(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$. Also, there exists a sequence $(\boldsymbol{S}_k)$ of $\mathbb{D}_s(\Omega)$ that converges to $\boldsymbol{S}$ in $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$. So, we have

$$\forall\, \boldsymbol{A} \in \mathbb{H}^p_s(\mathbf{Curl\,Curl},\,\Omega),\ \langle \boldsymbol{\ell}, \boldsymbol{A}\rangle = \lim_{k\longrightarrow\infty}\int_\Omega (\boldsymbol{A} : \mathbf{Curl\,Curl}\,\boldsymbol{S}_k - \boldsymbol{S}_k : \mathbf{Curl\,Curl}\,\boldsymbol{A})\,dx = 0,$$

then, the density of $\mathbb{D}_s(\overline{\Omega})$ in $\mathbb{H}^p_s(\mathbf{Curl\,Curl},\,\Omega)$ is true.

ii) a) Now, we assume that Ω is of class $\mathcal{C}^{1,1}$. Let us prove statement a). For all $\boldsymbol{A} \in \mathbb{D}(\overline{\Omega})$ and $\boldsymbol{E} \in \mathbb{W}^{2,p'}(\Omega) \cap \mathbb{W}^{1,p'}_0(\Omega)$ the following Green's formula holds true

$$\langle \boldsymbol{A}\times\boldsymbol{n},\, \mathbf{Curl}\,\boldsymbol{E}\rangle_{\mathbb{W}^{-\frac{1}{p},p}(\Gamma)\times\mathbb{W}^{\frac{1}{p},p'}(\Gamma)} = \int_\Omega \boldsymbol{A} :\ \mathbf{Curl\,Curl}\,\boldsymbol{E}\,dx - \int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{A} :\ \boldsymbol{E}\,dx. \tag{4.2.2}$$

Let $\boldsymbol{A}$ be in $\mathbb{D}_s(\overline{\Omega})$ and Let $\boldsymbol{M}$ be in $\mathbb{W}^{\frac{1}{p},p'}(\Gamma)$. Since Ω is of class $\mathcal{C}^{1,1}$, then the matrix $\boldsymbol{M}_\tau =: (\boldsymbol{M}\times\boldsymbol{n})\times\boldsymbol{n}$ belongs to $\mathbb{W}^{\frac{1}{p},p'}(\Gamma)$ and there exists $\boldsymbol{E} \in \mathbb{W}^{2,p'}(\Omega)$ such that:

$$\boldsymbol{E} = \mathbf{0} \quad \text{and} \quad \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{n}} = (\boldsymbol{M}^T_\tau \times \boldsymbol{n})^T \quad \text{on } \Gamma,$$

Furthermore, we have

$$\|\boldsymbol{E}\|_{\mathbb{W}^{2,p'}(\Omega)} \le C\|\boldsymbol{M}\|_{\mathbb{W}^{\frac{1}{p},p'}(\Gamma)},$$

(see for example the proof of Theorem 5.4 of [9].) Moreover, the relation (5.10) of [9] and (0.0.2) imply that

$$\mathbf{Curl}\,\boldsymbol{E} = -(\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{n}}\times\boldsymbol{n})^T = -((\boldsymbol{M}^T_\tau\times\boldsymbol{n})^T\times\boldsymbol{n})^T = -(\boldsymbol{M}_\tau\times\boldsymbol{n})^T\times\boldsymbol{n}.$$

define the vector line $(\boldsymbol{M}_\tau)^i$ by $(\boldsymbol{M}_\tau)^i = \boldsymbol{M}^i - (\boldsymbol{M}^i\cdot\boldsymbol{n})\boldsymbol{n}^T$. Since $\boldsymbol{M}_\tau\boldsymbol{n} = 0$ and $\boldsymbol{A}$ is symmetric, we can verify that

$$\langle \boldsymbol{A}\times\boldsymbol{n},\, \boldsymbol{M}_\tau\rangle_\Gamma = -\langle \boldsymbol{A}\times\boldsymbol{n}, (\boldsymbol{M}_\tau\times\boldsymbol{n})^T\times\boldsymbol{n}\rangle_\Gamma = \langle \boldsymbol{A}\times\boldsymbol{n},\, \mathbf{Curl}\,\boldsymbol{E}\rangle_\Gamma.$$

As

$$\langle \boldsymbol{A}\times\boldsymbol{n},\, \boldsymbol{M}\rangle_\Gamma = \langle \boldsymbol{A}\times\boldsymbol{n},\, \boldsymbol{M}_\tau\rangle_\Gamma,$$

we get from (4.2.2),

$$\begin{aligned} |\langle \boldsymbol{A}\times\boldsymbol{n},\, \boldsymbol{M}\rangle_\Gamma| &= |\langle \boldsymbol{A}\times\boldsymbol{n},\, \mathbf{Curl}\,\boldsymbol{E}\rangle_\Gamma| \\ &\le \|\boldsymbol{A}\|_{\mathbb{H}^p_s(\mathbf{Curl\,Curl},\,\boldsymbol{\Omega})}\|\boldsymbol{E}\|_{\mathbb{W}^{2,p'}(\Omega)} \\ &\le C_1\|\boldsymbol{A}\|_{\mathbb{H}^p_s(\mathbf{Curl\,Curl},\,\boldsymbol{\Omega})}\|\boldsymbol{M}\|_{\mathbb{W}^{\frac{1}{p},p'}(\Gamma)}. \end{aligned}$$

Which means that

$$\|\boldsymbol{A}\times\boldsymbol{n}\|_{\mathbb{W}^{-\frac{1}{p},p}(\Gamma)} \leq C_1\|\boldsymbol{A}\|_{\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)}. \tag{4.2.3}$$

Then, the linear mapping

$$\begin{array}{ccc} \mathbb{D}_s(\overline{\Omega}) & \longrightarrow & \mathbb{W}^{-\frac{1}{p},p}(\Gamma) \\ \boldsymbol{A} & \longrightarrow & \boldsymbol{A}\times\boldsymbol{n} \end{array}$$

is continuous on $\mathbb{D}_s(\overline{\Omega})$ equipped with the norm of $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$. Thanks to point i), it can be extended to a unique linear and continuous mapping from $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$ into $\mathbb{W}^{-\frac{1}{p},p}(\Gamma)$.

b) Let $\boldsymbol{A}\in\mathbb{D}_s(\overline{\Omega})$ and $\boldsymbol{M}\in\mathbb{W}^{1+\frac{1}{p},p'}(\Gamma)$. Then, there exists $\boldsymbol{E}\in\mathbb{W}^{2,p'}(\Omega)$ such that $\boldsymbol{E}|_\Gamma=\boldsymbol{M}$, with $\|\boldsymbol{E}\|_{\mathbb{W}^{2,p'}(\Omega)}\leq C\|\boldsymbol{M}\|_{\mathbb{W}^{1+\frac{1}{p},p'}(\Gamma)}$ and we have the following Green's formula:

$$\langle\boldsymbol{A}\times\boldsymbol{n},\,\mathbf{Curl}\,\boldsymbol{E}\rangle_\Gamma+\langle\mathbf{Curl}\,\boldsymbol{A}\times\boldsymbol{n},\,\boldsymbol{M}\rangle_\Gamma=\int_\Omega\boldsymbol{A}:\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E}\,dx-\int_\Omega\boldsymbol{E}:\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}\,dx. \tag{4.2.4}$$

Using the estimate (4.2.3), then

$$\begin{aligned} |\langle\mathbf{Curl}\,\boldsymbol{A}\times\boldsymbol{n},\,\boldsymbol{M}\rangle_\Gamma| &= |-\langle\boldsymbol{A}\times\boldsymbol{n},\,\mathbf{Curl}\,\boldsymbol{E}\rangle_\Gamma+\int_\Omega\boldsymbol{A}:\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E}-\int_\Omega\boldsymbol{E}:\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}| \\ &\leq (1+C_1)\|\boldsymbol{A}\|_{\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\Omega)}\|\boldsymbol{E}\|_{\mathbb{W}^{2,p'}(\Omega)} \\ &\leq C_2\|\boldsymbol{A}\|_{\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\Omega)}\|\boldsymbol{M}\|_{\mathbb{W}^{1+\frac{1}{p},p'}(\Gamma)}. \end{aligned}$$

That means that

$$\|\mathbf{Curl}\,\boldsymbol{A}\times\boldsymbol{n}\|_{\mathbb{W}^{-1-\frac{1}{p},p}(\Gamma)}\leq C_2\|\boldsymbol{A}\|_{\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)}.$$

Then, the linear mapping

$$\begin{array}{c} \mathbb{D}_s(\overline{\Omega})\longrightarrow\mathbb{W}^{-1-\frac{1}{p},p}(\Gamma) \\ \boldsymbol{A}\longrightarrow\mathbf{Curl}\,\boldsymbol{A}\times\boldsymbol{n} \end{array}$$

is continuous on $\mathbb{D}_s(\overline{\Omega})$ equipped with the norm of $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$. Thanks to point i), it can be extended to a unique linear and continuous mapping from $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)$ into $\mathbb{W}^{-1-\frac{1}{p},p}(\Gamma)$ and the Green's formula (4.2.1) holds true. □

Lemma 4.2.2. *Let* $\boldsymbol{S} \in \mathbb{H}^p_s(\mathbf{Curl\,Curl},\,\Omega)$*, then the following property*

$$\textit{for all } \boldsymbol{E} \in \mathbb{W}^{2,p'}_s(\Omega),\ \int_\Omega \boldsymbol{S} : \mathbf{Curl\,Curl}\,\boldsymbol{E}\,dx - \int_\Omega \boldsymbol{E} : \mathbf{Curl\,Curl}\,\boldsymbol{S}\,dx = 0, \tag{4.2.5}$$

is equivalent to the property

$$\textit{for all } \boldsymbol{M} \in \mathbb{W}^{2,p'}(\Omega),\ \int_\Omega \boldsymbol{S} : \mathbf{Curl\,Curl}\,\boldsymbol{M}\,dx - \int_\Omega \boldsymbol{M} : \mathbf{Curl\,Curl}\,\boldsymbol{S}\,dx = 0. \tag{4.2.6}$$

Proof. Let $\boldsymbol{S} \in \mathbb{H}^p_s(\mathbf{Curl\,Curl},\,\Omega)$ satisfying (4.2.5). For any $\boldsymbol{M} \in \mathbb{W}^{2,p'}(\Omega)$, we denote $\boldsymbol{M}^{sym}$ its symmetric part and $\boldsymbol{M}^{skw}$ its antisymmetric part. Recall that $\mathbf{Curl\,Curl}\,\boldsymbol{M}^{skw}$ is an antisymmetric matrix. Then,

$$\begin{aligned} & \int_\Omega \boldsymbol{S} : \mathbf{Curl\,Curl}\,\boldsymbol{M}\,dx - \int_\Omega \boldsymbol{M} : \mathbf{Curl\,Curl}\,\boldsymbol{S}\,dx \\ = \ & \int_\Omega \boldsymbol{S} : \mathbf{Curl\,Curl}\,(\boldsymbol{M}^{sym} + \boldsymbol{M}^{skw})\,dx - \int_\Omega (\boldsymbol{M}^{sym} + \boldsymbol{M}^{skw}) : \mathbf{Curl\,Curl}\,\boldsymbol{S}\,dx \\ = \ & \int_\Omega \boldsymbol{S} : \mathbf{Curl\,Curl}\,\boldsymbol{M}^{skw}\,dx - \int_\Omega \boldsymbol{M}^{skw} : \mathbf{Curl\,Curl}\,\boldsymbol{S}\,dx \\ = \ & 0. \end{aligned}$$

Then, (4.2.5) implies (4.2.6) and it is clear that (4.2.6) implies (4.2.5), which ends the proof. □

Remark 4.2.3. If Ω is of class $\mathcal{C}^{1,1}$, then Green's formula (4.2.1) and Lemma 4.2.2 imply that the condition (4.2.5) is equivalent to

$$\text{for all } \boldsymbol{M} \in \mathbb{W}^{2,p'}(\Omega),\ \langle \boldsymbol{S} \times \boldsymbol{n}, \mathbf{Curl}\,\boldsymbol{M}\rangle + \langle \mathbf{Curl}\,\boldsymbol{S} \times \boldsymbol{n},\,\boldsymbol{M}\rangle = 0. \tag{4.2.7}$$

As we have seen in proof of Proposition 4.2.1, condition (4.2.7) is equivalent to

$$\boldsymbol{S} \times \boldsymbol{n} = \boldsymbol{0} \quad \textit{and} \quad \mathbf{Curl}\,\boldsymbol{S} \times \boldsymbol{n} = \boldsymbol{0} \quad \textit{on} \quad \Gamma. \tag{4.2.8}$$

Consequently, condition (4.2.5) *is equivalent to* (4.2.8)*. Then, Theorem 4.1.2 implies the following identity*

$$\mathbb{H}^p_{0,s}(\mathbf{Curl\,Curl},\,\Omega) = \{\boldsymbol{S} \in \mathbb{H}^p_s(\mathbf{Curl\,Curl},\,\Omega),\ \boldsymbol{S} \times \boldsymbol{n} = \boldsymbol{0} \textit{ and } \mathbf{Curl}\,\boldsymbol{S} \times \boldsymbol{n} = \boldsymbol{0} \quad \textit{on} \quad \Gamma\}.$$

We define the kernel space

$$\mathbb{K}^p_{N,s}(\Omega) = \{\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)\,;\, \textbf{Div}\,\boldsymbol{S} = \boldsymbol{0} \text{ in } \Omega,\ \textbf{Curl}\,\textbf{Curl}\,\boldsymbol{S} = \boldsymbol{0} \text{ in } \Omega,\ \boldsymbol{S}\times\boldsymbol{n} = \textbf{Curl}\,\boldsymbol{S}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}.$$

The following theorem characterizes the space $\mathbb{K}^2_{N,s}(\Omega)$.

Theorem 4.2.4. *Assume that Ω is Lipschitz. The dimension of the space $\mathbb{K}^2_{N,s}(\Omega)$ is $6I$. It is spanned by the matrix $\boldsymbol{\nabla}_s\boldsymbol{v}^k_i$ and $\boldsymbol{\nabla}_s\boldsymbol{w}^k_i$ where $\boldsymbol{v}^k_i$ is the solution in $\boldsymbol{H}^1(\Omega)$ of the problem*

$$\begin{cases} -\textbf{Div}\,(\boldsymbol{\nabla}_s\boldsymbol{v}^k_i) = \boldsymbol{0} \quad in \quad \Omega, \\ \boldsymbol{v}^k_i|_{\Gamma_0} = \boldsymbol{0} \quad and \quad \boldsymbol{v}^k_i|_{\Gamma_\ell} = \boldsymbol{rig}, \quad 1 \le \ell \le I, \\ \langle(\boldsymbol{\nabla}_s\boldsymbol{v}^k_i)\,\boldsymbol{n},\, \boldsymbol{e}^j\rangle_{\Gamma_\ell} = \delta_{ij}\delta_{k\ell}, \quad 1 \le j \le 3, \quad and \quad 1 \le \ell \le I, \\ \langle(\boldsymbol{\nabla}_s\boldsymbol{v}^k_i)\,\boldsymbol{n},\, \boldsymbol{e}^j\rangle_{\Gamma_0} = -\delta_{ij}, \\ \langle(\boldsymbol{\nabla}_s\boldsymbol{v}^k_i)\,\boldsymbol{n},\, \boldsymbol{P}^j\rangle_{\Gamma_\ell} = 0, \quad 0 \le \ell \le I, \end{cases} \tag{4.2.9}$$

and $\boldsymbol{w}^k_i$ is the solution in $\boldsymbol{H}^1(\Omega)$ of the problem

$$\begin{cases} -\textbf{Div}\,(\boldsymbol{\nabla}_s\boldsymbol{w}^k_i) = \boldsymbol{0} \quad in \quad \Omega, \\ \boldsymbol{w}^k_i|_{\Gamma_0} = \boldsymbol{0} \quad and \quad \boldsymbol{w}^k_i|_{\Gamma_\ell} = \boldsymbol{rig}, \quad 1 \le \ell \le I, \\ \langle(\boldsymbol{\nabla}_s\boldsymbol{w}^k_i)\,\boldsymbol{n},\, \boldsymbol{P}^j\rangle_{\Gamma_\ell} = \delta_{ij}\delta_{k\ell}, \quad 1 \le j \le 3, \quad and \quad 1 \le \ell \le I, \\ \langle(\boldsymbol{\nabla}_s\boldsymbol{w}^k_i)\,\boldsymbol{n},\, \boldsymbol{P}^j\rangle_{\Gamma_0} = -\delta_{ij}, \\ \langle(\boldsymbol{\nabla}_s\boldsymbol{w}^k_i)\,\boldsymbol{n},\, \boldsymbol{e}^j\rangle_{\Gamma_\ell} = 0, \quad 0 \le \ell \le I. \end{cases} \tag{4.2.10}$$

Moreover, if Ω is of class $\mathcal{C}^{1,1}$, then $\boldsymbol{v}^k_i$ and $\boldsymbol{w}^k_i$ belong to $\boldsymbol{H}^2(\Omega)$ for any $1 < p < \infty$.

Proof. We consider here only the first problem, which is similar to the second one. Let

$$\boldsymbol{\mathcal{V}}^{\boldsymbol{\Gamma}}_{\boldsymbol{D}} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega),\, \boldsymbol{v}|_{\Gamma_0} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{v}|_{\Gamma_k} = \boldsymbol{rig},\ 1 \le k \le I\}.$$

For $1 \le i \le 3$ and $1 \le k \le I$, Lax-Milgram lemma implies that the problem:

find $\boldsymbol{v}^k_i \in \boldsymbol{\mathcal{V}}^{\boldsymbol{\Gamma}}_{\boldsymbol{D}}$ such that

$$\forall \boldsymbol{u} \in \boldsymbol{\mathcal{V}}^{\boldsymbol{\Gamma}}_{\boldsymbol{D}}, \quad \int_\Omega \boldsymbol{\nabla}_s\boldsymbol{v}^k_i : \boldsymbol{\nabla}_s\boldsymbol{u}\,dx = a^k_i(\boldsymbol{u}) \tag{4.2.11}$$

has unique solution $\boldsymbol{v}^k_i \in \boldsymbol{\mathcal{V}}^{\boldsymbol{\Gamma}}_{\boldsymbol{D}}$.

