Limit theorem for a multi-type branching process in a random environment

Erwan Pin

DOCTORAT BRETAGNE LOIRE MATHSTIC

Université Bretagne Sud 584

THESE DE DOCTORAT DE

L'UNIVERSITE BRETAGNE SUD

ECOLE DOCTORALE N° 601
Mathématiques et Sciences et Technologies
de l'Information et de la Communication
Spécialité : *Mathématiques et leurs Interactions*

Par
Erwan PIN

Théorèmes limites pour un processus de branchement multi-type dans un environnement aléatoire

Thèse présentée et soutenue à l'Université de Bretagne Sud, le 18 Décembre 2020
Unité de recherche : LMBA, UMR CNRS 6205

Thèse n°584

Composition du jury

Rapporteurs :	Götz Kersting	Professeur à Goethe-Universität, Allemagne
	Romain Abraham	Professeur à l'Université d'Orléans
Président :	Sylvie Méléard	Professeur à l'Ecole Polytechnique
Examinateurs :	Vincent Bansaye	Professeur à l'Ecole Polytechnique
	Loïc Chaumont	Professeur à l'Université d'Angers
	Sylvie Méléard	Professeur à l'Ecole Polytechnique
	Marc Peigné	Professeur à l'Université de Tours
Dir. de thèse :	Quansheng Liu	Professeur à l'Université de Bretagne Sud / HDR
	Ion Grama	Professeur à l'Université de Bretagne Sud / HDR

Table of Contents

Chapitre 1
Introduction

1.1 Contexte

Cela fait quelques décennies maintenant que les processus de branchements occupent une place importante dans la théorie des probabilités, dû principalement à un grand champs d'applications dans beaucoup de domaines (biologie, chimie, dynamique de population, physique nucléaire, etc). On cite tout particulièrement les livres de Harris [39] et Athreya et Ney [7] comme références sur les processus de Galton-Watson. La théorie des processus de branchement a bénéficiée d'un intérêt grandissant avec l'introduction d'un environnement aléatoire par Smith et Wilkinson [65], et Athreya et Karlin [5, 6]. Depuis lors et jusqu'à maintenant, ces processus sont beaucoup étudiés ; les biologistes, notamment, y prettent une attention toute particulière car des problèmes liés à la transformation génétique pourraient être étudiés à l'aide de ces processus (voir notamment Bansaye [8] sur la biologie cellulaire).

Le modèle de processus de branchement uni-type est aujourd'hui relativement bien compris, et de nombreux résultats important ont été établis : voir par exemple [52, 27, 38, 28, 37, 4, 3] pour l'étude de la probabilité de survie et la convergence conditionnelle de la population dans le cas critique et sous-critique, et [38, 9, 53, 43, 61, 44, 31, 29] pour l'étude des moments et de grandes déviations dans le cas surcritique. Voir aussi le livre récent de Kersting et Vatutin [50] et sa bibliographie. En ce qui concerne le cas multi-type en environnement aléatoire, le modèle est moins connu de part sa grande complexité, mais de récents progrès ont été faits, principalement dans les cas critique et sous-critique pour lesquels le processus s'éteind presque sûrement ; parmis beaucoup de travaux, il y a ceux de Le Page, Peigné, et Pham [57], Vatutin et Dyakonova [70], et Vatutin et Wachtel [73], qui principalement ont étudié la vitesse de convergence de la probabilité de survie du processus de branchement. Pour le régime surcritique, il existe très peu de résultats jusqu'à ce jour, le plus récent étant celui de Cohn [17] donnant un théorème limite de convergence L^2 ; on peut aussi citer les résultats de Jones [45] et Biggins, Cohn et Nerman [12] qui ont établis des théorèmes limites pour des processus de branchement en environnement variable, qui peuvent s'appliquer pour le cas d'environnements aléatoires pour obtenir des résultats de convergence presque sûre et de convergence conditionnelle dans L^p sachant l'environnement. Le principal objectif de cette thèse est de démontrer des propriétés asymptotiques des processus de branchements multi-type en environnement aléatoire en régime surcritique, en généralisant celles bien connus du modèle de Galton-Watson multi-type sans environnement aléatoire, et du cas uni-type avec ou sans environnement aléatoire.