A similar argument to that used in the proof of Theorem 4.1.9 shows that $\boldsymbol{v}_i^k$ satisfies (4.2.9). Moreover, if Ω is of class $\mathcal{C}^{1,1}$, then $\boldsymbol{v}_i^k$ belongs to $\boldsymbol{H}^2(\Omega)$. Now, we prove that $\boldsymbol{E} = \{\boldsymbol{\nabla}_s \boldsymbol{v}_i^k\}_i \cup \{\boldsymbol{\nabla}_s \boldsymbol{w}_i^k\}_i$ is a basis of $\mathbb{K}^2_{N,s}(\Omega)$. The elements of $\boldsymbol{E}$ are linearly independent and belong to $\mathbb{K}^2_{N,s}(\Omega)$. Let $\boldsymbol{S} \in \mathbb{K}^2_{N,s}(\Omega)$, we set

$$\boldsymbol{A} = \boldsymbol{S} - \sum_{i=1}^{3} \left(\sum_{k=1}^{I} \left(\langle \boldsymbol{Sn},\, \boldsymbol{e}^i \rangle_{\Gamma_k} \boldsymbol{\nabla}_s \boldsymbol{v}_i^k + \langle \boldsymbol{Sn},\, \boldsymbol{P}^i \rangle_{\Gamma_k} \boldsymbol{\nabla}_s \boldsymbol{w}_i^k \right) \right).$$

It is clear that $\boldsymbol{A}$ satisfies the compatibility conditions of Theorem 4.1.10. Then, there exists a symmetric matrix field $\boldsymbol{A}_0 \in \mathbb{H}^2_s(\Omega)$ such that $\boldsymbol{A} = \mathbf{Curl\,Curl}\, \boldsymbol{A}_0$. Then, we have

$$\int_\Omega \boldsymbol{A} : \boldsymbol{A}\, dx = \int_\Omega \boldsymbol{A} : \mathbf{Curl\,Curl}\, \boldsymbol{A}_0\, dx = \int_\Omega \boldsymbol{A}_0 : \mathbf{Curl\,Curl}\, \boldsymbol{A}\, dx = 0.$$

Then, $\boldsymbol{A} = \mathbf{0}$, which is the required result.

$H^2(\Omega)$ regularity is immediate. □

We will show, now, that the vector fields belonging to the kernel spaces $\mathbb{K}^p_{N,s}(\Omega)$ are more regular and this regularity does not depend on p. For that, we need to establish some auxiliary results. The first one gives some equivalence properties to inf-sup condition (see [33]).

Theorem 4.2.5. *Let X and M be two reflexive Banach space and X' and M' their dual spaces. Let a be a continuous bilinear form defined on $X \times M$, let $A \in \mathcal{L}(X,\, M')$ and $A' \in \mathcal{L}(M,\, X')$ be the operators defined by*

$$\forall v \in X,\, \forall w \in M, \quad a(v,w) = \langle Av,\, w\rangle = \langle v,\, A'w\rangle,$$

and $V = \operatorname{Ker} A$. The following statements are equivalent:

(i) There exists $\beta > 0$ such that

$$\inf_{w \in M,\, w \neq 0} \sup_{v \in X,\, v \neq 0} \frac{a(v,w)}{\|v\|_X \|w\|_M} \geq \beta.$$

(ii) The operator $A : X/V \longrightarrow M'$ is an isomorphism and $\frac{1}{\beta}$ is the continuity constant of A^{-1}.

(iii) The operator $A' : M \longrightarrow X' \perp V$ is an isomorphism and $\frac{1}{\beta}$ is the continuity constant of $(A')^{-1}$.

The second one gives an inf-sup condition of the **Curl Curl** operator.

Lemma 4.2.6. *Assume that* Ω *is of class* $\mathcal{C}^{1,1}$. *Then, the following inf-sup condition holds true: there exists a constant* $\alpha > 0$, *such that*

$$\inf_{\substack{\boldsymbol{\Psi}\in\mathbb{V}^{p'}_{T,s}(\Omega),\\ \boldsymbol{\Psi}\neq\mathbf{0}}} \ \sup_{\substack{\boldsymbol{E}\in\mathbb{V}^{p}_{T,s}(\Omega),\\ \boldsymbol{E}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E} : \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{\Psi}\,dx}{\|\boldsymbol{E}\|_{\mathbb{X}^p_s(\Omega)}\|\boldsymbol{\Psi}\|_{\mathbb{X}^{p'}_s(\Omega)}} \geq \alpha, \tag{4.2.12}$$

where

$$\mathbb{V}^p_{T,s}(\Omega) = \{\boldsymbol{S}\in\mathbb{X}^p_{T,s}(\Omega),\ \mathbf{Div}\,\boldsymbol{S}=\mathbf{0}\quad \textit{and}\quad \langle\boldsymbol{Sn},\,\boldsymbol{e}^i\rangle_{\Sigma_j} = \langle\boldsymbol{Sn},\,\boldsymbol{P}^i\rangle_{\Sigma_j} = 0\}.$$

Proof. Let $\boldsymbol{A}\in\mathbb{L}^{p'}_s(\Omega)$ and $\boldsymbol{v}$ be the solution in $\boldsymbol{W}^{1,p'}_0(\Omega)$ of the homogeneous Dirichlet problem $\mathbf{Div}\,\boldsymbol{\nabla}_s\,\boldsymbol{v} = \mathbf{Div}\,\boldsymbol{A}$ which satisfies the estimate

$$\|\boldsymbol{\nabla}_s\,\boldsymbol{v}\|_{\mathbb{L}^{p'}(\Omega)} \leq C\|\boldsymbol{A}\|_{\mathbb{L}^{p'}(\Omega)}. \tag{4.2.13}$$

We set $\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{\nabla}_s\boldsymbol{v}$ and let $\boldsymbol{E}\in\mathbb{V}^p_{T,s}(\Omega)$. Due to Corollary 4.1.15, we obtain

$$\|\boldsymbol{E}\|_{\mathbb{X}^p_s(\Omega)} \leq C\|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} = C \sup_{\substack{\boldsymbol{A}\in\mathbb{L}^{p'}_s(\Omega)\\ \boldsymbol{A}\neq\mathbf{0}}} \frac{|\displaystyle\int_\Omega \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E} : \boldsymbol{A}\,dx|}{\|\boldsymbol{A}\|_{\mathbb{L}^{p'}(\Omega)}}. \tag{4.2.14}$$

Now, setting

$$\widetilde{\boldsymbol{F}} = \boldsymbol{F} - \left[\sum_{i=1}^{3}\sum_{k=1}^{I}\big(\langle\boldsymbol{Fn},\,\boldsymbol{e}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{v}^k_i + \langle\boldsymbol{Fn},\,\boldsymbol{P}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{w}^k_i\big)\right],$$

then, $\widetilde{\boldsymbol{F}}\in\mathbb{L}^{p'}_s(\Omega)$, $\mathbf{Div}\,\widetilde{\boldsymbol{F}} = \mathbf{0}$ in Ω, $\langle\widetilde{\boldsymbol{F}}\boldsymbol{n},\,\boldsymbol{e}^i\rangle_{\Gamma_k} = \langle\widetilde{\boldsymbol{F}}\boldsymbol{n},\,\boldsymbol{P}^i\rangle_{\Gamma_k} = \mathbf{0}$ for any $1\leq i\leq 3$ and any $1\leq k\leq I$ and we have

$$\int_\Omega \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E} : \boldsymbol{A}\,dx = \int_\Omega \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E} : \boldsymbol{F}\,dx = \int_\Omega \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E} : \widetilde{\boldsymbol{F}}\,dx.$$

Moreover, we have

$$\begin{aligned}
\|\widetilde{\boldsymbol{F}}\|_{\mathbb{L}^{p'}(\Omega)} &\leq \|\boldsymbol{F}\|_{\mathbb{L}^{p'}(\Omega)} + \sum_{i=1}^{3}\sum_{k=1}^{I}\Big(|\langle\boldsymbol{Fn},\,\boldsymbol{e}^i\rangle|\|\boldsymbol{\nabla}_s\,\boldsymbol{v}^k_i\|_{\mathbb{L}^{p'}(\Omega)} + |\langle\boldsymbol{Fn},\,\boldsymbol{P}^i\rangle|\|\boldsymbol{\nabla}_s\,\boldsymbol{w}^k_i\|_{\mathbb{L}^{p'}(\Omega)}\Big)\\
&\leq \|\boldsymbol{F}\|_{\mathbb{L}^{p'}(\Omega)} + C\|\boldsymbol{Fn}\|_{\boldsymbol{W}^{-\frac{1}{p'},p'}(\Omega)}.
\end{aligned}$$

Since $\boldsymbol{F}$ belongs to $\mathbb{H}^p_s(\mathbf{Div},\,\Omega)$ and $\mathbf{Div}\,\boldsymbol{F} = \mathbf{0}$ in Ω, then

$$\|\widetilde{\boldsymbol{F}}\|_{\mathbb{L}^{p'}(\Omega)} \leq \|\boldsymbol{F}\|_{\mathbb{L}^{p'}(\Omega)} + C\|\boldsymbol{F}\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p'},p'}(\Omega)} \leq C\|\boldsymbol{F}\|_{\mathbb{L}^{p'}(\Omega)}. \tag{4.2.15}$$

Using (4.2.13) and (4.2.15), we obtain

$$\|\widetilde{\boldsymbol{F}}\|_{\mathbb{L}^{p'}(\Omega)} \leq C\|\boldsymbol{A}\|_{\mathbb{L}^{p'}(\Omega)}.$$

From Theorem 4.1.16, there exists $\boldsymbol{\Psi} \in \mathbb{V}^{p'}_{T,s}(\Omega)$ such that $\widetilde{\boldsymbol{F}} = \mathbf{Curl\,Curl}\,\boldsymbol{\Psi}$, and due to Corollary 4.1.15, we have

$$\|\boldsymbol{\Psi}\|_{\mathbb{X}^{p'}(\Omega)} \leq C\|\widetilde{\boldsymbol{F}}\|_{\mathbb{L}^{p'}(\Omega)}.$$

Finally,

$$\frac{|\displaystyle\int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{E} : \boldsymbol{A}|}{\|\boldsymbol{A}\|_{\mathbb{L}^{p'}(\Omega)}} \leq C\frac{|\displaystyle\int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{E} : \widetilde{\boldsymbol{F}}|}{\|\widetilde{\boldsymbol{F}}\|_{\mathbb{L}^{p'}(\Omega)}} \leq C\frac{|\displaystyle\int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{E} : \mathbf{Curl\,Curl}\,\boldsymbol{\Psi}|}{\|\boldsymbol{\Psi}\|_{\mathbb{X}^{p'}_s(\Omega)}}.$$

As a matter of fact, (4.2.14) implies that the inf-sup condition (4.2.12) holds true. □

Using the inf-sup condition (4.2.12), we solve the following elliptic problem:

Proposition 4.2.7. *Assume that Ω is of class $\mathcal{C}^{1,1}$ and let $\boldsymbol{B} \in \mathbb{L}^p_s(\Omega)$. Then, the elliptic problem*

$$\begin{cases} \triangle^2\boldsymbol{E} = \mathbf{Curl\,Curl}\,\boldsymbol{B} \quad \textit{and} \quad \mathbf{Div}\,\boldsymbol{E} = \mathbf{0} \quad \textit{in} \quad \Omega, \\ \boldsymbol{E}\boldsymbol{n} = \mathbf{0}, \quad (\mathbf{Curl\,Curl}\,\boldsymbol{E} - \boldsymbol{B}) \times \boldsymbol{n} = (\mathbf{Curl\,Curl\,Curl\,E} - \mathbf{Curl}\,\boldsymbol{B}) \times \boldsymbol{n} = \mathbf{0} \quad \textit{on} \quad \Gamma, \\ \langle \boldsymbol{E}\boldsymbol{n},\, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \boldsymbol{E}\boldsymbol{n},\, \boldsymbol{P}^i\rangle_{\Sigma_j} = 0, \quad 1 \leq i \leq 3, \quad 1 \leq j \leq J. \end{cases} \tag{4.2.16}$$

has a unique solution in $\mathbb{W}^{1,p}_s(\Omega)$ and we have the estimate

$$\|\boldsymbol{E}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C_1\|\boldsymbol{B}\|_{\mathbb{L}^p(\Omega)}. \tag{4.2.17}$$

Moreover, if Ω is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{E}$ belongs to $\mathbb{W}^{2,p}_s(\Omega)$ and we have the estimate

$$\|\boldsymbol{E}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C_2\|\boldsymbol{B}\|_{\mathbb{L}^p(\Omega)}. \tag{4.2.18}$$

Proof. **Step 1. Existence and uniqueness**. Thanks to Lemma 4.2.6 and Theorem 4.2.5, the following problem: find $\boldsymbol{E} \in \mathbb{V}^{p}_{T,s}(\Omega)$ such that for all $\boldsymbol{\Psi} \in \boldsymbol{V}^{p'}_{T,s}(\Omega)$

$$\int_{\Omega} \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E} : \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{\Psi}\,dx = \int_{\Omega} \boldsymbol{B} : \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{\Psi}\,dx, \tag{4.2.19}$$

has a unique solution in $\boldsymbol{V}^{p}_{T,s}(\Omega)$. We want to extend (4.2.19) to any test function in $\mathbb{X}^{p}_{s}(\Omega)$. Given $\widetilde{\boldsymbol{\Psi}} \in \mathbb{X}^{p}_{s}(\Omega)$, we know that there exists $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ solution of the problem

$$\begin{cases} \mathbf{Div}\,\boldsymbol{\nabla}_s \boldsymbol{v} & = & \mathbf{Div}\,\widetilde{\boldsymbol{\Psi}} & \text{in} \quad \Omega, \\ (\boldsymbol{\nabla}_s \boldsymbol{v} - \widetilde{\boldsymbol{\Psi}})\boldsymbol{n} & = & \mathbf{0} & \text{on} \quad \Gamma \end{cases}$$

and satisfying the following estimate

$$\|\nabla_s \boldsymbol{v}\|_{\mathbb{L}^p(\Omega)} \leq C \|\widetilde{\Psi}\|_{\mathbb{L}^p(\Omega)}.$$

Setting now

$$\boldsymbol{\Psi} = \widetilde{\boldsymbol{\Psi}} - \boldsymbol{\nabla}_s \boldsymbol{v} - \sum_{i=1}^{3} \sum_{j=1}^{J} \left(\langle (\widetilde{\boldsymbol{\Psi}} - \boldsymbol{\nabla}_s \boldsymbol{v})\boldsymbol{n},\, \boldsymbol{e}^i \rangle_{\Sigma_j} \widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}_i^j} + \langle (\widetilde{\boldsymbol{\Psi}} - \boldsymbol{\nabla}_s \boldsymbol{v})\boldsymbol{n},\, \boldsymbol{P}^i \rangle_{\Sigma_j} \widetilde{\boldsymbol{\nabla}_s \boldsymbol{r}_i^j} \right),$$

we note that $\boldsymbol{\Psi} \in \mathbb{V}^{p'}_{T,s}(\Omega)$ and that $\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{\Psi} = \mathbf{Curl}\,\mathbf{Curl}\,\widetilde{\boldsymbol{\Psi}}$. So, the problem (4.2.19) becomes: find $\boldsymbol{E} \in \boldsymbol{V}^{p}_{T,s}(\Omega)$ such that for all $\widetilde{\boldsymbol{\Psi}} \in \mathbb{X}^{p'}_{s}(\Omega)$

$$\int_{\Omega} \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E} : \mathbf{Curl}\,\mathbf{Curl}\,\widetilde{\boldsymbol{\Psi}}\,dx = \int_{\Omega} \boldsymbol{B} : \mathbf{Curl}\,\mathbf{Curl}\,\widetilde{\boldsymbol{\Psi}}\,dx. \tag{4.2.20}$$

Every solution of (4.2.16) solves (4.2.20). Conversely, let $\boldsymbol{E}$ be the solution of (4.2.20). Let us apply twice the following relation, which holds for any symmetric matrix $\boldsymbol{S}$,

$$\triangle\,\boldsymbol{S} = -\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S} - \boldsymbol{\nabla}^2(\mathrm{tr}(\boldsymbol{S})) + 2\boldsymbol{\nabla}_s\,\mathbf{Div}\,\boldsymbol{S} + [\triangle(\mathrm{tr}\,\boldsymbol{S}) - \mathrm{div}\,\mathbf{Div}\,\boldsymbol{S}]\,\mathbf{I}. \tag{4.2.21}$$

We get

$$\triangle^2 \boldsymbol{E} = \mathbf{Curl}\,\mathbf{Curl}\,(\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E}) \quad \text{in} \quad \Omega$$

and then from (4.2.20),

$$\triangle^2 \boldsymbol{E} = \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{B} \quad \text{in} \quad \Omega.$$

As the matrix $\mathbf{Curl\,Curl}\,\boldsymbol{E} - \boldsymbol{B}$ belongs to $\mathbb{H}^p_s(\mathbf{Curl\,Curl}, \Omega)$ and using (4.2.20) for all $\boldsymbol{\Psi} \in \mathbb{W}^{2,p}_s(\Omega)$, we obtain

$$\int_\Omega \mathbf{Curl\,Curl}\,(\mathbf{Curl\,Curl}\,\boldsymbol{E} - \boldsymbol{B}) : \boldsymbol{\Psi} dx = \int_\Omega (\mathbf{Curl\,Curl}\,\boldsymbol{E} - \boldsymbol{B}) : \mathbf{Curl\,Curl}\,\boldsymbol{\Psi}\, dx = 0. \tag{4.2.22}$$

Remark 4.2.3 implies then that

$$(\mathbf{Curl\,Curl}\,\boldsymbol{E} - \boldsymbol{B}) \times \boldsymbol{n} = (\mathbf{Curl\,Curl\,Curl}\,\mathbf{E} - \mathbf{Curl}\,\boldsymbol{B}) \times \boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma.$$

Consequently, $\boldsymbol{E}$ solves (4.2.16). Using Remark 4.2 iii) of [9], then there exists C such that

$$\|\boldsymbol{E}\|_{\mathbb{X}^p_s(\Omega)} \le C\|\boldsymbol{B}\|_{\mathbb{L}^p(\Omega)}. \tag{4.2.23}$$

Step 2. Regularity. Thanks to Theorem 4.1.13, since Ω is of class $\mathcal{C}^{1,1}$, then $\boldsymbol{E}$ belongs to $\mathbb{W}^{1,p}(\Omega)$ and from (4.1.33) and (4.2.23), we have

$$\|\boldsymbol{E}\|_{\mathbb{W}^{1,p}(\Omega)} \le C\|\boldsymbol{E}\|_{\mathbb{X}^p_s(\Omega)} \le C\|\boldsymbol{B}\|_{\mathbb{L}^p(\Omega)}.$$

If moreover Ω is of class $\mathcal{C}^{2,1}$, using again Theorem 4.1.13 we find $\boldsymbol{E}$ belonging to $\mathbb{W}^{2,p}(\Omega)$ and due to Corollary 4.1.15, we have

$$\|\boldsymbol{E}\|_{\mathbb{W}^{2,p}(\Omega)} \le C_2\|\mathbf{Curl\,Curl}\,\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)}.$$

consequently, we obtain the required estimate (4.2.18). □

Now, we will use Proposition 4.2.7 to show that for any $1 < p < \infty$ the kernel space $\mathbb{K}^p_{N,s}(\Omega)$ is independent of p.