1.2 Description du modèle et notations

Soit $d \geq 1$ un entier fixé. On considère l'espace $\mathbb{R}^d$ muni de la norme L^1 et du produit scalaire définis par :

$$\|x\| := \sum_{i=1}^{d} x(i), \quad \text{et} \quad \langle x, y\rangle = \sum_{i=1}^{d} x(i)y(i), \quad x, y \in \mathbb{R}^d.$$

Soit $\mathcal{M}_d(\mathbb{R})$ l'espace des matrices de taille $d \times d$ dont les entrées sont réelles. On équipe $\mathcal{M}_d(\mathbb{R})$ avec la norme subordonnée à la norme $\|\cdot\|$:

$$\|M\| = \sup_{\|x\|=1} \|Mx\|, \quad M \in \mathcal{M}_d(\mathbb{R}).$$

Un processus de branchement $Z_n = (Z_n(1), \cdots, Z_n(d))$ multi-type en environnement aléatoire est défini de la façon suivante. On note $\xi = (\xi_n)_{n\geq 0}$ l'environnement aléatoire ; c'est une suite stationnaire ergodique de variables aléatoires prennnant leurs valeurs dans un espace abstrait $\mathbb{X}$. A chaque réalisation de ξ_n est associé d lois de probabilités sur $\mathbb{N}^d$ que l'on définit par leurs fonctions génératrices :

$$f_n^r(s) = \sum_{k_1,\cdots,k_d=0}^{\infty} p^r_{k_1,\cdots,k_d}(\xi_n) s_1^{k_1} \cdots s_d^{k_d}, \quad s = (s_1, \cdots, s_d) \in [0,1]^d, \quad 1 \leq r \leq d,$$

où les quantités $p^r_{k_1,\cdots,k_d}(\xi_n)$ sont positives et sont tels que $\sum_{k_1,\cdots,k_d=0}^{\infty} p^r_{k_1,\cdots,k_d}(\xi_n) = 1$. Le processus de branchement $(Z_n)_{n\geq 0}$ dans l'environnement aléatoire ξ est une suite à valeurs dans $\mathbb{N}^d$ tel que sa valeur initiale $Z_0 \in \mathbb{N}^d$ est fixée, et pour tout $n \geq 0$,

$$Z_{n+1} = \sum_{r=1}^{d} \sum_{l=1}^{Z_n(r)} N^r_{l,n}, \tag{1.1}$$

où chaque coordonnée $N^r_{l,n}(j)$ du vecteur aléatoire $N^r_{l,n} = (N^r_{l,n}(1), \cdots, N^r_{l,n}(d))$ représente le nombre de particules de type j produites à la génération $n+1$ par la l-ième particule de type r à la génération n, $Z_n(j)$ est le nombre total de particules de type j à la génération n. Sachant l'environnement ξ, les vecteurs $N^r_{l,n} = (N^r_{l,n}(1), \cdots, N^r_{l,n}(d))$ indexés par $l \geq 1$, $n \geq 0$ et $1 \leq r \leq d$ sont indépendants, et chaque $N^r_{l,n}$ admet la fonction génératrice f^r_n.

On distingue deux aléas différents dans le processus (Z_n) : l'environnement ξ, et celui venant des lois de reproduction f^r_n à ξ fixé. Notons que ce modèle généralise celui de Galton-Watson (où l'environnement est déterministe), ainsi que le modèle avec un seul type de particule. On note par $\mathbb{P}_\xi$ la probabilité sous-jacente lorsque l'environnement ξ est fixé. Soit τ la loi de l'environnement ξ ; on note par $\mathbb{P}$ la loi totale de (Z_n), définie par $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$. La probabilité $\mathbb{P}_\xi$, peut être vue comme la mesure $\mathbb{P}$ sachant l'environnement ξ. On définit $\mathbb{E}_\xi$ et $\mathbb{E}$ respectivement les espérences selon les mesures $\mathbb{P}_\xi$ et $\mathbb{P}$. Par ces définitions, on peut écrire la fonction génératrice f^r_n du vecteur aléatoire $N^r_{l,n}$ sous la forme suivante :

$$f^r_n(s) = \mathbb{E}_\xi\left(\prod_{j=1}^d s_j^{N^r_{l,n}(j)}\right), \quad s = (s_1, \cdots, s_d) \in [0,1]^d.$$

On associe au processus de branchement (Z_n) la suite des matrices moyennes conditionnée à l'environnement ξ, notée $(M_n)_{n\geq 0}$. Pour tout $n \geq 0$, M_n est une matrice aléatoire de taille $d \times d$ dont les coefficients sont

$$M_n(i,j) := \frac{\partial f^i_n}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi\Big[Z_{n+1}(j)\Big|Z_n = e_i\Big], \quad 1 \leq i, j \leq d,$$

où $\frac{\partial f^i_n}{\partial s_j}(\mathbf{1})$ est la dérivée à gauche en $\mathbf{1}$ par rapport à s_j de f^i_n, et e_i est le vecteur de taille d avec 1 comme i-ème coordonnée et 0 ailleurs ; $M_n(i,j)$ représente la moyenne conditionnelle du nombre d'enfants de type j produits par une particule de type i au temps n. La matrice M_n dépend seulement de l'environnement ξ_n, en particulier (M_n) est une suite stationnaire ergodique de matrices aléatoires puisque $\xi = (\xi_n)$ est stationnaire ergodique. Pour tout $0 \leq k \leq n$, on définit également le produit des matrices moyennes $M_{k,n} := M_k \cdots M_n$; il est facile de voir que la composante (i,j) de la matrice $M_{k,n}$ est

$$M_{k,n}(i,j) = \mathbb{E}_\xi\Big[Z_{n+1}(j)\Big|Z_k = e_i\Big]. \tag{1.2}$$

Par la suite, on notera $(Z^i_n)_{n\geq 0}$ le processus de branchement $(Z_n)_{n\geq 0}$ qui démarre avec une particule de type i à la génération 0, ce qui correspond à $Z_0 = e_i$. En prenant $k = 0$ dans (1.2), on obtient la moyenne conditionnelle de la population totale au temps $n+1$:

$$\mathbb{E}_\xi Z^i_{n+1}(j) = M_{0,n}(i,j), \quad n \geq 0, \quad 1 \leq i,j \leq d. \tag{1.3}$$

Ce que l'on cherche principalement à connaître du processus de branchement (Z^i_n), c'est son comportement asymptotique lorsque $n \to +\infty$. Grâce à la relation (1.3), on peut obtenir certaines informations à partir de celui du produit de matrices aléatoires $M_{0,n-1}$. Notamment, dans [26], Furstenberg et Kesten établissent une loi des grands nombres pour la norme du produit $\|M_{0,n-1}\|$: sous la condition $\mathbb{E}\log^+\|M_0\| < +\infty$,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-p.s.}, \tag{1.4}$$

où γ est une constante appelée exposant de Lyapunov de la suite de matrices (M_n) définie par

$$\gamma := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\| = \inf_{n\geq 1}\frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\|. \tag{1.5}$$

La loi des grands nombres (1.4) permet de distinguer deux comportements différents pour le produit de matrices $M_{0,n-1}$, suivant la valeur de γ : $\|M_{0,n-1}\| \to 0$ presque sûrement (p.s.) quand $\gamma < 0$, et $\|M_{0,n-1}\| \to +\infty$ p.s. quand $\gamma > 0$. Du fait que la ligne i de $M_{0,n-1}$ est la moyenne conditionnelle $\mathbb{E}_\xi Z^i_n$ définie en (1.3) et par la loi des grands nombres (1.4), l'exposant de Lyapunov γ permet une classification des processus de branchement multi-type en environnement aléatoire (PBMEA). On dira qu'un PBMEA est :

(1) sous-critique si $\gamma < 0$;

(2) critique si $\gamma = 0$;

(3) surcritique si $\gamma > 0$.

Par les travaux de Athreya et Karlin [5, Théorème 12], complétés par Kaplan [46, Théorème 2], si $\mathbb{E}|\log\sum_{i=1}^d(1-\mathbb{P}(\|Z^i_1\|=0))| < +\infty$ et s'il existe des constantes strictement positives C_1, C_2 et C_3 telles que $C_1 \leq M_0(i,j) \leq C_2$ et $0 < \mathbb{E}_\xi(Z^i_1(j))^2 \leq C_3$ presque surement pour tout $i,j \in \{1,\cdots,d\}$, alors :

(1) $\|Z^i_n\| \underset{n\to+\infty}{\to} 0$ p.s. quand $(Z^i_n)_{n\geq 0}$ est sous-critique ou critique ;

(2) $\mathbb{P}\Big(\|Z_n^i\| \underset{n\to+\infty}{\to} +\infty\Big) > 0$ quand $(Z_n^i)_{n\geq 0}$ est surcritique.

Ce résultat a aussi été démontré sous de plus faibles conditions, voir par exemple [68, Théorème 9.6] par Tanny, et aussi [65, Théorème 3.1] par Smith et Wilkinson pour le cas uni-type $d = 1$.