Proposition 4.2.8. *Assume that Ω is of class $\mathcal{C}^{1,1}$. Then, for all $p \in]1, \infty[$, we have*

$$\mathbb{K}^p_{N,s}(\Omega) = \mathbb{K}^2_{N,s}(\Omega), \tag{4.2.24}$$

which means, in particular, that each vector field of $\mathbb{K}^2_{N,s}(\Omega)$ belongs to $\mathbb{W}^{1,p}_s(\Omega)$ for any $p > 1$.

Proof. **Step 1.** We show that for any $1 < p < \infty$, $\mathbb{K}^2_{N,s}(\Omega) \subset \mathbb{K}^p_{N,s}(\Omega)$.

Indeed, as for any $1 \leq i \leq 3$ and $1 \leq k \leq I$, the vector fields $\boldsymbol{v}_i^k$ and $\boldsymbol{w}_i^k$ belong to $\mathbb{W}^{2,p}(\Omega)$ for all $1 \leq p < \infty$, we obviously have our inclusion.

Step 2. We show that for all $1 < p < \infty$, $\mathbb{K}^p_{N,s}(\Omega) \subset \mathbb{K}^2_{N,s}(\Omega)$.

Let $\boldsymbol{S}$ be in $\mathbb{K}^p_{N,s}(\Omega)$ and set

$$\boldsymbol{A} = \boldsymbol{S} - \sum_{i=1}^{3}\left(\sum_{k=1}^{I}\left(\langle \boldsymbol{Sn},\, \boldsymbol{e}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{v}_i^k + \langle \boldsymbol{Sn},\, \boldsymbol{P}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{w}_i^k\right)\right).$$

Step 1 implies that $\boldsymbol{A} \in \mathbb{K}^p_{N,s}(\Omega)$ besides, it satisfies the conditions (4.1.15) and (4.1.16). From Lemma 4.2.6 we deduce that $\boldsymbol{A} = \boldsymbol{0}$ and hence, we get the required inclusion. □

Remark 4.2.9. From now, we will use the notation $\mathbb{K}_{N,s}(\Omega)$ instead of $\mathbb{K}^2_{N,s}(\Omega)$ in the rest of the paper.

Here, we will show the embedding of $\mathbb{X}^p_{N,s}(\Omega)$ in $\mathbb{W}^{1,p}(\Omega)$ if Ω is of class $\mathcal{C}^{1,1}$ and the embedding of $\mathbb{Y}^p_{N,s}(\Omega)$ in $\mathbb{W}^{2,p}(\Omega)$ if Ω is of class $\mathcal{C}^{2,1}$. First, we recall that any matrix field $\boldsymbol{S}$ of $\mathbb{K}_{T,s}(\Omega)$ can be written by $\boldsymbol{S} = \widetilde{\boldsymbol{\nabla}_s\boldsymbol{v}}$ with $\boldsymbol{v} = \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{J}\left(\langle \boldsymbol{Sn},\, \boldsymbol{e}^i\rangle_{\Sigma_j}\boldsymbol{u}_i^j + \langle \boldsymbol{Sn},\, \boldsymbol{P}^i\rangle_{\Sigma_j}\boldsymbol{r}_i^j\right)$, and we show the following lemma:

Lemma 4.2.10. *Assume that Ω is Lipschitz. For any $1 \leq i \leq 3$ and $1 \leq j \leq J$, there exist two matrix fields $\widetilde{\boldsymbol{\nabla}_s\boldsymbol{t}_i^j}$ and $\widetilde{\boldsymbol{\nabla}_s\boldsymbol{z}_i^j}$ in $\mathbb{K}_{T,s}(\Omega)$ such that*

$$[\boldsymbol{t}_i^j]_k = \delta_{i\ell}\delta_{kj}\,\boldsymbol{e}^\ell \quad \textit{and} \quad [\boldsymbol{z}_i^j]_k = \delta_{i\ell}\delta_{kj}\,\boldsymbol{p}^\ell, \quad 1 \leq \ell \leq 3,\ 1 \leq k \leq J.$$

Proof. To simplify the proof, we consider $J = 1$. For any matrix field $\boldsymbol{S} = \widetilde{\boldsymbol{\nabla}_s\boldsymbol{v}}$ of $\mathbb{K}_{T,s}(\Omega)$, we denote $[\boldsymbol{v}]_\Sigma = \displaystyle\sum_{i=1}^{3}(a_i^1(\boldsymbol{v})\,\boldsymbol{e}^i + b_i^1(\boldsymbol{v})\,\boldsymbol{P}^i)$. We define the operator

$$\begin{array}{rcl} \boldsymbol{T} : \mathbb{K}_{T,s}(\Omega) & \longrightarrow & \mathbb{R}^6 \\ \widetilde{\boldsymbol{\nabla}_s\boldsymbol{v}} & \mapsto & (\boldsymbol{T}(\widetilde{\boldsymbol{\nabla}_s\boldsymbol{v}}))_k = \begin{cases} a_k^1(\boldsymbol{v}), & 1 \leq k \leq 3, \\ b_{k-3}^1(\boldsymbol{v}), & 4 \leq k \leq 6. \end{cases} \end{array}$$

We will show that the operator $\boldsymbol{T}$ is onto. We use a contradiction, we suppose that $\boldsymbol{T}$ is not onto, that means that for any $\widetilde{\boldsymbol{\nabla}_s \boldsymbol{v}} \in \mathbb{K}_{T,s}(\Omega)$ there exist k', $1 \leq k' \leq 6$ and $\{\lambda_k\}_{\substack{1\leq k\leq 6\\ k\neq k'}}$ such that

$$(\boldsymbol{T}(\widetilde{\boldsymbol{\nabla}_s \boldsymbol{v}}))_{k'} = \sum_{\substack{k=1\\ k\neq k'}}^{6} \lambda_k (\boldsymbol{T}(\widetilde{\boldsymbol{\nabla}_s \boldsymbol{v}}))_k.$$

Let us suppose that $1 \leq k' \leq 3$, by using formulas (4.1.10) and (4.1.11), we obtain for any $\widetilde{\boldsymbol{\nabla}_s \boldsymbol{v}} \in \mathbb{K}_{T,s}(\Omega)$

$$\int_{\Omega^\circ} (\boldsymbol{\nabla}_s \boldsymbol{u}^1_{k'} - (\sum_{\substack{i=1\\ i\neq k'}}^{3} \lambda_i \boldsymbol{\nabla}_s \boldsymbol{u}^1_i + \sum_{i=1}^{3} \lambda_{i+3} \boldsymbol{\nabla}_s \boldsymbol{r}^1_i)) : \boldsymbol{\nabla}_s \boldsymbol{v}\, dx = 0. \tag{4.2.25}$$

By choosing $\boldsymbol{v} = \boldsymbol{u}^1_{k'} - (\sum_{\substack{i=1\\ i\neq k'}}^{3} \lambda_i \boldsymbol{u}^1_i + \sum_{i=1}^{3} \lambda_{i+3} \boldsymbol{r}^1_i)$ and using (4.2.25), we obtain

$$\widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}^1_{k'}} = \sum_{\substack{i=1\\ i\neq k'}}^{3} \lambda_i \widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}^1_i} + \sum_{i=1}^{3} \lambda_{i+3} \widetilde{\boldsymbol{\nabla}_s \boldsymbol{r}^1_i},$$

which is a contradiction with the fact that the matrix fields $\widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}^1_i}$ and $\widetilde{\boldsymbol{\nabla}_s \boldsymbol{r}^1_i}$ are linearly independent, this ends the proof. $\square$

Theorem 4.2.11. *i) Assume that Ω is of class $\mathcal{C}^{1,1}$. Then space $\mathbb{X}^p_{N,s}(\Omega)$ is continuously embedded in $\mathbb{W}^{1,p}(\Omega)$.*
ii) If Ω is of class $\mathcal{C}^{2,1}$, then the space $\mathbb{Y}^p_{N,s}(\Omega)$ is continuously embedded in $\mathbb{W}^{2,p}(\Omega)$.

Proof. **i)** Let $\boldsymbol{A} \in \mathbb{X}^p_{N,s}(\Omega)$ and $\boldsymbol{S} = \mathbf{Curl\,Curl}\,\boldsymbol{A}$. So, we have $\mathbf{Div}\,\boldsymbol{S} = \mathbf{0}$ in Ω and for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\overline{\Omega})$,

$$\begin{aligned} {}_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\langle \boldsymbol{S}\boldsymbol{n},\, \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)} &= \int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{A} : \boldsymbol{\nabla}_s \boldsymbol{\varphi}\, dx \\ &= \int_\Omega \boldsymbol{A} : \mathbf{Curl\,Curl}\,\boldsymbol{\nabla}_s \boldsymbol{\varphi}\, dx \\ &= 0. \end{aligned}$$

Then, $\boldsymbol{Sn} = \boldsymbol{0}$ on Γ. According to Lemma 4.1.4, the quantities $\langle \boldsymbol{Sn}, \boldsymbol{e}^i\rangle_{\Sigma_j}$ and $\langle \boldsymbol{Sn}, \boldsymbol{P}^i\rangle_{\Sigma_j}$ make sense for any $1 \leq i \leq 3$ and $1 \leq j \leq J$ and furthermore, from Lemma 4.2.10 we get

$$\langle \boldsymbol{Sn}, \boldsymbol{e}^i\rangle_{\Sigma_j} = \int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{A} : \widetilde{\boldsymbol{\nabla}_s \boldsymbol{t}_i^j}\, dx.$$

As the matrix $\boldsymbol{A}$ belongs to $\mathbb{H}^p_{0,s}(\mathbf{Curl\,Curl},\, \Omega)$, then there exists a sequence $(\boldsymbol{A}_n)$ in $\mathbb{D}_s(\Omega)$ which converges to $\boldsymbol{A}$ in $\mathbb{H}^p_s(\mathbf{Curl\,Curl},\, \Omega)$. Then,

$$\langle \boldsymbol{Sn}, \boldsymbol{e}^i\rangle_{\Sigma_j} = \int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{A} : \widetilde{\boldsymbol{\nabla}_s \boldsymbol{t}_i^j}\, dx = \lim_n \int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{A}_n : \widetilde{\boldsymbol{\nabla}_s \boldsymbol{t}_i^j}\, dx = 0.$$

By the same, we conclude that

$$\langle \boldsymbol{Sn}, \boldsymbol{P}^i\rangle_{\Sigma_j} = 0$$

and then, the matrix $\boldsymbol{S}$ belongs to the space $\mathbb{U}^{0,p}_s(\Omega)$. In Chapter 3, we have shown that the operator

$$\mathbf{Curl\,Curl} :\ \mathbb{W}^{2,p}_{0,s}(\Omega) \longrightarrow \mathbb{U}^{0,p}_s(\Omega) \tag{4.2.26}$$

is onto. Then, there exists $\boldsymbol{B} \in \mathbb{W}^{2,p}_{0,s}(\Omega)$ such that $\mathbf{Curl\,Curl}\,\boldsymbol{B} = \boldsymbol{S}$ in Ω and we have the estimate

$$\|\boldsymbol{B}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C\|\mathbf{Curl\,Curl}\,\boldsymbol{A}\|_{\mathbb{L}^p(\Omega)}. \tag{4.2.27}$$

Setting now $\boldsymbol{D} = \boldsymbol{A} - \boldsymbol{B}$ and let us consider the solution $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ of the problem

$$\begin{cases} \mathbf{Div}(\boldsymbol{\nabla}_s \boldsymbol{v}) = \mathbf{Div}\,\boldsymbol{D} & \text{in} \quad \Omega, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on} \quad \Gamma. \end{cases}$$

which satisfies the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C\|\mathbf{Div}\,\boldsymbol{D}\|_{\boldsymbol{L}^p(\Omega)} \leq C(\|\mathbf{Div}\,\boldsymbol{A}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{Curl\,Curl}\,\boldsymbol{A}\|_{\mathbb{L}^p(\Omega)}). \tag{4.2.28}$$

Also, for all $\boldsymbol{M} \in \mathbb{W}^{2,p'}_s(\Omega)$, we have

$$\int_\Omega \boldsymbol{\nabla}_s \boldsymbol{v} : \mathbf{Curl\,Curl}\,\boldsymbol{M}\, dx = {}_{\boldsymbol{W}^{-\frac{1}{p'},p'}(\Gamma)}\langle (\mathbf{Curl\,Curl}\,\boldsymbol{M})\,\boldsymbol{n},\, \boldsymbol{v}\rangle_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)} = 0.$$

Hence, Remark 4.2.3 implies that

$$\boldsymbol{\nabla}_s \boldsymbol{v} \times \boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{\nabla}_s \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma.$$

Thus, the matrix $\boldsymbol{E} = \boldsymbol{D} - \boldsymbol{\nabla}_s \boldsymbol{v}$ belongs to $\mathbb{K}_{N,s}(\Omega) \subset \mathbb{W}^{1,p}(\Omega)$ and consequently $\boldsymbol{A} \in \mathbb{W}^{1,p}(\Omega)$. Moreover, since all norms are equivalent in finite dimension, we have

$$\|\boldsymbol{E}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} \leq C\left(\|\boldsymbol{A}\|_{\mathbb{L}^p(\Omega)} + \|\mathbf{Div}\,\boldsymbol{A}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}\|_{\mathbb{L}^p(\Omega)}\right). \quad (4.2.29)$$

From (4.2.27)-(4.2.29), there exists a constant C_1 such that

$$\|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C_1\|\boldsymbol{A}\|_{\mathbb{X}^p(\Omega)}.$$

ii) Assume that Ω is of class $\mathcal{C}^{2,1}$ and $\boldsymbol{A} \in \mathbb{Y}^p_{N,s}(\Omega)$. Then $\boldsymbol{v} \in \boldsymbol{W}^{3,p}(\Omega)$ and $\boldsymbol{E} \in \mathbb{W}^{2,p}(\Omega)$. Finally we get $\boldsymbol{A} \in \mathbb{W}^{2,p}(\Omega)$ with the estimate

$$\|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C_2\|\boldsymbol{A}\|_{\mathbb{Y}^p_s(\Omega)}.$$

□

Using Theorem 4.2.11 and the fact that the embedding of $W^{1,p}(\Omega)$ in $L^p(\Omega)$ is compact, then the following result holds:

Lemma 4.2.12. *Assume that Ω is of class $\mathcal{C}^{1,1}$. Then, the embedding of $\mathbb{X}^p_{N,s}(\Omega)$ in $\mathbb{L}^p_s(\Omega)$ is compact.*

Lemma 4.2.12 together with Peetre-Tartar theorem, allow us to prove the following corollary.

Corollary 4.2.13. *Assume that Ω is of class $\mathcal{C}^{1,1}$. On the space $\mathbb{X}^p_{N,s}(\Omega)$, the semi-norm*

$$\boldsymbol{S} \longrightarrow \|\mathbf{Div}\,\boldsymbol{S}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \sum_{i=1}^{3}\sum_{k=1}^{I}\left(|\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{e}^i\rangle_{\Gamma_k}| + |\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Gamma_k}|\right), \quad (4.2.30)$$

is equivalent to the norm $\|\cdot\|_{\mathbb{W}^{1,p}(\Omega)}$. In particular, we have the following Friedrich's inequality type for every matrix $\boldsymbol{S} \in \mathbb{X}^p_{N,s}(\Omega)$:

$$\|\boldsymbol{S}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C(\|\mathbf{Div}\,\boldsymbol{S}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \sum_{i=1}^{3}\sum_{k=1}^{I}(|\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{e}^i\rangle_{\Gamma_k}| + |\langle \boldsymbol{S}\boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Gamma_k}|)). \quad (4.2.31)$$

Moreover, if Ω *is of class* $\mathcal{C}^{2,1}$*, then the semi-norm:*

$$\boldsymbol{S} \longrightarrow \|\mathbf{Div}\, \boldsymbol{S}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \sum_{i=1}^{3}\sum_{k=1}^{I} \left(|\langle \boldsymbol{Sn}, \boldsymbol{e}^i\rangle_{\Gamma_k}| + |\langle \boldsymbol{Sn}, \boldsymbol{P}^i\rangle_{\Gamma_k}| \right), \quad (4.2.32)$$

is equivalent to the norm $\|\cdot\|_{\mathbb{W}^{2,p}(\Omega)}$.