1.3 Résultats antérieurs

Nous présentons dans cette section d'importants résultats déjà établis sur les processus de branchement en environnement aléatoire qui sont en lien avec la thèse. En première partie, on énonce des résultats récents concernant les cas critique et sous-critique. Dans les parties suivantes, on présente les propriétés fondamentales du processus de branchement en régime surcritique, respectivement pour le modèle de Galton-Watson multi-type et pour le cas uni-type en environnement aléatoire, connus dans la littérature.

1.3.1 Les cas critique et sous-critique

Lorsque le processus de branchement (Z_n^i) est en régime critique ou sous-critique, sous certaines conditions il s'éteint presque sûrement (cf. section 1.2), autrement dit $\|Z_n^i\| \to 0$ quand $n \to +\infty$. Nous ne citerons que les travaux récents dans ce domaine, focalisés sur l'étude asymptotique de la probabilité de survie du processus : on aimerait savoir à quelle vitesse se produit l'extinction, c'est à dire on cherche à expliciter une suite $(a_n)_{n\geq 0}$ telle que

$$\mathbb{P}(\|Z_n^i\| > 0) \underset{n\to+\infty}{\sim} a_n.$$

Cas sous-critique

Le régime sous-critique est défini par la condition $\gamma < 0$. Différentes vitesses peuvent y être observées pour la probabilité de survie, ce qui conduit à une sous classification des processus. Soit $I = \{s \geq 0 : \mathbb{E}\|M_0\|^s < +\infty\}$ un intervalle de $\mathbb{R}_+$. Par le théorème ergodique de sous-additivité, pour tout $s \in I$ on définit la limite

$$\kappa(s) := \lim_{n\to+\infty} \Big(\mathbb{E}\|M_{0,n-1}\|^s\Big)^{1/n}.$$

Soit $\Lambda(s) := \log \kappa(s)$, $s \in I$. Sous de bonnes conditions, la fonction Λ est analytique sur l'intérieur de I, noté $\mathrm{int}(I)$ (cf. [14]). On suppose que $1 \in \mathrm{int}(I)$. Alors, suivant le signe de la quantité $\Lambda'(1)$, on observe ce que l'on appelle des transitions de phases dans l'asymptotique de la probabilité de survie $\mathbb{P}(\|Z_n^i\| > 0)$.

1. Quand $\Lambda'(1) < 0$, le régime est dit fortement sous-critique. Soit Γ le semi-groupe engendré par le support de la loi de M_0. On dit que Γ est fortement irréductible s'il n'existe aucune réunion finie de sous espaces propres de $\mathbb{R}^d$ invariante par l'action de Γ. Pour tout $1 \leq i \leq d$, On définit $B_0^i \in \mathcal{M}_d(\mathbb{R})$ la matrice Hessienne aléatoire de coefficients

$$B_0^i(r,j) := \frac{\partial^2 f_0^i}{\partial s_r \partial s_j}(\mathbf{1}) \quad 1 \leq r, j \leq d. \tag{1.6}$$

On pose

$$T_0 := \frac{1}{\|M_0\|^2} \sum_{i=1}^{d} \|B_0^i\|.$$

Vatutin et Wachtel [73] ont établi l'asymptotique exacte de la probabilité de survie dans le cas fortement sous-critique : si Γ est fortement irréductible, $M_0 > 0$ p.s., et s'il existe des constantes $\varepsilon > 0$ et $C > 0$ telles que $\mathbb{E}\Big[\|M_0\|| \log T_0|^{1+\varepsilon}\Big] < +\infty$ et

$$1 \leq \frac{\max_{1 \leq i,j \leq d} M_0(i,j)}{\min_{1 \leq i,j \leq d} M_0(i,j)} \leq C \quad \mathbb{P}\text{-p.s.},$$

alors pour chaque $1 \leq i \leq d$ il existe une constante $c_i > 0$ telle que

$$\mathbb{P}(\|Z_n^i\| > 0) \underset{n \to +\infty}{\sim} c_i \kappa(1)^n.$$

2. Lorsque $\Lambda'(1) = 0$, le processus (Z_n^i) est dit moyennement sous-critique. Soit $\mathcal{S} = \{x \in \mathbb{R}^d : x \geq 0, \|x\| = 1\}$. La vitesse de convergence de la probabilité de survie en régime moyennement sous-critique est donnée par le résultat de Vatutin et Dyakonova [71] : sous les conditions que Γ est fortement irréductible et qu'il existe des constantes $\varepsilon > 0$, $\delta > 0$ et $C > 0$ telles que $\mathbb{E}\Big[\|M_0\|| \log T_0|^{1+\varepsilon}\Big] < +\infty$, $\inf_{x \in \mathcal{S}} \mathbb{P}\Big(\log \|M_0 x\| > \delta\Big) > 0$ et