Now, we show our third extension of Beltrami's completeness:

Theorem 4.2.14. *Assume that* Ω *is of class* $\mathcal{C}^{1,1}$*. Then a matrix* $\boldsymbol{S}$ *in* $\mathbb{L}^p_s(\Omega)$ *satisfies*

$$\begin{aligned} \mathbf{Div}\, \boldsymbol{S} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \boldsymbol{Sn} &= \mathbf{0} \quad \text{on} \quad \Gamma, \\ \langle \boldsymbol{Sn},\, \boldsymbol{e}^i\rangle_{\Sigma_j} = \langle \boldsymbol{Sn},\, \boldsymbol{P}^i\rangle_{\Sigma_j} &= 0, \quad \text{for any} \;\; 1 \le i \le 3,\ \forall\, 1 \le j \le J, \end{aligned} \quad (4.2.33)$$

if and only if there exists a matrix $\boldsymbol{A} \in \mathbb{Y}^p_s(\Omega)$ *such that*

$$\begin{aligned} \mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{A} = \boldsymbol{S} \text{ in} \quad \Omega, &\qquad \mathbf{Div}\, \boldsymbol{A} = \mathbf{0} \quad \text{in} \quad \Omega, \\ \boldsymbol{A} \times \boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma, &\qquad \mathbf{Curl}\, \boldsymbol{A} \times \boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma, \\ \langle \boldsymbol{Sn},\, \boldsymbol{e}^i\rangle_{\Gamma_k} = \langle \boldsymbol{Sn},\, \boldsymbol{P}^i\rangle_{\Gamma_k} = 0, &\qquad \text{for any} \;\; 1 \le i \le 3,\ \forall\, 1 \le k \le I. \end{aligned} \quad (4.2.34)$$

Moreover, this matrix $\boldsymbol{A}$ *is unique and satisfies the estimate*

$$\|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)} \le C_1 \|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \quad (4.2.35)$$

In addition, if Ω *is of class* $\mathcal{C}^{2,1}$*, then* $\boldsymbol{A} \in \mathbb{W}^{2,p}_s(\Omega)$ *and we have the estimate*

$$\|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} \le C_2 \|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \quad (4.2.36)$$

Proof. Let $\boldsymbol{A}$ be in $\mathbb{Y}^p_s(\Omega)$ and satisfies (4.2.34)). The matrix $\boldsymbol{S} = \mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{A}$ satisfies (4.2.33) (see proof of Theorem 4.2.11). Conversely, let $\boldsymbol{S}$ be in $\mathbb{H}^p(\mathbf{Div}, \Omega)$ and satisfies (4.2.33). $\boldsymbol{A}_0$ is the matrix field in $\mathbb{W}^{2,p}_s(\Omega)$ given by Theorem 4.1.10. Due to Lemma 4.2.6, the following problem:

find $\boldsymbol{E} \in \mathbb{V}^p_{T,s}(\Omega)$ such that for any $\boldsymbol{\Psi} \in \mathbb{V}^{p'}_{T,s}(\Omega)$,

$$\int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{E} : \mathbf{Curl\,Curl}\,\boldsymbol{\Psi}\,dx = \int_\Omega \boldsymbol{A}_0 : \mathbf{Curl\,Curl}\,\boldsymbol{\Psi}\,dx - \int_\Omega \boldsymbol{\Psi} : \mathbf{Curl\,Curl}\,\boldsymbol{A}_0\,dx, \tag{4.2.37}$$

has a unique solution.

We want to extend (4.2.37) to any matrix test in $\mathbb{X}^{p'}_s(\Omega)$. For that, let $\widetilde{\boldsymbol{\Psi}} \in \mathbb{X}^{p'}_s(\Omega)$ and $\boldsymbol{v}$ be the solution of the problem

$$\begin{cases} \mathbf{Div}(\boldsymbol{\nabla}_s\boldsymbol{v}) = \mathbf{Div}\,\widetilde{\boldsymbol{\Psi}} & \text{in} \quad \Omega, \\ (\boldsymbol{\nabla}_s\boldsymbol{v} - \widetilde{\boldsymbol{\Psi}})\,\boldsymbol{n} = \mathbf{0} & \text{on} \quad \Gamma. \end{cases}$$

We can check that the matrix defined by

$$\boldsymbol{\Psi} = \widetilde{\boldsymbol{\Psi}} - \boldsymbol{\nabla}_s\boldsymbol{v} - \sum_{i=1}^3\sum_{j=1}^J \left(\langle(\widetilde{\boldsymbol{\Psi}} - \boldsymbol{\nabla}_s\boldsymbol{v})\boldsymbol{n},\, \boldsymbol{e}^i\rangle_{\Sigma_j} \widetilde{\boldsymbol{\nabla}_s\boldsymbol{u}_i^j} + \langle(\widetilde{\boldsymbol{\Psi}} - \boldsymbol{\nabla}_s\boldsymbol{v})\boldsymbol{n},\, \boldsymbol{P}^i\rangle_{\Sigma_j} \widetilde{\boldsymbol{\nabla}_s\boldsymbol{r}_i^j} \right).$$

belongs to $\mathbb{V}^{p'}_{T,s}(\Omega)$. From (4.2.33), we have

$$\int_\Omega \boldsymbol{\nabla}_s\boldsymbol{v} : \mathbf{Curl\,Curl}\,\boldsymbol{A}_0\,dx = \langle \boldsymbol{S}\boldsymbol{n},\, \boldsymbol{v}\rangle - \int_\Omega \boldsymbol{v}\cdot\mathbf{Div}\,\boldsymbol{S}\,dx = 0,$$

$$\int_\Omega \widetilde{\boldsymbol{\nabla}_s\boldsymbol{u}_i^j} : \mathbf{Curl\,Curl}\,\boldsymbol{A}_0\,dx = \int_{\Omega^\circ} \boldsymbol{\nabla}_s\boldsymbol{u}_i^j : \boldsymbol{S}\,dx = \sum_{k=1}^I \langle \boldsymbol{S}\boldsymbol{n},\, [\boldsymbol{u}_i^j]_k\rangle_{\Sigma_k} = 0,$$

and

$$\int_\Omega \widetilde{\boldsymbol{\nabla}_s\boldsymbol{r}_i^j} : \mathbf{Curl\,Curl}\,\boldsymbol{A}_0\,dx = \int_{\Omega^\circ} \boldsymbol{\nabla}_s\boldsymbol{r}_i^j : \boldsymbol{S}\,dx = \sum_{k=1}^I \langle \boldsymbol{S}\boldsymbol{n},\, [\boldsymbol{r}_i^j]_k\rangle_{\Sigma_k} = 0.$$

Then, for all $\widetilde{\boldsymbol{\Psi}} \in \mathbb{X}^{p'}_s(\Omega)$, we have

$$\int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{E} : \mathbf{Curl\,Curl}\,\widetilde{\boldsymbol{\Psi}}\,dx = \int_\Omega \boldsymbol{A}_0 : \mathbf{Curl\,Curl}\,\widetilde{\boldsymbol{\Psi}}\,dx - \int_\Omega \widetilde{\boldsymbol{\Psi}} : \mathbf{Curl\,Curl}\,\boldsymbol{A}_0\,dx.$$

It follows from this relation that the matrix

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A}_0 - \mathbf{Curl\,Curl}\,\boldsymbol{E} - \sum_{i=1}^3\sum_{k=1}^I \big(\langle(\boldsymbol{A}_0 - \mathbf{Curl\,Curl}\,\boldsymbol{E})\,\boldsymbol{n},\, \boldsymbol{e}^i\rangle_{\Gamma_k} \boldsymbol{\nabla}_s\boldsymbol{v}_i^k \\ &+ \langle(\boldsymbol{A}_0 - \mathbf{Curl\,Curl}\,\boldsymbol{E})\,\boldsymbol{n},\, \boldsymbol{P}^i\rangle_{\Gamma_k} \boldsymbol{\nabla}_s\boldsymbol{w}_i^k\big), \end{aligned}$$

satisfies (4.2.34). Hence, $\boldsymbol{A} \in \mathbb{Y}^p_{N,s}(\Omega) \subset \mathbb{W}^{1,p}_s(\Omega)$ and due to Corollary 4.2.13 the estimate (4.2.35) holds true. If Ω is of class $\mathcal{C}^{2,1}$, then Theorem 4.2.11 implies that $\boldsymbol{A} \in \mathbb{W}^{2,p}_s(\Omega)$ and Corollary 4.2.13 implies that the estimate (4.2.36) holds true, also.
The uniqueness of $\boldsymbol{A}$ is due to the characterization of the space $\mathbb{K}_{N,s}(\Omega)$. □

Remark 4.2.15. We can give another proof of Theorem 4.2.14. Indeed, Let $\boldsymbol{S}$ be in $\mathbb{H}^p_s(\mathbf{Div},\,\Omega)$ and satisfies (4.2.33). As the operator (4.2.26) is onto, then there exists $\boldsymbol{B} \in \mathbb{W}^{2,p}_{0,s}(\Omega)$ such that $\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{B} = \boldsymbol{S}$ in Ω. Setting now $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(\Omega)$ the solution of the homogeneous Dirichlet problem $\mathbf{Div}\,\boldsymbol{\nabla}_s\boldsymbol{v} = \mathbf{Div}\,\boldsymbol{B}$ in Ω. We define the matrix $\boldsymbol{A}$ by

$$\boldsymbol{A} = \boldsymbol{B} - \boldsymbol{\nabla}_s\boldsymbol{v} - \sum_{i=1}^{3}\sum_{k=1}^{I}\left[\langle(\boldsymbol{B} - \boldsymbol{\nabla}_s\boldsymbol{v})\,\boldsymbol{n},\,\boldsymbol{e}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{v}^k_i + \langle(\boldsymbol{B} - \boldsymbol{\nabla}_s\boldsymbol{v})\,\boldsymbol{n},\,\boldsymbol{P}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{w}^k_i\right].$$

Note that $\boldsymbol{A}$ belongs to $\mathbb{Y}^p_s(\Omega)$ and satisfies (4.2.34). Therefore $\boldsymbol{A}$ belongs to $\mathbb{W}^{1,p}_s(\Omega)$ and Corollary 4.2.13 implies that the estimate (4.2.35) is valid. Moreover, if Ω is of class $\mathcal{C}^{2,1}$, then $\boldsymbol{A}$ belongs to $\mathbb{W}^{2,p}_s(\Omega)$ and Corollary 4.2.13 implies that the estimate (4.2.36) is true.

4.3 Beltrami's type decomposition

In this section, we will use the previous extensions of Beltrami's completeness to show three versions of Beltrami's type decomposition for matrix fields in $\mathbb{L}^p_s(\Omega)$.

Theorem 4.3.1. *Assume that Ω is of class $\mathcal{C}^{1,1}$.*
i) Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$, then there exist $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$, $\boldsymbol{A} \in \mathbb{W}^{1,p}_{\sigma,s}(\Omega) \cap \mathbb{Y}^p_{N,s}(\Omega)$ and $\boldsymbol{E} \in \mathbb{K}_{T,s}(\Omega)$ such that

$$\boldsymbol{S} = \boldsymbol{\nabla}_s\boldsymbol{v} + \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} + \boldsymbol{E}, \tag{4.3.1}$$

where $\boldsymbol{v}$ is unique up to an additive rigid displacement, $\boldsymbol{A}$ is unique up to an element of $\mathbb{K}_{N,s}(\Omega)$, and $\boldsymbol{E}$ is unique, in addition, we have the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{R}(\Omega)} + \|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)/\mathbb{K}_{N,s}(\Omega)} + \|\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \tag{4.3.2}$$

Moreover, if Ω *is of class* $\mathcal{C}^{2,1}$*, thus* $\boldsymbol{A} \in \mathbb{W}^{2,p}_{\sigma,s}(\Omega) \cap \mathbb{Y}^{p}_{N,s}(\Omega)$ *and we have the estimate*

$$\|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)/\mathbb{K}_{N,s}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \tag{4.3.3}$$

ii) Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$*, thus there exist* $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$ *,* $\boldsymbol{A} \in \mathbb{W}^{2,p}_{0,s}(\Omega)$ *and* $\boldsymbol{E} \in \mathbb{K}_{T,s}(\Omega)$ *such that*

$$\boldsymbol{S} = \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl\,Curl}\,\boldsymbol{A} + \boldsymbol{E}, \tag{4.3.4}$$

where $\boldsymbol{v}$ *is unique up to an additive rigid displacement,* $\boldsymbol{A}$ *is an unique element of* $\mathbb{W}^{2,p}_s(\Omega)$*,* $\boldsymbol{E}$ *is unique and we have the estimate*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{R}(\Omega)} + \|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} + \|\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \tag{4.3.5}$$

iii) Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$*, then there exist* $\boldsymbol{v} \in \boldsymbol{W}^{1,p}_0(\Omega)$*,* $\boldsymbol{A} \in \mathbb{W}^{1,p}_{\sigma,s}(\Omega) \cap \mathbb{X}^p_{T,s}(\Omega)$ *and* $\boldsymbol{E} \in \mathbb{K}_{N,s}(\Omega)$ *such that*

$$\boldsymbol{S} = \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl\,Curl}\,\boldsymbol{A} + \boldsymbol{E}, \tag{4.3.6}$$

where $\boldsymbol{v}$ *and* $\boldsymbol{E}$ *are unique,* $\boldsymbol{A}$ *is unique to an additive element of* $\mathbb{K}_{T,s}(\Omega)$ *and we have the estimate*

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)/\mathbb{K}_{T,s}(\Omega)} + \|\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \tag{4.3.7}$$

Moreover, if Ω *is of class* $\mathcal{C}^{2,1}$*, then* $\boldsymbol{A} \in \mathbb{W}^{2,p}_{\sigma,s}(\Omega) \cap \mathbb{X}^p_{T,s}(\Omega)$*; and thus we have the estimate*

$$\|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)/\mathbb{K}_{T,s}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}. \tag{4.3.8}$$

Proof. i) Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$. We set $\boldsymbol{v}$ the unique solution in $\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{R}(\Omega)$ of the problem

$$\begin{cases} \mathbf{Div}\,\boldsymbol{\nabla}_s \boldsymbol{v} = \mathbf{Div}\,\boldsymbol{S} \ \text{ in } \Omega, \\ (\boldsymbol{\nabla}_s \boldsymbol{v} - \boldsymbol{S})\,\boldsymbol{n} = \mathbf{0} \ \text{ on } \Gamma. \end{cases}$$

The vector $\boldsymbol{v}$ is unique up to an rigid displacement and we have the estimate

$$\|\boldsymbol{v}\|_{\mathbb{W}^{1,p}(\Omega)/\boldsymbol{R}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}.$$

Set $\boldsymbol{B} = \boldsymbol{S} - \boldsymbol{\nabla}_s \boldsymbol{v}$, $\boldsymbol{B}$ satisfies $\mathbf{Div}\,\boldsymbol{B} = \mathbf{0}$ in Ω and $\boldsymbol{B}\boldsymbol{n} = \mathbf{0}$ on Γ. We define the matrix $\boldsymbol{E}$ by

$$\boldsymbol{E} = \sum_{i=1}^{3}\sum_{j=1}^{J} \langle \boldsymbol{B}\boldsymbol{n}, \boldsymbol{e}^i\rangle_{\Sigma_j} \widetilde{\boldsymbol{\nabla}_s \boldsymbol{r}_i^j} + \langle \boldsymbol{B}\boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Sigma_j} \widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}_i^j}.$$

Note that the matrix $\boldsymbol{C}$ defined by $\boldsymbol{C} = \boldsymbol{B} - \boldsymbol{E}$ satisfies the compatibility condition (4.2.33), hence Theorem 4.2.14 implies that there exists $\boldsymbol{A} \in \mathbb{W}^{1,p}_{\sigma,s}(\Omega) \cap \mathbb{Y}^p_{N,s}(\Omega)$ such that

$$\boldsymbol{C} = \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} \quad \text{and} \quad \|\boldsymbol{A}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{C}\|_{\mathbb{L}^p(\Omega)}.$$

So

$$\boldsymbol{S} = \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} + \boldsymbol{E},$$

and the estimate (4.3.2) holds true. Moreover, if Ω is of class $\mathcal{C}^{2,1}$, then Theorem 4.2.14 implies that $\boldsymbol{A} \in \mathbb{W}^{2,p}_{\sigma,s}(\Omega) \cap \mathbb{Y}^p_{N,s}(\Omega)$ and satisfies

$$\|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C\|\boldsymbol{C}\|_{\mathbb{L}^p(\Omega)}.$$

Consequently, the estimate (4.3.3) is valid.
ii) Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$ and $\boldsymbol{C}$ be defined as in the proof of the point i). Note that $\boldsymbol{C} \in \mathbb{U}^{0,p}_s(\Omega)$, then there exists $\boldsymbol{A} \in \mathbb{W}^{2,p}_{0,s}(\Omega)$ such that

$$\boldsymbol{C} = \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} \quad \text{and} \quad \|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C\|\boldsymbol{C}\|_{\mathbb{L}^p(\Omega)}.$$

Henceforth

$$\boldsymbol{S} = \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} + \boldsymbol{E},$$

and the estimate (4.3.5) is valid.
iii) Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$. Denote by, $\boldsymbol{v}$ the unique solution in $\mathbb{W}^{1,p}_0(\Omega)$ of the Dirichlet problem

$$\begin{cases} \mathbf{Div}\,\boldsymbol{\nabla}_s \boldsymbol{v} = \mathbf{Div}\,\boldsymbol{S} & \text{in} \quad \Omega, \\ \boldsymbol{v} = \mathbf{0} & \text{on} \quad \Gamma. \end{cases}$$

Then, $\boldsymbol{B} = \boldsymbol{S} - \boldsymbol{\nabla}_s \boldsymbol{v}$ satisfies $\mathbf{Div}\,\boldsymbol{B} = \mathbf{0}$.

We set

$$\boldsymbol{E} = \sum_{i=1}^{3}\sum_{k=1}^{I} \langle \boldsymbol{B}\boldsymbol{n}, \boldsymbol{e}^i\rangle_{\Gamma_k} \boldsymbol{\nabla}_s \boldsymbol{v}_i^k + \langle \boldsymbol{B}\boldsymbol{n}, \boldsymbol{P}^i\rangle_{\Gamma_k} \boldsymbol{\nabla}_s \boldsymbol{w}_i^k.$$

So, the matrix $\boldsymbol{C} = \boldsymbol{B} - \boldsymbol{E}$ satisfies the compatibility conditions (4.1.14)-(4.1.16), then Theorem (4.1.16) implies that there exists a unique matrix $\boldsymbol{A} \in \mathbb{W}^{1,p}_{\sigma,s}(\Omega) \cap \mathbb{Y}^{p}_{T,s}(\Omega)/\mathbb{K}_{T,s}(\Omega)$ such that $\boldsymbol{C} = \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}$. Consequently,

$$\boldsymbol{S} = \boldsymbol{\nabla}_s \boldsymbol{v} + \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} + \boldsymbol{E},$$

and the estimate (4.3.7) is valid. Moreover, if Ω is of class $\mathcal{C}^{2,1}$, then $\boldsymbol{A} \in \mathbb{W}^{2,p}_{\sigma,s}(\Omega)$ and the estimate (4.3.8) is true. □

Remark 4.3.2. *The decomposition 4.3.4 is still true if $p = 2$ and Ω is only Lipschitz.*

Geymonat et al [32] have shown a Hodge decomposition of $\mathbb{L}^2_s(\Omega)$. Here, we will show a Hodge decomposition of $\mathbb{L}^p_s(\Omega)$ when $1 < p < \infty$.

Corollary 4.3.3. *Assume that Ω is of class $\mathcal{C}^{1,1}$, then the following direct sum is true:*

$$\mathbb{L}^p_s(\Omega) = \mathbb{K}_{N,s}(\Omega) \oplus \mathbb{H}^p_{1,s}(\Omega) \oplus \mathbb{H}^p_{2,s}(\Omega) \oplus \mathbb{K}_{T,s}(\Omega) \oplus \mathbb{U}^{0,p}_s(\Omega), \tag{4.3.9}$$

where

$$\mathbb{H}^p_{1,s}(\Omega) = \{\boldsymbol{\nabla}_s \boldsymbol{v},\ \boldsymbol{v} \in \boldsymbol{W}^{1,p}_0(\Omega)\},$$

$$\mathbb{H}^p_{2,s}(\Omega) = \{\boldsymbol{\nabla}_s \boldsymbol{v},\ \boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega),\ \mathbf{Div}\,\boldsymbol{\nabla}_s \boldsymbol{v} = \mathbf{0},\ \langle(\boldsymbol{\nabla}_s \boldsymbol{v})\, n,\ \boldsymbol{e}^i\rangle_{\Gamma_k} = \langle(\boldsymbol{\nabla}_s \boldsymbol{v})\, n,\ \boldsymbol{P}^i\rangle_{\Gamma_k} = 0\}.$$

Proof. Let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$, then Theorem 4.3.1 implies that $\boldsymbol{S}$ is composed as in (4.3.6). We set $\boldsymbol{w}$ the solution of the problem

$$\begin{cases} \mathbf{Div}\,\boldsymbol{\nabla}_s \boldsymbol{w} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ (\boldsymbol{\nabla}_s \boldsymbol{w})\, \boldsymbol{n} &= (\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A})\, \boldsymbol{n} \quad \text{on} \quad \Gamma, \end{cases}$$

Then, for any $1 \le i \le 3$ and $1 \le k \le I$, we have

$$\begin{cases} \langle(\boldsymbol{\nabla}_s \boldsymbol{w})\, \boldsymbol{n},\ \boldsymbol{e}^i\rangle_{\Gamma_k} = \langle(\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A})\, \boldsymbol{n},\ \boldsymbol{e}^i\rangle_{\Gamma_k} = 0, \\ \langle(\boldsymbol{\nabla}_s \boldsymbol{w})\, \boldsymbol{n},\ \boldsymbol{P}^i\rangle_{\Gamma_k} = \langle(\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A})\, \boldsymbol{n},\ \boldsymbol{P}^i\rangle_{\Gamma_k} = 0. \end{cases}$$

So, $\boldsymbol{\nabla}_s \boldsymbol{w} \in \mathbb{H}_{2,s}(\Omega)$. Setting now, $\boldsymbol{C} = \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A} - \boldsymbol{\nabla}_s \boldsymbol{u}$ which belongs to $\mathbb{H}^p_{0,s}(\mathbf{Div},\,\Omega)$. Therefore,

$$\begin{aligned}\boldsymbol{C} &= \left(\boldsymbol{C} - \sum_{i=1}^{3}\sum_{j=1}^{J}\left[\langle \boldsymbol{Cn},\, \boldsymbol{e}^i\rangle_{\Sigma_j}\widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}^i_j} + \langle \boldsymbol{Cn},\, \boldsymbol{P}^i\rangle_{\Sigma_j}\widetilde{\boldsymbol{\nabla}_s \boldsymbol{r}^i_j}\right]\right)\\ &+ \sum_{i=1}^{3}\sum_{j=1}^{J}\left[\langle \boldsymbol{Cn},\, \boldsymbol{e}^i\rangle_{\Sigma_j}\widetilde{\boldsymbol{\nabla}_s \boldsymbol{u}^i_j} + \langle \boldsymbol{Cn},\, \boldsymbol{P}^i\rangle_{\Sigma_j}\widetilde{\boldsymbol{\nabla}_s \boldsymbol{r}^i_j}\right]\\ &= \boldsymbol{D} + \boldsymbol{F},\end{aligned}$$

where $\boldsymbol{D}$ is unique in $\mathbb{U}^{0,p}_s(\Omega)$ and $\boldsymbol{F}$ is unique in $\mathbb{K}_{T,s}(\Omega)$. Henceforth,

$$\boldsymbol{S} = \boldsymbol{E} + \boldsymbol{\nabla}_s \boldsymbol{v} + \boldsymbol{\nabla}_s \boldsymbol{w} + \boldsymbol{D} + \boldsymbol{F},$$

which is the required result. □

4.4 The bi-Laplacian problem for symmetric matrix with normal boundary conditions

In section (4.2), we have used the inf-sup condition (4.2.12) to solve the elliptic problem (4.2.16) in $\mathbb{V}^p_{T,s}(\Omega)$. Here, we will use a similar argument to solve the following bi-Laplacian problem

$$\begin{cases} \triangle^2\,\boldsymbol{E} + \boldsymbol{\nabla}_s \boldsymbol{v} = \boldsymbol{B} \quad \text{and} \quad \mathbf{Div}\,\boldsymbol{E} = \boldsymbol{0} \quad \text{in} \quad \Omega,\\ \boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{D}\times\boldsymbol{n}, \quad \boldsymbol{v} = \boldsymbol{v}_0 \quad \text{on} \quad \Gamma,\\ \mathbf{Curl}\,\boldsymbol{E}\times\boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n} \quad \text{on} \quad \Gamma,\\ \langle \boldsymbol{Sn},\, \boldsymbol{e}^i\rangle_{\Gamma_k} = \langle \boldsymbol{Sn},\, \boldsymbol{P}^i\rangle_{\Gamma_k} = 0, \quad 1\le k\le I, \quad 1\le i\le 3, \end{cases} \tag{4.4.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{v}$ are unknowns, $\boldsymbol{B}$, $\boldsymbol{v}_0$ and $\boldsymbol{D}$ are are given data. This problem represents a matrix analog of Stokes problem with pressure boundary conditions (see [9]) , where the Laplacian operator is replaced by the bi-Laplacian ones and the gradient operator is replaced by the linearised strain tensor.

Using Corollary 4.2.13 and similar argument as in proof of Lemma 4.2.6, we can establish the following inf-sup condition:

Lemma 4.4.1. *Assume that* Ω *is of class* $\mathcal{C}^{1,1}$*. Then, the following inf-sup condition holds true: there exists a constant* $\beta > 0$*, such that*

$$\inf_{\substack{\boldsymbol{\Psi}\in\mathbb{V}^{p'}_{N,s}(\Omega),\\ \boldsymbol{\Psi}\neq\mathbf{0}}} \sup_{\substack{\boldsymbol{E}\in\mathbb{V}^{p}_{N,s}(\Omega),\\ \boldsymbol{E}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{E} : \mathbf{Curl\,Curl}\,\boldsymbol{\Psi}\,dx}{\|\boldsymbol{E}\|_{\mathbb{X}^p_s(\Omega)}\|\boldsymbol{\Psi}\|_{\mathbb{X}^{p'}_s(\Omega)}} \geq \beta, \tag{4.4.2}$$

where

$$\mathbb{V}^p_{N,s}(\Omega) = \{\boldsymbol{S}\in\mathbb{X}^p_{N,s}(\Omega),\ \mathbf{Div}\,\boldsymbol{S} = \mathbf{0} \quad in \quad \Omega \quad and \quad \{\boldsymbol{Sn},\,\boldsymbol{e}^i\rangle_{\Gamma_i} = \{\boldsymbol{Sn},\,\boldsymbol{P}^i\rangle_{\Gamma_i} = 0\}.$$

The inf-sup condition (4.4.2) allows us to solve the following elliptic problem.

Theorem 4.4.2. *Assume that* Ω *is of class* $\mathcal{C}^{1,1}$ *and* $\boldsymbol{B}\in(\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\,\Omega))'$ *such that* $\mathbf{Div}\,\boldsymbol{B} = \mathbf{0}$ *in* Ω *and satisfying the following compatibility condition*

$$\forall\,\boldsymbol{M}\in\mathbb{K}_{N,s}(\Omega),\quad \langle\boldsymbol{B},\,\boldsymbol{M}\rangle_{[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\Omega)]'\times[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\Omega)]} = 0. \tag{4.4.3}$$

Then, the problem

$$\begin{cases} \triangle^2\boldsymbol{E} = \boldsymbol{B} \quad and \quad \mathbf{Div}\,\boldsymbol{E} = \mathbf{0} \quad in\,\Omega,\\ \boldsymbol{E}\times\boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{E}\times\boldsymbol{n} = \mathbf{0} \quad on\,\Gamma,\\ \langle\boldsymbol{En},\,\boldsymbol{e}^i\rangle_{\Gamma_k} = \langle\boldsymbol{En},\,\boldsymbol{P}^i\rangle_{\Gamma_k} = 0,\quad 1\leq i\leq 3,\quad 1\leq k\leq I. \end{cases} \tag{4.4.4}$$

has a unique solution in $\mathbb{W}^{1,p}_s(\Omega)$ *which satisfies the following estimate*

$$\|\boldsymbol{E}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C_1\|\boldsymbol{B}\|_{(\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\Omega))'}. \tag{4.4.5}$$

Moreover, if Ω *is of class* $\mathcal{C}^{2,1}$*, then* $\boldsymbol{E}\in\mathbb{W}^{2,p}_s(\Omega)$ *and*

$$\|\boldsymbol{E}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C_2\|\boldsymbol{B}\|_{(\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\Omega))'}. \tag{4.4.6}$$

Remark 4.4.3. Using the same argument has in proof of [15, *Proposition* 8.14], we can show that for any matrix $\boldsymbol{S}$ of $[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\,\Omega)]'$, there exist two matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ of $\mathbb{L}^p_s(\Omega)$ such that

$$\boldsymbol{S} = \boldsymbol{A} + \mathbf{Curl\,Curl}\,\boldsymbol{B}.$$

Proof. Due to the inf-sup condition of Lemma 4.4.1, the problem:

find $\boldsymbol{E} \in \mathbb{V}^p_{N,s}(\Omega)$ such that for all $\boldsymbol{\Psi} \in \mathbb{V}^{p'}_{N,s}(\Omega)$,

$$\int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{E} : \mathbf{Curl\,Curl}\,\boldsymbol{\Psi}\,dx = \langle \boldsymbol{B},\,\boldsymbol{\Psi}\rangle_{[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\,\Omega)]'\times[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\,\Omega)]} \tag{4.4.7}$$

has a unique solution $\boldsymbol{E} \in \mathbb{V}^p_{N,s}(\Omega) \subset \mathbb{W}^{1,p}(\Omega)$.

Let $\widetilde{\boldsymbol{\Psi}} \in \mathbb{X}^{p'}_{N,s}(\Omega)$ and let $\boldsymbol{v}$ be the unique solution in $\mathbb{W}^{2,p}_s(\Omega)\cap\mathbb{W}^{1,p}_0(\Omega)$ satisfying $\mathbf{Div}\,\boldsymbol{\nabla}_s\boldsymbol{v} = \mathbf{Div}\,\widetilde{\boldsymbol{\Psi}}$ in Ω. Setting

$$\boldsymbol{\Psi} = \widetilde{\boldsymbol{\Psi}} - \boldsymbol{\nabla}_s\boldsymbol{v} - \Big(\sum_{i=1}^{3}\sum_{k=1}^{I}\langle(\widetilde{\boldsymbol{\Psi}} - \boldsymbol{\nabla}_s\boldsymbol{v})\,\boldsymbol{n},\,\boldsymbol{e}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{v}^k_i + \langle(\widetilde{\boldsymbol{\Psi}} - \boldsymbol{\nabla}_s\boldsymbol{v})\,\boldsymbol{n},\,\boldsymbol{P}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{w}^k_i\Big)$$

we see that $\boldsymbol{\Psi} \in \mathbb{V}^{p'}_{N,s}(\Omega)$ and using the compatibility condition (4.4.3), the problem (4.4.7) becomes:

For all $\widetilde{\boldsymbol{\Psi}} \in \mathbb{X}^{p'}_{N,s}(\Omega)$,

$$\int_\Omega \mathbf{Curl\,Curl}\,\boldsymbol{E} : \mathbf{Curl\,Curl}\,\widetilde{\boldsymbol{\Psi}}\,dx = \langle \boldsymbol{B},\,\widetilde{\boldsymbol{\Psi}}\rangle_{[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\,\Omega)]'\times[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\,\Omega)]}$$

which is equivalent with the problem (4.4.4). Then, the problem (4.4.4) has unique solution $\boldsymbol{E} \in \mathbb{W}^{1,p}(\Omega)$. Remark 4.2 iii) of [9] implies that

$$\|\mathbf{Curl\,Curl}\,\boldsymbol{E}\|_{\mathbb{L}^p(\Omega)} \le C\|\boldsymbol{B}\|_{(\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\,\Omega))'}.$$

The estimate (4.4.5) is a consequence of Corollary 4.2.13. Moreover, if Ω is of class $\mathcal{C}^{2,1}$, then $\boldsymbol{E}$ belongs to $\mathbb{W}^{2,p}_s(\Omega)$ and Corollary 4.2.13 implies that the estimate (4.4.6) holds true. □

Now, we consider the case of the inhomogeneous boundary conditions.

Corollary 4.4.4. *Assume that* Ω *is of class* $\mathcal{C}^{2,1}$. *Let* $\boldsymbol{B} \in [\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\, \Omega)]'$ *such that* $\mathbf{Div}\,\boldsymbol{B} = \mathbf{0}$ *and satisfying the compatibility condition* (4.4.3) *and* $\boldsymbol{D}$ *belongs to* $\mathbb{H}^p_s(\mathbf{Curl\,Curl},\, \Omega)$ *with* $(\boldsymbol{D}\times\boldsymbol{n},\, \mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n}) \in \mathbb{W}^{2-\frac{1}{p},p}(\Gamma)\times\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)$. *Then, the following problem*

$$\begin{cases} \triangle^2\boldsymbol{E} = \boldsymbol{B} \quad \textit{and} \quad \mathbf{Div}\,\boldsymbol{E} = \mathbf{0} \quad \textit{in}\,\Omega, \\ \boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{D}\times\boldsymbol{n} \quad \textit{and} \quad \mathbf{Curl}\,\boldsymbol{E}\times\boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n} \quad \textit{on}\,\Gamma, \\ \langle \boldsymbol{E}\boldsymbol{n},\, \boldsymbol{e}^i\rangle_{\Gamma_k} = \langle \boldsymbol{E}\boldsymbol{n},\, \boldsymbol{P}^i\rangle_{\Gamma_k} = 0, \quad 1\le i\le 3, \quad 1\le k\le I \end{cases} \tag{4.4.8}$$

has a unique solution in $\mathbb{W}^{2,p}_s(\Omega)$ *which satisfies the following estimate*

$$\|\boldsymbol{E}\|_{\mathbb{W}^{2,p}(\Omega)} \le C\left(\|\boldsymbol{B}\|_{[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl\,Curl},\,\Omega)]'} + \|\boldsymbol{D}\times\boldsymbol{n}\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)} + \|\mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}\right). \tag{4.4.9}$$

Proof. **Step 1.** We show the existence of a divergence free matrix $\boldsymbol{E}_0 \in \mathbb{W}^{2,p}_s(\Omega)$ such that $\boldsymbol{E}_0\times\boldsymbol{n} = \boldsymbol{D}\times\boldsymbol{n}$ and $\mathbf{Curl}\,\boldsymbol{E}_0\times\boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n}$.