$$1 \leq \frac{\max_{1 \leq i,j \leq d} M_0(i,j)}{\min_{1 \leq i,j \leq d} M_0(i,j)} \leq C \quad \mathbb{P}\text{-p.s.},$$

alors il existe deux constantes $c > 0$ et $C > 0$ telles que pour tout $n \geq 1$ et tout

$1 \leq i \leq d$,

$$\frac{c}{\sqrt{n}}\kappa(1)^n \leq \mathbb{P}(\|Z_n^i\| > 0) \leq \frac{C}{\sqrt{n}}\kappa(1)^n.$$

Le cas $\Lambda'(1) > 0$, appelé régime faiblement sous critique, est jusqu'à présent trop peu étudié pour le modèle multi-type en environnement aléatoire, et il n'existe aucun résultat significatif.

Cas critique

On rappelle que le processus (Z_n^i) est en régime critique si la condition $\gamma = 0$ est satisfaite. Dans ce cas, il a été établi par Vatutin et Dyakonova dans [70] l'asymptotique de la probabilité de survie du PBMEA. Plus précisément, si Γ est fortement irréductible, $\mathrm{int}(I)$ est non vide, $\sup_{x\in\mathcal{S}} \mathbb{E}\|M_0^T x\|^{-1} < +\infty$, et s'il existe des constantes $\varepsilon > 0$, $\delta > 0$ et $C > 0$ telles que $\mathbb{E}\Big[T_0^{1+\varepsilon}\Big] < +\infty$, $\inf_{x\in\mathcal{S}} \mathbb{P}\Big(\log\|M_0^T x\| > \delta\Big) > 0$ et

$$1 \leq \frac{\max_{1\leq i,j\leq d} M_0(i,j)}{\min_{1\leq i,j\leq d} M_0(i,j)} \leq C \quad \mathbb{P}\text{-p.s.},$$

alors, pour tout $1 \leq i \leq d$ il existe une constante $c_i > 0$ telle que

$$\mathbb{P}(\|Z_n^i\| > 0) \underset{n\to+\infty}{\sim} \frac{c_i}{\sqrt{n}}.$$

Ce résultat est une généralisation de celui de Le Page, Peigné et Pham [57], qui l'avaient démontré dans le cas où la fonction génératrice $f_0 := (f_0^1, \cdots, f_n^d)$ est linéaire fractionnaire, c'est à dire qu'elle peut s'écrire sous la forme

$$f_0(s) = \mathbf{1} - \frac{1}{1 + \alpha_0\langle \mathbf{1}, \mathbf{1} - s\rangle} M_0(\mathbf{1} - s), \quad s \in [0,1]^d,$$

où α_0 est une variable aléatoire avec $\alpha_0 > 0$ p.s. ; ils avaient aussi montrer que, sous la condition supplémentaire $B_0^i(j,k) \leq C M_0(i,j)$ pour tout $1 \leq i,j,k \leq d$ où $C > 0$ est une constante, on a

$$\frac{C_1}{\sqrt{n}} \leq \mathbb{P}(\|Z_n^i\| > 0) \leq \frac{C_2}{\sqrt{n}}, \quad 1 \leq i \leq d,$$

avec $C_1 > 0$ et $C_2 > 0$ des constantes.

1.3.2 Le cas surcritique

Contrairement aux cas critique et sous-critique de la section précédente, en régime surcritique le processus de branchement (Z_n^i) a une chance de ne jamais s'éteindre. Plus précisément, sous certaines conditions on aura $\mathbb{P}\Big(\|Z_n^i\| \to 0\Big) < 1$. Ainsi, le problème fondamental en régime surcritique est de décrire la taille du processus (Z_n^i) sur l'évènement de survie $\{\|Z_n^i\| > 0, \ \ \forall n \geq 0\}$.