We define the matrix $\boldsymbol{D}_\tau = (\boldsymbol{D}\times\boldsymbol{n})\times\boldsymbol{n}$ which belongs to $\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)$. Then, there exists a divergence free matrix field $\boldsymbol{A}_1 \in \mathbb{W}^{2,p}(\Omega)$ such that $\boldsymbol{A}_{1|\Gamma} = \boldsymbol{D}_\tau$ and satisfies the estimate

$$\|\boldsymbol{A}_1\|_{\mathbb{W}^{2,p}(\Omega)} \le C\|\boldsymbol{D}_\tau\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)} \le C\|\boldsymbol{D}\times\boldsymbol{n}\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)}. \tag{4.4.10}$$

Furthermore, $\boldsymbol{A}_1\times\boldsymbol{n} = \boldsymbol{D}\times\boldsymbol{n}$ on Γ. Now, we set $\boldsymbol{C} = \boldsymbol{A}_1^{sym} - \boldsymbol{D}$, we have for any $\boldsymbol{\Psi} \in \mathbb{W}^{2,p'}_s(\Omega)\cap\mathbb{W}^{1,p'}_0(\Omega)$,

$$\begin{aligned} \int_\Gamma(\boldsymbol{C}\times\boldsymbol{n}) : \mathbf{Curl}\,\boldsymbol{\Psi} &= \int_\Omega \boldsymbol{C} : \mathbf{Curl\,Curl}\,\boldsymbol{\Psi} - \int_\Omega \boldsymbol{\Psi} : \mathbf{Curl\,Curl}\,\boldsymbol{C} \\ &= \int_\Omega(\boldsymbol{A}_1 - \boldsymbol{D}) : \mathbf{Curl\,Curl}\,\boldsymbol{\Psi} - \int_\Omega \boldsymbol{\Psi} : \mathbf{Curl\,Curl}\,(\boldsymbol{A}_1 - \boldsymbol{D}) \\ &= 0. \end{aligned}$$

Then, Remark 4.2.3 implies that

$$\boldsymbol{A}_1^{sym}\times\boldsymbol{n} = \boldsymbol{D}\times\boldsymbol{n} \quad \text{on} \quad \Gamma.$$

We set $\mathbf{Curl}\,\boldsymbol{C}_\tau = (\mathbf{Curl}\,\boldsymbol{C}\times\boldsymbol{n})\times\boldsymbol{n}$ which belongs to $\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)$. As we have seen in the proof of Proposition 4.2.1, there exists $\boldsymbol{A}_2 \in \mathbb{W}^{2,p}(\Omega)\cap\mathbb{W}^{1,p}_0(\Omega)$ such that $\mathbf{Curl}\,\boldsymbol{A}_2 = \mathbf{Curl}\,\boldsymbol{C}_\tau$

and satisfies the estimate

$$\|\boldsymbol{A}_2\|_{\mathbb{W}^{2,p}(\Omega)} \le C\|\mathbf{Curl}\,\boldsymbol{C}_\tau\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)} \le C(\|\boldsymbol{D}\times\boldsymbol{n}\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)}+\|\mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}). \quad (4.4.11)$$

Furthermore, $\mathbf{Curl}\,\boldsymbol{A}_2\times\boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{C}\times\boldsymbol{n}$ on Γ. Also, for any $\boldsymbol{\Psi}\in\mathbb{W}^{2,p'}_s(\Omega)$, we have

$$\begin{aligned}\int_\Gamma(\mathbf{Curl}\,(\boldsymbol{A}_2^{sym}-\boldsymbol{C})\times\boldsymbol{n}):\boldsymbol{\Psi} &= \int_\Omega(\boldsymbol{A}_2^{sym}-\boldsymbol{C}):\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{\Psi}-\int_\Omega\boldsymbol{\Psi}:\mathbf{Curl}\,\mathbf{Curl}\,(\boldsymbol{A}_2^{sym}-\boldsymbol{C})\\ &= \int_\Omega(\boldsymbol{A}_2-\boldsymbol{C}):\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{\Psi}-\int_\Omega\boldsymbol{\Psi}:\mathbf{Curl}\,\mathbf{Curl}\,(\boldsymbol{A}_2-\boldsymbol{C})\\ &= 0.\end{aligned}$$

Then, Remark 4.2.3 implies that

$$\mathbf{Curl}\,\boldsymbol{A}_2^{sym}\times\boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{C}\times\boldsymbol{n} \quad \text{on} \quad \Gamma.$$

We define $\boldsymbol{A} = \boldsymbol{A}_1^{sym}-\boldsymbol{A}_2^{sym}$ which belongs to $\mathbb{W}^{2,p}_s(\Omega)$ and satisfies $\boldsymbol{A}\times\boldsymbol{n} = \boldsymbol{D}\times\boldsymbol{n}$ and $\mathbf{Curl}\,\boldsymbol{A}\times\boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n}$ on Γ. We set $\boldsymbol{v}$ the solution in $\boldsymbol{W}^{3,p}(\Omega)\cap\boldsymbol{W}^{1,p}_0(\Omega)$ of the homogeneous Dirichlet problem $\mathbf{Div}\,\boldsymbol{\nabla}_s\boldsymbol{v} = \mathbf{Div}\,\boldsymbol{A}$ in Ω. Finally, we define $\boldsymbol{E}_0$ by

$$\boldsymbol{E}_0 = \boldsymbol{A}-\boldsymbol{\nabla}_s\boldsymbol{v}-\sum_{i=1}^{3}\sum_{k=1}^{I}\left(\langle(\boldsymbol{A}-\boldsymbol{\nabla}_s\boldsymbol{v})\,\boldsymbol{n},\boldsymbol{e}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{v}_i^k+\langle(\boldsymbol{A}-\boldsymbol{\nabla}_s\boldsymbol{v})\,\boldsymbol{n},\boldsymbol{P}^i\rangle_{\Gamma_k}\boldsymbol{\nabla}_s\boldsymbol{w}_i^k\right).$$

Note that $\boldsymbol{E}_0$ satisfies

$$\begin{cases}\boldsymbol{E}_0\in\mathbb{W}^{2,p}_s(\Omega), \quad \mathbf{Div}\,\boldsymbol{E}_0 = \mathbf{0} \quad \text{in} \quad \Omega,\\ \boldsymbol{E}_0\times\boldsymbol{n} = \boldsymbol{D}\times\boldsymbol{n} \quad \text{and} \quad \mathbf{Curl}\,\boldsymbol{E}_0\times\boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n} \quad \text{on} \quad \Gamma,\\ \langle\boldsymbol{E}_0\boldsymbol{n},\boldsymbol{e}^i\rangle_{\Gamma_k} = \langle\boldsymbol{E}_0\boldsymbol{n},\boldsymbol{P}^i\rangle_{\Gamma_k} = 0, \text{ for any } 1\le i\le 3, \quad 1\le k\le I.\end{cases}$$

Also, from (4.4.10) and (4.4.11), we have

$$\|\boldsymbol{E}_0\|_{\mathbb{W}^{2,p}(\Omega)} \le C(\|\boldsymbol{D}\times\boldsymbol{n}\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)}+\|\mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}). \quad (4.4.12)$$

Step 2. We solve the elliptic problem (4.4.8).
We set $\boldsymbol{F} = \boldsymbol{B}-\mathbf{Curl}\,\mathbf{Curl}(\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{E}_0)$. Note that $\boldsymbol{F}$ belongs to $[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl}\,\mathbf{Curl},\,\Omega)]'$ and satisfies the estimate

$$\|\boldsymbol{F}\|_{[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl}\,\mathbf{Curl},\Omega)]'} \le C(\|\boldsymbol{B}\|_{[\mathbb{H}^{p'}_{0,s}(\mathbf{Curl}\,\mathbf{Curl},\Omega)]'}+\|\boldsymbol{D}\times\boldsymbol{n}\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)}+\|\mathbf{Curl}\,\boldsymbol{D}\times\boldsymbol{n}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}). \quad (4.4.13)$$

Let $\boldsymbol{E}_1$ be the solution of the homogeneous problem (4.4.4), when the right hand side is equal to $\boldsymbol{F}$. Then, $\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_1$ is the unique solution in $\mathbb{W}^{2,p}_s(\Omega)$ of the problem (4.4.8) and we deduce from (4.4.13) that $\boldsymbol{E}$ satisfies the estimate (4.4.9). □

In Section 4.2, we have shown that if Ω is of class $\mathcal{C}^{2,1}$, then the space $\mathbb{Y}^p_{N,s}(\Omega)$ is continuously embedded in $\mathbb{W}^{2,p}(\Omega)$. This result can be generalized in the case when the homogeneous boundary conditions are replaced by inhomogeneous ones. We define the space $\mathbb{Y}^{m,p}_s(\Omega)$ by

$$\begin{gathered}\mathbb{Y}^{m,p}_s(\Omega) = \{\boldsymbol{S} \in \mathbb{L}^p_s(\Omega),\ \mathbf{Div}\,\boldsymbol{S} \in \mathbb{W}^{m-1,p}_s(\Omega),\ \mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S} \in \mathbb{W}^{m-2,p}_s(\Omega),\\ \boldsymbol{S} \times \boldsymbol{n} \in \mathbb{W}^{m-\frac{1}{p},p}(\Gamma),\ \ \mathbf{Curl}\,\boldsymbol{S} \times \boldsymbol{n} \in \mathbb{W}^{m-1-\frac{1}{p},p}(\Gamma) \quad \text{on} \quad \Gamma\}.\end{gathered}$$

Theorem 4.4.5. *Assume that Ω is of class $\mathcal{C}^{m,1}$, where m is an integer such that $m \geq 2$, then the space $\mathbb{Y}^{m,p}_s(\Omega)$ is continuously embedded in $\mathbb{W}^{m,p}_s(\Omega)$.*

Proof. i) First of all, we suppose that $m = 2$. Let $\boldsymbol{S} \in \mathbb{Y}^{2,p}_s(\Omega)$, let $\boldsymbol{E}$ in $\mathbb{W}^{2,p}_s(\Omega)$ be the solution of the inhomogeneous problem (4.4.8) when $\boldsymbol{D} = \boldsymbol{F} = \boldsymbol{S}$ and the right hand side $\boldsymbol{B}$ is equal to $\mathbf{0}$. Now, we set $\boldsymbol{A} = \boldsymbol{S} - \boldsymbol{E}$ which belongs to $\mathbb{Y}^p_{N,s}(\Omega)$. Due to Theorem 4.2.11, the matrix $\boldsymbol{A}$ belongs to $\mathbb{W}^{2,p}_s(\Omega)$. Consequently, $\boldsymbol{S}$ belongs to $\mathbb{W}^{2,p}_s(\Omega)$ and we have

$$\begin{aligned}\|\boldsymbol{S}\|_{\mathbb{W}^{2,p}(\Omega)} &\leq \|\boldsymbol{A}\|_{\mathbb{W}^{2,p}(\Omega)} + \|\boldsymbol{E}\|_{\mathbb{W}^{2,p}(\Omega)}\\ &\leq C\big(\|\boldsymbol{A}\|_{\mathbb{L}^p(\Omega)} + \|\mathbf{Div}\,\boldsymbol{A}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{A}\|_{\mathbb{L}^p(\Omega)}\\ &+ \ \|\boldsymbol{S}\times\boldsymbol{n}\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)} + \|\mathbf{Curl}\,\boldsymbol{S}\times\boldsymbol{n}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}\Big)\\ &\leq C\big(\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \|\mathbf{Div}\,\boldsymbol{S}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\,\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)}\\ &+ \ \|\boldsymbol{S}\times\boldsymbol{n}\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)} + \|\mathbf{Curl}\,\boldsymbol{S}\times\boldsymbol{n}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}\Big).\end{aligned}$$

ii) We suppose that $m > 3$. We introduce the space of vector fields

$$\boldsymbol{Y}^{m,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div}\boldsymbol{v} \in W^{m-1,p}(\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{m-1,p}(\Omega),\ \boldsymbol{v}\times\boldsymbol{n} \in \boldsymbol{W}^{m-\frac{1}{p}}(\Gamma)\}.$$

Amrouche et al [9] have shown that if Ω is of class $\mathcal{C}^{m,1}$, then $\boldsymbol{Y}^{m,p}(\Omega)$ is continuously embedded in $\boldsymbol{W}^{m,p}(\Omega)$. Let $\boldsymbol{S} \in \mathbb{Y}^{m,p}_s(\Omega)$, we have shown that $\boldsymbol{S}$ belongs to $\mathbb{W}^{2,p}_s(\Omega)$, then $\mathbf{Curl}\,\boldsymbol{S}$ belongs to $\mathbb{W}^{1,p}(\Omega)$. Consequently, for any $1 \leq i \leq 3$, the vector field $\mathbf{Curl}\, \boldsymbol{S}^i$ belongs to $\boldsymbol{Y}^{m-1,p}(\Omega)$, then $\mathbf{Curl}\, \boldsymbol{S}$ belongs to $\mathbb{W}^{m-1,p}(\Omega)$ and we have the estimate

$$\begin{aligned} \|\mathbf{Curl}\, \boldsymbol{S}\|_{\mathbb{W}^{m-1,p}(\Omega)} &\leq C(\|\mathbf{Curl}\, \boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \|\mathbf{Div}\, \boldsymbol{S}\|_{\boldsymbol{W}^{m-1,p}(\Omega)} \\ &+ \|\mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{S}\|_{\mathbb{W}^{m-2}(\Omega)} + \|\mathbf{Curl}\, \boldsymbol{S} \times \boldsymbol{n}\|_{\mathbb{W}^{m-1-\frac{1}{p},p}(\Gamma)}) \\ &\leq C(\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \|\mathbf{Div}\, \boldsymbol{S}\|_{\boldsymbol{W}^{m-1,p}(\Omega)} + \|\mathbf{Curl}\,\mathbf{Curl}\, \boldsymbol{S}\|_{\mathbb{W}^{m-2}(\Omega)} \\ &+ \|\boldsymbol{S} \times \boldsymbol{n}\|_{\mathbb{W}^{m-\frac{1}{p},p}(\Gamma)} + \|\mathbf{Curl}\, \boldsymbol{S} \times \boldsymbol{n}\|_{\mathbb{W}^{m-1-\frac{1}{p},p}(\Gamma)}) \end{aligned} \quad (4.4.14)$$

Again, for any $1 \leq i \leq 3$, the vector $\boldsymbol{S}^i$ belongs to $\boldsymbol{Y}^{m,p}(\Omega)$, then $\boldsymbol{S}$ belongs to $\mathbb{W}^{m,p}_s(\Omega)$ and we have

$$\|\boldsymbol{S}\|_{\mathbb{W}^{m,p}(\Omega)} \leq C(\|\boldsymbol{S}\|_{\mathbb{L}^p(\Omega)} + \|\mathbf{Div}\, \boldsymbol{S}\|_{\boldsymbol{W}^{m-1,p}(\Omega)} + \|\mathbf{Curl}\, \boldsymbol{S}\|_{\mathbb{W}^{m-1,p}(\Omega)} + \|\boldsymbol{S} \times \boldsymbol{n}\|_{\mathbb{W}^{m-\frac{1}{p},p}(\Gamma)}). \quad (4.4.15)$$

From (4.4.14) and (4.4.15), we conclude

$$\|\boldsymbol{S}\|_{\mathbb{W}^{m,p}(\Omega)} \leq C\|\boldsymbol{S}\|_{\mathbb{Y}^{m,p}_s(\Omega)},$$

which is the required result. □

Theorem 4.4.6. *Assume that Ω is of class $\mathcal{C}^{2,1}$. Let $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{v}_0$ such that $\boldsymbol{B} \in [\mathbb{H}^p_{0,s}(\mathbf{Curl}\,\mathbf{Curl}, \Omega)]'$, $\boldsymbol{D}$ belongs to $\mathbb{H}^p_s(\mathbf{Curl}\,\mathbf{Curl}, \Omega)$ with $(\boldsymbol{D} \times \boldsymbol{n}, \mathbf{Curl}\, \boldsymbol{D} \times \boldsymbol{n}) \in \mathbb{W}^{2-\frac{1}{p},p}(\Gamma) \times \mathbb{W}^{1-\frac{1}{p},p}(\Gamma)$ and $\boldsymbol{v}_0 \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$ satisfying*

$$\forall \boldsymbol{M} \in \mathbb{K}_{N,s}(\Omega),\ \langle \boldsymbol{B}, \boldsymbol{M}\rangle_\Omega - \int_\Gamma \boldsymbol{v}_0 \cdot (\boldsymbol{M}\boldsymbol{n})\, ds = 0. \quad (4.4.16)$$

Then, The problem (4.4.1) has unique solution $(\boldsymbol{E}, \boldsymbol{v}) \in \mathbb{W}^{2,p}_s(\Omega) \times \boldsymbol{W}^{1,p}(\Omega)$ which satisfies the estimate

$$\begin{aligned} \|\boldsymbol{E}\|_{\mathbb{W}^{2,p}(\Omega)} + \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} &\leq C\Big(\|\boldsymbol{B}\|_{[\mathbb{H}^p_{0,s}(\mathbf{Curl}\,\mathbf{Curl},\Omega)]'} + \|\boldsymbol{D} \times \boldsymbol{n}\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)} \\ &+ \|\mathbf{Curl}\, \boldsymbol{D} \times \boldsymbol{n}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{v}_0\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)}\Big) \end{aligned}$$

Proof. We consider the problem

$$\mathbf{Div}\, \boldsymbol{\nabla}_s \boldsymbol{v} = \mathbf{Div}\, \boldsymbol{B} \ \text{ in } \Omega,\ \boldsymbol{v} = \boldsymbol{v}_0 \ \text{ on } \Gamma.$$

Since $\mathbf{Div}\,\boldsymbol{B}$ belongs to $\boldsymbol{W}^{-1,p}(\Omega)$, it has an unique solution in $\boldsymbol{W}^{1,p}(\Omega)$ and satisfies the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\|\boldsymbol{B}\|_{[\mathbb{H}^p_{0,s}(\mathbf{Curl}\,\mathbf{Curl},\Omega)]'} + \|\boldsymbol{v}_0\|_{\boldsymbol{W}^{1-\frac{1}{p},p}(\Omega)}).$$

Note that the matrix $\boldsymbol{F} = \boldsymbol{B} - \boldsymbol{\nabla}_s \boldsymbol{v}$ belongs to $[\mathbb{H}^p_{0,s}(\mathbf{Curl}\,\mathbf{Curl},\Omega)]'$ and satisfies the compatibility condition (4.4.3). Then, the problem (4.4.1) becomes

$$\begin{cases} \triangle^2 \boldsymbol{E} = \boldsymbol{F}, \quad \mathbf{Div}\,\boldsymbol{E} = \mathbf{0} \quad \text{in} \quad \Omega, \\ \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{D} \times \boldsymbol{n}, \quad \mathbf{Curl}\,\boldsymbol{E} \times \boldsymbol{n} = \mathbf{Curl}\,\boldsymbol{D} \times \boldsymbol{n} \quad \text{on} \quad \Gamma, \\ \langle \boldsymbol{E}\boldsymbol{n},\, \boldsymbol{e}^i \rangle_{\Gamma_k} = \langle \boldsymbol{E}\boldsymbol{n},\, \boldsymbol{P}^i \rangle_{\Gamma_k}, \;\; 1 \leq i \leq 3, \\ \qquad 1 \leq k \leq I \end{cases} \tag{4.4.17}$$

We have shown that the regularity of the domain Ω implies that the problem (4.4.17) has a unique solution in $\mathbb{W}^{2,p}(\Omega)$ with the estimate

$$\|\boldsymbol{E}\|_{\mathbb{W}^{2,p}(\Omega)} \leq C(\|\boldsymbol{F}\|_{[\mathbb{H}^p_{0,s}(\mathbf{Curl}\,\mathbf{Curl},\Omega)]'} + \|\boldsymbol{D} \times \boldsymbol{n}\|_{\mathbb{W}^{2-\frac{1}{p},p}(\Gamma)} + \|\mathbf{Curl}\,\boldsymbol{D} \times \boldsymbol{n}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}),$$

which ends the proof. □

Chapter 5

Traces Characterizations for Sobolev Spaces on Lipschitz Domains of $\mathbb{R}^2$

In [29], Geymonat et al have used the Airy's function (which represents the 2-dimensional case of the Beltrami's representation) to characterize the range of the trace operator

$$\begin{aligned} \gamma: \quad H^2(\Omega) &\longrightarrow H^1(\Gamma) \times L^2(\Gamma) \\ f &\longrightarrow (f_{|\Gamma}, \frac{\partial f}{\partial \boldsymbol{n}}), \end{aligned} \tag{5.0.1}$$

where Ω is a Lipschitz domain of $\mathbb{R}^2$. Duràn et al have used the same argument to generalize this result in the Sobolev spaces $W^{2,p}(\Omega)$, $1 < p < \infty$. Later, Geymonat et al in [16] have used a different technic to generalize the above result in the three dimensional case.

In this chapter, we will use another characterization of L^p-symmetric matrix fields to characterize the range of the operator:

$$\begin{aligned} \gamma: \quad W^{3,p}(\Omega) &\longrightarrow W^{1,p}(\Gamma) \times L^p(\Gamma) \times L^p(\Gamma) \\ f &\longrightarrow (f_{|\Gamma}, \frac{\partial f}{\partial \boldsymbol{n}}, \frac{\partial^2 f}{\partial \boldsymbol{n}^2}), \end{aligned} \tag{5.0.2}$$

where Ω is Lipschitz domain of $\mathbb{R}^2$. .

5.1 Homogeneous Bi-Laplacian problem

In this section, we will consider the following homogeneous Bi-Laplacian problem:

$$(\mathcal{P}_B) \quad \begin{cases} \Delta^2 u \ = \ 0 \quad \text{in} \ \ \Omega, \\ u \ = \ g_0 \quad \text{on} \ \Gamma, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} \ = \ g_1 \quad \text{on} \ \ \Gamma. \end{cases} \tag{5.1.1}$$

Recall the following result (see [22]).

Theorem 5.1.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$ of class of $\mathcal{C}^{0,1}$, with $N \geq 2$ and let*

$$g_0 \in H^1(\Gamma) \quad \text{and} \quad g_1 \in L^2(\Gamma). \tag{5.1.2}$$

Then there exists a unique $u \in H^{3/2}(\Omega)$ solution to Problem $(\mathcal{P}_B)$ with the estimate

$$\|u\|_{H^{3/2}(\Omega)} \leq \ C(\|g_0\|_{H^1(\Gamma)} + \|g_1\|_{L^2(\Gamma)}). \tag{5.1.3}$$

On the other hand, we know that if Ω is a bounded open subset of $\mathbb{R}^N$ of class of $\mathcal{C}^{0,1}$ and $f \in L^2(\Omega)$, then there exists a unique solution $u \in H^2_0(\Omega)$ satisfying $\Delta^2 u = f$ in Ω with the estimate

$$\|u\|_{H^2(\Omega)} \leq \ C \, \|f\|_{L^2(\Omega)} \, . \tag{5.1.4}$$

We know that if $g_0 \in H^1(\Gamma)$ and $g_1 \in L^2(\Gamma)$ verify the condition (1.0.13) with $p = 2$, then there exists a function $u \in H^2(\Omega)$ satisfying $u = g_0$ and $\frac{\partial u}{\partial \boldsymbol{n}} \ = \ g_1$ on Γ with the estimate

$$\|u\|_{H^2(\Omega)} \leq \ C \left\| \frac{\partial g_0}{\partial \boldsymbol{t}} \boldsymbol{t} + g_1 \boldsymbol{n} \right\|_{\boldsymbol{H}^{1/2}(\Gamma)} . \tag{5.1.5}$$

The question that interests us here is to find such a function u in addition biharmonic in Ω.

Theorem 5.1.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ of class $\mathcal{C}^{0,1}$, with $N \geq 2$. Let g_0 and g_1 be satisfy the conditions (5.1.2) and (1.0.13). Then there exists a unique biharmonic function $u \in H^2(\Omega)$ satisfying $u = g_0$ and $\frac{\partial u}{\partial \boldsymbol{n}} \ = \ g_1$ on Γ with the estimate (5.1.5).*

Proof. Let $w \in H^2(\Omega)$ such that $w = g_0$ and $\frac{\partial w}{\partial \boldsymbol{n}} = g_1$ on Γ. We know that there exists a unique solution $z \in H^2_0(\Omega)$ satisfying $\Delta^2 z = \Delta^2 w$ in Ω. The required function is given by $u = w - z$. □

Remark 5.1.3. *Let us introduce the following Hilbert space*

$$\boldsymbol{H}^{1/2}_T(\Gamma) = \{\boldsymbol{v} \in \boldsymbol{H}^{1/2}(\Gamma);\ \boldsymbol{v}_\tau = \boldsymbol{0}\}.$$

Clearly

$$\boldsymbol{v} \in \boldsymbol{H}^{1/2}_T(\Gamma) \Longleftrightarrow \boldsymbol{v} = g\boldsymbol{n} \quad \text{with} \quad g \in L^2(\Gamma) \quad \textit{and} \quad g\boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma)$$

The above result asserts that for any

$$g \in L^2(\Gamma) \quad \text{such that} \quad g\boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma)$$

there exists a function $u \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $\frac{\partial u}{\partial \boldsymbol{n}} = g$ on Γ. Moreover among all functions satisfying these conditions, there is one that is biharmonic.

We will see now an interested consequence of this result which will allow us to establish the existence of very weak solutions in domains which are only Lipschitz. Before that, recall that if Ω is of class $\mathcal{C}^{1,1}$ and $g \in H^{-1/2}(\Gamma)$, then there exists a unique harmonic function $u \in L^2(\Omega)$ satisfying $u = g$ on Γ. When Ω is not sufficiently regular, there is not possible in general to define the trace of harmonic function $u \in L^2(\Omega)$ in $H^{-s}(\Gamma)$ for some $s > 0$. So, let us introduce the following Hilbert space:

$$M(\Omega) = \left\{v \in L^2(\Omega);\ \Delta v \in L^2(\Omega)\right\}.$$

We denote its norm by

$$\|v\|_{M(\Omega)} = (\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)})^{1/2}.$$

It is easy to prove that $\mathcal{D}\big(\overline{\Omega}\big)$ is dense in $M(\Omega)$.

As a consequence of this density result and of Theorem 5.1.2, we can prove the following lemma.

Lemma 5.1.4. *Let Ω be a bounded open set of $\mathbb{R}^N$ of class $\mathcal{C}^{0,1}$, with $N \geq 2$. The linear mapping $v \longmapsto (v\boldsymbol{n})|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$M(\Omega) \longrightarrow [\boldsymbol{H}_T^{1/2}(\Gamma)]'.$$

Moreover, we have the Green formula: For all $v \in M(\Omega)$ and $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\int_\Omega v\Delta\varphi\, dx - \int_\Omega \varphi\Delta v\, dx = \langle (v\boldsymbol{n})_\Gamma, \nabla\varphi\rangle. \tag{5.1.6}$$

Remark 5.1.5. *When Ω is of class $\mathcal{C}^{1,1}$, then the linear mapping $v \longmapsto v|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$M(\Omega) \longrightarrow H^{-1/2}(\Gamma)$$

and we have the Green formula: For all $v \in M(\Omega)$ and $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\int_\Omega v\Delta\varphi\, dx - \int_\Omega \varphi\Delta v\, dx = \langle v, \frac{\partial\varphi}{\partial\boldsymbol{n}}\rangle. \tag{5.1.7}$$

We now can solve the Laplace equation with singular boundary condition.

Theorem 5.1.6. *Let Ω be a bounded open set of $\mathbb{R}^N$ of class $\mathcal{C}^{0,1}$, with $N \geq 2$. For any*

$$g \in H^{-1/2}(\Gamma) \quad \textit{such that} \quad g\boldsymbol{n} \in [\boldsymbol{H}_T^{1/2}(\Gamma)]'$$

there exists a unique function $u \in L^2(\Omega)$ solution to the problem

$$\Delta u = 0 \quad \textit{in } \Omega \quad \textit{and} \quad u\boldsymbol{n} = g\boldsymbol{n} \quad \textit{on } \Gamma, \tag{5.1.8}$$

with the estimate

$$\|u\|_{L^2(\Omega)} \leq \ C\|g\boldsymbol{n}\|_{[\boldsymbol{H}_T^{1/2}(\Gamma)]'}.$$

Proof. Thanks to the Green formula (5.1.7), it is easy to verify that $u \in L^2(\Omega)$ is solution to Problem (5.1.8) is equivalent to the following variational formulation: Find $u \in L^2(\Omega)$ such that for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\int_\Omega u\Delta\varphi\, dx = \langle g\boldsymbol{n}, \nabla\varphi\rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}^{1/2}(\Gamma)}\,. \tag{5.1.9}$$

Indeed, let $u \in L^2(\Omega)$ be a solution to (5.1.8). Then, the Green formula (5.1.7) yields (5.1.9).

Conversely, let $u \in L^2(\Omega)$ be a solution to (5.1.9). Taking φ in $\mathcal{D}(\Omega)$, we obtain $\Delta u = 0$ in Ω and $u \in M(\Omega)$. Using this last relation and again the Green formula (5.1.7), we deduce that for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\langle u\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}^{1/2}(\Gamma)} = \langle g\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)} .$$

Let $\boldsymbol{\mu} \in \boldsymbol{H}_T^{1/2}(\Gamma)$. By Remark 5.1.3, we know that there exits $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\boldsymbol{\mu} = \nabla\varphi$ on Γ. Thus,

$$\langle u\boldsymbol{n}, \boldsymbol{\mu} \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)} = \langle u\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)} = \langle g\boldsymbol{n}, \boldsymbol{\mu} \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)} .$$

and $u\boldsymbol{n} = g\boldsymbol{n}$ on Γ.

Let's then solve Problem (5.1.9). We know that for all $F \in L^2(\Omega)$, there exists a unique $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying $-\Delta\varphi = F$ in Ω, with the estimate

$$\|v\|_{H^2(\Omega)} \leq C\|F\|_{L^2(\Omega)}.$$

Using estimate (5.1.4) we get

$$|\langle g\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)}| \leq \|g\boldsymbol{n}\|_{[\boldsymbol{H}_T^{1/2}(\Gamma)]'} \|\nabla\varphi\|_{\boldsymbol{H}^{1/2}(\Gamma)} \leq \ C \ \|g\boldsymbol{n}\|_{[\boldsymbol{H}_T^{1/2}(\Gamma)]'} \|F\|_{L^2(\Omega)}.$$

In other words, we can say that the linear mapping

$$T : F \longmapsto \langle g\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)}$$

is continuous on $L^2(\Omega)$, and according to the Riesz representation theorem, there exists a unique $u \in L^2(\Omega)$, such that

$$\forall F \in L^2(\Omega), \ T(F) = \int_\Omega u\,F,$$

i.e u is solution of Problem (5.1.9).

□

5.2 An Hessian representation for L^p-symmetric matrix fields

In the rest of this chapter, Ω is a bounded and connected open of $\mathbb{R}^2$ with Lipschitz-continuous boundary. In this section, we will present an new characterization of the L^p-symmetric by using the Hessian matrix. For that, we need some results, the first one is the following vector potential theorem which have been presented by Duràn and Muschietti in [23]:

Lemma 5.2.1. *Let $1 < p < \infty$, $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$ in Ω and satisfying the compatibility condition*

$$< \boldsymbol{v} \cdot \boldsymbol{n}, 1 >_{\Gamma_j} = 0 \quad for \quad j = 0, \cdots, J.$$

Then there exists a function $\psi \in W^{1,p}(\Omega)$ such that $\mathbf{curl}\, \psi = \boldsymbol{v}$ in Ω.

The previous lemma is the key to generalize the Airy's function theorem in $\mathbb{L}^p(\Omega)$. In fact it suffices to follow the same steps of proof of Theorem 2 of [1] to obtain the following result:

Lemma 5.2.2. *Given $\boldsymbol{S} = (s_{ij})_{i,j=1,2} \in \mathbb{L}^p_s(\Omega)$, then $\boldsymbol{S}$ fulfills the following statements :*

$$\mathbf{Div}\, \boldsymbol{S} = \mathbf{0} \quad in\, \Omega, \tag{5.2.1}$$

$$\langle \boldsymbol{S}_i \cdot \boldsymbol{n},\, 1\rangle_{\Gamma_j} = 0 \quad for \quad i = 1, 2 \quad and \quad j = 0, \cdots, J, \tag{5.2.2}$$

$$\langle \boldsymbol{S}_1 \cdot \boldsymbol{n},\, x_2\rangle_{\Gamma_j} = \langle \boldsymbol{S}_2 \cdot \boldsymbol{n},\, x_1\rangle_{\Gamma_j} \quad for \quad j = 0, \cdots, J, \tag{5.2.3}$$

if and only if there exists an Airy's function $w \in W^{2,p}(\Omega)$ such that

$$s_{11} = \frac{\partial^2 w}{\partial x_2^2}, \quad s_{12} = -\frac{\partial^2 w}{\partial x_1 \partial x_2} \quad and \quad s_{22} = \frac{\partial^2 w}{\partial x_1^2}. \tag{5.2.4}$$

We are now in position to give a characterization of L^p-symmetric matrix field as a Hessian of a scalar field belonging to $W^{2,p}(\Omega)$.

Theorem 5.2.3. *Given $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$, then $\boldsymbol{S}$ fulfills the following statements:*

$$\operatorname{curl} \boldsymbol{S}_i = 0 \quad in\ \Omega, \quad for \quad i = 1,\, 2 \tag{5.2.5}$$

$$\langle \boldsymbol{St},\, \boldsymbol{e}^i\rangle_{\Gamma_j} = 0 \quad for \quad i = 1,\, 2 \quad and \quad j = 0, \cdots, J, \tag{5.2.6}$$

$$\langle \boldsymbol{St},\, \boldsymbol{x}\rangle_{\Gamma_j} = 0 \quad for \quad j = 0, \cdots, J, \tag{5.2.7}$$

if and only if there exists $w \in W^{2,p}(\Omega)$ such that

$$\boldsymbol{S} = \mathbf{Hess}\, w \quad in \quad \Omega. \tag{5.2.8}$$

Proof. **i)** First, let $\boldsymbol{S} = \mathbf{Hess}\, w$ with $w \in W^{2,p}(\Omega)$. It is clear that $\boldsymbol{S}$ belongs to $\mathbb{L}^p_s(\Omega)$ and satisfies (5.2.5). It rest to show that $\boldsymbol{S}$ satisfies the compatibility conditions (5.2.6) and (5.2.7). Lemma 5.2.2 implies that the following compatibility conditions hold true

$$\langle \boldsymbol{S}^\star \boldsymbol{n}, e^i\rangle_{\Gamma_j} = 0 \quad for \quad i = 1,\, 2 \quad and \quad j = 0, \cdots, J, \tag{5.2.9}$$

$$\langle \boldsymbol{S}^\star_1 \cdot \boldsymbol{n}, x_2\rangle_{\Gamma_j} = \langle \boldsymbol{S}^\star_2 \cdot \boldsymbol{n}, x_1\rangle_{\Gamma_j} \quad for \quad j = 0, \cdots, J. \tag{5.2.10}$$

Let us observe the following equalities

$$\boldsymbol{S}^\star_1 \cdot \boldsymbol{n} = \boldsymbol{S}_2 \cdot \boldsymbol{t} \quad \text{and} \quad \boldsymbol{S}^\star_2 \cdot \boldsymbol{n} = -\boldsymbol{S}_1 \cdot \boldsymbol{t}.$$

So, we have the relations (5.2.6) and (5.2.7).

ii) Conversely, let $\boldsymbol{S} \in \mathbb{L}^p_s(\Omega)$ satisfies the compatibility conditions (5.2.5)-(5.2.7). Then, the matrix $\boldsymbol{S}^\star \in \mathbb{L}^p_s(\Omega$ satisfies (5.2.9) and (5.2.10). Moreover, as $\mathbf{curl}\, \boldsymbol{S} = \mathbf{0}$ in Ω, then $\mathbf{div}\, \boldsymbol{S}^\star = \mathbf{0}$ in Ω. Due to Lemma 5.2.2, there exists $w \in W^{2,p}(\Omega)$ such that

$$\boldsymbol{S}^\star = \begin{pmatrix} \dfrac{\partial^2 w}{\partial x_2^2} & -\dfrac{\partial^2 w}{\partial x_1 \partial x_2} \\ -\dfrac{\partial^2 w}{\partial x_2 \partial x_1} & \dfrac{\partial^2 w}{\partial x_1^2} \end{pmatrix}.$$

Consequently,

$$\boldsymbol{S} = \mathbf{Hess}\, w \quad \text{in} \quad \Omega.$$

□

5.3 The range of the traces of $W^{3,p}(\Omega)$

Geymonat [28] proved that if Ω is a Lipschitz domain of $\mathbb{R}^2$ and $(g_0,\, g_1,\, g_2) \in W^{1,p}(\Gamma) \times L^p(\Gamma) \times L^p(\Gamma)$ belongs to the range of the operator $(\gamma_0,\, \gamma_1,\, \gamma_2)$, then it must satisfy the following conditions

$$\boldsymbol{q} := \frac{\partial g_0}{\partial \boldsymbol{t}}\boldsymbol{t} + g_1 \boldsymbol{n} \in \boldsymbol{W}^{1,p}(\Gamma), \tag{5.3.1}$$

and

$$\boldsymbol{H} := [(\boldsymbol{\nabla} \boldsymbol{q}\, \boldsymbol{t}) \cdot \boldsymbol{t}]\, \boldsymbol{t} \otimes \boldsymbol{t} + [(\boldsymbol{\nabla} \boldsymbol{q}\, \boldsymbol{t}) \cdot \boldsymbol{n}]\, (\boldsymbol{t} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{t}) + g_2\, \boldsymbol{n} \otimes \boldsymbol{n} \in \mathbb{W}^{1-\frac{1}{p},p}(\Gamma). \tag{5.3.2}$$

In this section, we will show that the necessary conditions (5.3.1) and (5.3.2) are sufficient. First, we will show the following results.

Lemma 5.3.1. *The operator*

$$\textbf{Div} : \mathbb{W}^{1,p}_{0,s}(\Omega) \to \boldsymbol{L}^p_0(\Omega), \tag{5.3.3}$$

is onto. Consequently, for each vector field $\boldsymbol{v} \in \boldsymbol{L}^p_0(\Omega)$*, there exists a symmetric matrix field* $\boldsymbol{S}$ *in* $\mathbb{W}^{1,p}_{0,s}(\Omega)$ *such that*

$$\textbf{Div}\, \boldsymbol{S} = \boldsymbol{v} \quad \text{in } \Omega,$$

and there exists a constant C *depends only on* p *and* Ω *such that*

$$\|\boldsymbol{S}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}.$$

Proof. The proof is based on Theorem 3 of [20] and it is composed on three steps.

Step 1. We show a vector version of J. L. Lions lemma. Here, we follow the same steps of proof of Theorem 3.1 of [6]. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ be such that $\boldsymbol{\nabla}_s\, \boldsymbol{v} \in \mathbb{W}^{-1,p}_s(\Omega)$. The identity

$$\partial_j(\partial_k v_i) = \partial_j(\boldsymbol{\nabla}_s v)_{ik} + \partial_k(\boldsymbol{\nabla}_s v)_{ij} - \partial_i(\boldsymbol{\nabla}_s v)_{jk}$$

implies that for any $k, i = 1, 2$, the distribution $\partial_k v_i$ has a gradient in $\boldsymbol{W}^{-2,p}(\Omega)$. Then Proposition 2.1 of [8] implies that $\partial_k v_i$ is in $W^{-1,p}(\Omega)$. In other words, $\boldsymbol{\nabla}\, v_i$ belongs to $\boldsymbol{W}^{-1,p}(\Omega)$ for each $i = 1, 2$. Again Proposition 2.1 of [8] implies that $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$.

Step 2. We show an extension of Donati's theorem. Let p' be the conjugate of p and $\boldsymbol{S} \in \mathbb{W}_s^{-1,p'}(\Omega)$ be such that

$$_{\mathbb{W}^{-1,p'}(\Omega)}\langle \boldsymbol{S},\, \boldsymbol{E}\rangle_{\mathbb{W}_0^{1,p}(\Omega)} = 0 \quad \text{for all } \boldsymbol{E} \in \mathbb{V}_{0,s}^{1,p}(\Omega). \tag{5.3.4}$$

Moreau's theorem [41] implies that there exists $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ such that $\boldsymbol{\nabla}_s \boldsymbol{v} = \boldsymbol{S}$ in Ω. By Step 1, we get $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$.

Step 3. We show that the operator (5.3.3) **is onto**. As consequence of Step 2, we deduce that the following operator

$$\boldsymbol{\nabla}_s :\ \boldsymbol{L}^{p'}(\Omega)/\boldsymbol{R}(\Omega) \to [\mathbb{V}_s^{1,p}(\Omega)]^{\circ}. \tag{5.3.5}$$

is an isomorphism. Above the polar set is defiend as follow:

$$[\mathbb{V}_s^{1,p}(\Omega)]^{\circ} = \{\boldsymbol{S} \in \mathbb{W}_s^{-1,p'}(\Omega) \quad \text{satisfying } (5.3.4)\}.$$

So, the dual operator

$$\textbf{Div} :\ \mathbb{W}_{0,s}^{1,p}(\Omega)/\mathbb{V}_s^{1,p}(\Omega) \to \boldsymbol{L}_0^p(\Omega), \tag{5.3.6}$$

is an isomorphism. □

Lemma 5.3.2. *Let $\boldsymbol{A} \in \mathbb{W}_s^{1-\frac{1}{p},p}(\Gamma)$ satisfies the compatibility conditions* (5.2.6) *and* (5.2.7) *of Theorem 5.2.3. Then, there exists $\boldsymbol{S} \in \mathbb{W}_s^{1,p}(\Omega)$ such that*

$$\textbf{curl}\, \boldsymbol{S} = \boldsymbol{0} \quad \textit{in} \quad \Omega \quad \textit{and} \quad \boldsymbol{S} = \boldsymbol{A} \quad \textit{on}\, \Gamma. \tag{5.3.7}$$

Moreover, there exists a constant C depends only on p and Ω such that

$$\|\boldsymbol{S}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{A}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}. \tag{5.3.8}$$

Proof. Let $\boldsymbol{A}$ be as in the statement of Lemma 5.3.2 and $\boldsymbol{M} \in \mathbb{W}_s^{1,p}(\Omega)$ be such that $\boldsymbol{M}_{|\Gamma} = \boldsymbol{A}$ on Γ and satisfies the estimate

$$\|\boldsymbol{M}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{A}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}. \tag{5.3.9}$$

Let us observe that

$$\mathbf{Div}\,\boldsymbol{M}^\star = \begin{pmatrix} \dfrac{\partial m_{22}}{\partial x_1} - \dfrac{\partial m_{21}}{\partial x_2} \\ -\dfrac{\partial m_{12}}{\partial x_1} + \dfrac{\partial m_{11}}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \operatorname{curl} \boldsymbol{M}_2 \\ -\operatorname{curl}\ \boldsymbol{M}_1 \end{pmatrix}.$$

Now, setting $\boldsymbol{v} = \mathbf{Div}\,\boldsymbol{M}^\star$. We search $\boldsymbol{R} \in \mathbb{W}^{1,p}_{0,s}(\Omega)$ such that $\mathbf{div}\,\boldsymbol{R} = \boldsymbol{v}$ in Ω. By using (5.2.6), we get

$$\int_\Omega \boldsymbol{v}\cdot\boldsymbol{e}^1\,dx = \int_\Omega (\frac{\partial m_{22}}{\partial x_1} - \frac{\partial m_{21}}{\partial x_2})\,dx = \int_\Gamma \boldsymbol{M}_2\cdot\boldsymbol{t}\,d\sigma = 0.$$

By the same, we get

$$\int_\Omega \boldsymbol{v}\cdot\boldsymbol{e}^2\,dx = -\int_\Gamma \boldsymbol{M}_1\cdot\boldsymbol{t}\,d\sigma = 0.$$

And by using (5.2.7), we get

$$\int_\Omega \boldsymbol{v}\cdot\boldsymbol{x}^\perp\,dx = -\int_\Omega \boldsymbol{M}^\star : \boldsymbol{\nabla}\boldsymbol{x}^\perp\,dx + \int_\Gamma (\boldsymbol{M}^\star\boldsymbol{n})\cdot\boldsymbol{x}^\perp\,d\sigma = -\int_\Gamma (\boldsymbol{M}\boldsymbol{t})\cdot\boldsymbol{x}\,d\sigma = 0.$$

The second integral above is equal to zero since $\boldsymbol{M}^\star$ is symmetric and also the third on the boundary by using (5.2.7) Then, Lemma 5.3.1 implies that there exists $\boldsymbol{R} \in W^{1,p}_{0,s}(\Omega)$ such that $\mathbf{div}\,\boldsymbol{R} = \mathbf{div}\,\boldsymbol{M}^\star$ and satisfies the estimate

$$\|\boldsymbol{R}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{A}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}. \tag{5.3.10}$$

The symmetric matrix $\boldsymbol{S} = \boldsymbol{M} - \boldsymbol{R}^\star \in \mathbb{W}^{1,p}_s(\Omega)$ and satisfies

$$\boldsymbol{S}_{|\Gamma} = \boldsymbol{M}_{|\Gamma} = \boldsymbol{A} \quad \text{with} \quad \|\boldsymbol{S}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{A}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}.$$

Observe that $\mathbf{Div}\,\boldsymbol{S}^\star = \mathbf{0}$, then $\mathbf{curl}\,\boldsymbol{S} = \mathbf{0}$. Moreover, (5.3.9) and (5.3.10) implies that the estimate (5.3.8) holds, which ends the proof. □

Lemma 5.3.3. *Let $g_0 \in W^{1,p}(\Gamma)$, g_1, g_2 in $L^p(\Gamma)$ be such that the vector field $\boldsymbol{q} = \dfrac{\partial g_0}{\partial \boldsymbol{t}}\boldsymbol{t} + g_1\boldsymbol{n}$ be in $W^{1,p}(\Gamma)$. Then, the matrix field $\boldsymbol{H}$ defined by*

$$\boldsymbol{H} = [(\boldsymbol{\nabla}\boldsymbol{q}\,\boldsymbol{t})\cdot\boldsymbol{t}]\,\boldsymbol{t}\otimes\ \boldsymbol{t} + [(\boldsymbol{\nabla}\boldsymbol{q}\,\boldsymbol{t})\cdot\boldsymbol{n}]\,(\boldsymbol{t}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{t}) + g_2\,\boldsymbol{n}\otimes\boldsymbol{n},$$

satisfies

$$\langle \boldsymbol{H}\,\boldsymbol{t},\, \boldsymbol{e}^1\rangle_{\Gamma_j} = \langle \boldsymbol{H}\,\boldsymbol{t},\, \boldsymbol{e}^2\rangle_{\Gamma_j} = \langle \boldsymbol{H}\,\boldsymbol{t},\, \boldsymbol{x}\rangle_{\Gamma_j} = 0,\ j = 1,\cdots,J. \tag{5.3.11}$$

Proof. As $\boldsymbol{q} \in W^{1,p}(\Gamma)$, there exists $w \in W^{2,p}(\Omega)$ such that $w_{|\Gamma} = g_0$, $\dfrac{\partial w}{\partial \boldsymbol{n}} = g_1$ and $\boldsymbol{q} = (\boldsymbol{\nabla} w)_{|\Gamma}$ (see [1]). By definition of tangential derivatives, we get $(\boldsymbol{\nabla q})\,\boldsymbol{t} = \partial_{\boldsymbol{t}}\boldsymbol{q} = (\boldsymbol{\nabla}^2 w)\boldsymbol{t}$. A simple calculus gives

$$(\boldsymbol{n}\otimes\boldsymbol{n})\,\boldsymbol{t} = \boldsymbol{0},\ (\boldsymbol{n}\otimes\boldsymbol{t})\ \boldsymbol{t} = \boldsymbol{n},\ (\boldsymbol{t}\otimes\boldsymbol{n})\,\boldsymbol{t} = \boldsymbol{0},\ (\boldsymbol{t}\otimes\boldsymbol{t}\,)\ \boldsymbol{t} = \boldsymbol{t}. \tag{5.3.12}$$

Then, we get

$$\begin{aligned} \boldsymbol{H}\,\boldsymbol{t} &= [(\boldsymbol{\nabla} q\,\boldsymbol{t})\cdot\boldsymbol{t}]\,[(\boldsymbol{t}\otimes\boldsymbol{t})\,\boldsymbol{t}] + [((\boldsymbol{\nabla} q\,\boldsymbol{t})\cdot\boldsymbol{n}]\,[(\boldsymbol{t}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{t})\,\boldsymbol{t}] + g_2\,[(\boldsymbol{n}\otimes\boldsymbol{n})\,\boldsymbol{t}] \\ &= [(\boldsymbol{\nabla} q\,\boldsymbol{t})\cdot\boldsymbol{t}]\ \boldsymbol{t} + [(\boldsymbol{\nabla} q\,\boldsymbol{t})\cdot\boldsymbol{n}]\ \boldsymbol{n} \\ &= (\boldsymbol{\nabla} q)\,\boldsymbol{t} = \boldsymbol{\nabla}^2 w\,\boldsymbol{t}. \end{aligned}$$

Finally, Theorem 5.2.3 implies that

$$\langle \boldsymbol{H}\,\boldsymbol{t},\,\boldsymbol{e}^i\rangle_{\Gamma_j} = \langle \boldsymbol{\nabla}^2\,w\,\boldsymbol{t},\,\boldsymbol{e}^i\rangle_{\Gamma_j} = 0,\ i=1,2,\ j=1,\cdots,J,$$

$$\langle \boldsymbol{H}\,\boldsymbol{t},\,\boldsymbol{x}\rangle_{\Gamma_j} = \langle \boldsymbol{\nabla}^2\,w\,\boldsymbol{t},\,\boldsymbol{x}\rangle_{\Gamma_j} = 0,\ j=1,\cdots,J.$$

□

By using the same argument of proof of Corollary 3.7 of [21], the following results holds true:

Proposition 5.3.4. *The following linear operator*

$$\partial_{\boldsymbol{t}} :\ W^{1-\frac{1}{p},p}(\Gamma) \to W^{-\frac{1}{p},p}(\Gamma),$$

is continuous and

$$\mathrm{Ker}\,\partial_{\boldsymbol{t}} = \mathbb{R}.$$

We are now in position to characterize the range of the trace operator in $W^{3,p}(\Omega)$.

Theorem 5.3.5. *Let $g_0 \in W^{1,p}(\Gamma)$, g_1, $g_2 \in L^p(\Gamma)$ be given. Then, there exists $w \in W^{3,p}(\Omega)$ such that*

$$w = g_0,\quad \frac{\partial w}{\partial \boldsymbol{n}} = g_1 \quad\text{and}\quad \frac{\partial^2 w}{\partial \boldsymbol{n}^2} = g_2 \quad \textit{on } \Gamma, \tag{5.3.13}$$

if and only if g_0, g_1 and g_2 satisfy the conditions (5.3.1) and (5.3.2).

Proof. **i**) First, let $w \in W^{3,p}(\Omega)$, $g_0 = w_{|\Gamma}$, $g_1 = \dfrac{\partial w}{\partial \boldsymbol{n}}$ and $g_2 = \dfrac{\partial^2 w}{\partial \boldsymbol{n}^2}$. By definition of tangential derivatives, then the vector field $\boldsymbol{q}$ and the matrix field $\boldsymbol{H}$ defined in Lemma 5.3.3 satisfy the conditions (5.3.1) and (5.3.2):

$$\boldsymbol{q} = (\boldsymbol{\nabla} w)_{|\Gamma} \in \boldsymbol{W}^{1,p}(\Gamma) \quad \text{and} \quad \boldsymbol{H} = (\boldsymbol{\nabla}^2 w)_{|\Gamma} \in \mathbb{W}_s^{1-\frac{1}{p},p}(\Gamma).$$

ii) Conversely, Lemma 5.3.3 implies that $\boldsymbol{H}$ satisfies the compatibility conditions (5.2.6) and (5.2.7), then Lemma 5.3.2 implies that there exists $\boldsymbol{S} \in \mathbb{W}_s^{1,p}(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{S} = \mathbf{0}$ in Ω and $\boldsymbol{S} = \boldsymbol{H}$ on Γ. As the matrix $\boldsymbol{S}$ satisfies the conditions (5.2.5)-(5.2.7), then there exists $w_0 \in W^{2,p}(\Omega)$ such that $\boldsymbol{\nabla}^2 w_0 = \boldsymbol{S}$ in Ω. Consequently, $w_0 \in W^{3,p}(\Omega)$ and $(\boldsymbol{\nabla}^2 w_0)_{|\Gamma} = \boldsymbol{H}$. A simple calculus gives

$$(\boldsymbol{n} \otimes \boldsymbol{n})\, \boldsymbol{n} = \boldsymbol{n}, \ (\boldsymbol{n} \otimes \boldsymbol{t})\, \boldsymbol{n} = \mathbf{0}, \ (\boldsymbol{t} \otimes \boldsymbol{n})\, \boldsymbol{n} = \boldsymbol{t}, \ (\boldsymbol{t} \otimes \boldsymbol{t})\, \boldsymbol{n} = \mathbf{0}.$$

Then,

$$(\boldsymbol{H}\boldsymbol{n}) \cdot \boldsymbol{n} = ((\boldsymbol{\nabla}^2 w_0)\boldsymbol{n}) \cdot \boldsymbol{n} = \frac{\partial^2 w_0}{\partial \boldsymbol{n}^2} = g_2 \ \text{ on } \Gamma. \tag{5.3.14}$$

Also, using (5.3.12), we get

$$(\boldsymbol{\nabla}^2 w_0)\boldsymbol{t} = \boldsymbol{H}\boldsymbol{t} = (\boldsymbol{\nabla} \boldsymbol{q})\, \boldsymbol{t} \ \text{ on } \Gamma.$$

Hence, Proposition 5.3.4 implies that $\boldsymbol{q} = (\boldsymbol{\nabla} w_0)_{|\Gamma} + \boldsymbol{c_0}$ where $\boldsymbol{c_0} \in \mathbb{R}^2$. Let us observe that the following fonction $w_1 = w_0 + \boldsymbol{c_0} \cdot \boldsymbol{x}$ satisfies

$$\frac{\partial^2 w_1}{\partial \boldsymbol{n}^2} = \frac{\partial^2 w_0}{\partial \boldsymbol{n}^2} = g_2, \quad \boldsymbol{q} = \boldsymbol{\nabla} w_1 \quad \text{and} \quad \frac{\partial w_1}{\partial \boldsymbol{n}} = g_1.$$

Moreover,

$$(\boldsymbol{\nabla}\, w_1) \cdot \boldsymbol{t} = \boldsymbol{q} \cdot \boldsymbol{t} = \boldsymbol{\nabla} g_0 \cdot \boldsymbol{t}.$$

Again, Proposition 5.3.4 implies that $g_0 = (w_1)_{|\Gamma} + c_1$ where $c_1 \in \mathbb{R}$. Finally, the function $w = w_1 + c_1$ answers to our question since

$$w = g_0, \quad \frac{\partial w}{\partial \boldsymbol{n}} = \frac{\partial w_1}{\partial \boldsymbol{n}} = g_1 \quad \text{and} \quad \frac{\partial^2 w}{\partial \boldsymbol{n}^2} = \frac{\partial^2 w_0}{\partial \boldsymbol{n}^2} = g_2$$

which ends the proof. □

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