Le modèle de Galton-Watson multi-type

Un des modèles les plus connus et les plus simples de processus de branchement, le processus de Galton-Watson correspond au modèle de processus de branchement multi-type sans envionnement aléatoire (tous les environnements ξ_n sont réduits à un seul et même environnement déterministe). Beaucoup de résultats ont été établis sur ce modèle, on cite nottament le livre d'Athreya et Ney [7] qui est une bonne réference sur le sujet ; nous nous concentrerons ici sur les plus importants résultats concernant le régime surcritique. Pour une étude récente sur des sujets divers sur ce modèle, voir par exemple le travail de Abraham et Delmas [1] pour la convergence conditionnelle des arbres multi-types de Galton-Watson dans le cas critique, ainsi que celui de Chaumont et Liu [15] sur le codage des forêts multi-types .

Le theorème fondamental pour les processus de Galton-Watson en surcritique est celui que l'on appelle théorème de Kesten-Stigum, résultat démontré par Kesten et Stigum dans [49]. Ce théorème nous dit notamment quand le processus (Z_n^i) croît vers $+\infty$ a une vitesse exponentielle, à travers un critère de non-dégénérescence de la limite de la martingale fondamentale du processus de branchement. Du fait que les environnements soient déterministes, les matrices moyennes M_n, $n \geq 0$, sont toutes elles aussi déterministes, et sont égales à une seule et même matrice M. On dit que M est une matrice primitive s'il existe $n \geq 1$ tel que $M^n > 0$ (toutes les entrées de M^n sont strictement positives). Soit ρ le rayon spectral de M. Par le théorème de Perron Frobénius, ρ est une valeur propre simple de la matrice M, et il existe $u = (u(1), \cdots, u(d))$ et $v = (v(1), \cdots, v(d))$ respectivement les vecteurs propres à droite et à gauche associés à ρ, que l'on choisit avec les normalisations $\|u\| = 1$ et $\langle v, u \rangle = 1$; de plus, lorsque M est primitive, les vecteurs u et v sont strictement positifs. Notons que, par définition de γ, on a $\gamma = \log \rho$ (voir (1.4)) ; ainsi, la condition surcritique $\gamma > 0$ peut se réduire en $\rho > 1$. Le théorème de

Kesten-Stigum est le suivant : si $\rho > 1$ et M est primitive, alors pour tout $1 \leq i, j \leq d$,

$$\frac{Z_n^i(j)}{\rho^n} \underset{n \to +\infty}{\to} W^i u(i) v(j) \quad \mathbb{P}\text{-p.s.}, \tag{1.7}$$

où W^i est une variable aléatoire positive ; la limite W^i est non-dégénérée (i.e. $\mathbb{P}(W^i > 0) > 0$) pour tout $1 \leq i \leq d$ si et seulement si

$$\mathbb{E}\left(Z_1^i(j) \log^+ Z_1^i(j)\right) < +\infty, \quad \forall 1 \leq i, j \leq d. \tag{1.8}$$

De plus, si chaque W^i est non-dégénérée, alors

$$\mathbb{E}W^i = 1 \quad \text{et} \quad \mathbb{P}(W^i = 0) = \mathbb{P}(\|Z_n^i\| \to 0). \tag{1.9}$$

Concrètement, le théorème de Kesten-Stigum (correspondant aux relations (1.7)-(1.9)) annonce que sous la condition de moment (1.8), soit le processus (Z_n^i) s'éteind, soit chacunes de ses composantes explosent vers $+\infty$ avec la vitesse exponentielle ρ^n.

L'objet fondamental pour la démontration de ce théorème, et de façon générale pour l'étude du régime surcritique, est la martingale (W_n^i) définie par

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, u \rangle}{\rho^n u(i)}, \quad n \geq 1. \tag{1.10}$$

La suite (W_n^i) est une martingale positive sous la filtration

$$\mathcal{F}_n = \sigma(N_{l,k}^r,\ 0 \leq k \leq n-1, 1 \leq r \leq d, l \geq 1), \quad n \geq 1,$$

et $\mathcal{F}_0$ est la tribu grossière (cf. [7]). Alors (W_n^i) converge presque sûrement, et sa limite est la variable aléatoire W^i.

Le modèle uni-type en environnement aléatoire

Le cas uni-type ($d = 1$) est celui qu'on a le plus étudié dans la famille des processus de branchements en environnement aléatoire, et c'est le modèle pour lequel on a le plus de résultats, notamment en régime surcritique (cf. [50]). On notera (Z_n) le processus de branchement uni-type, et $m_n := M_n$, $n \geq 0$, les moyennes conditionnellement à l'environnement ξ qui sont dans ce cas scalaires, ce qui rend l'étude plus simple qu'en multi-type. On peut remarquer que $\gamma = \mathbb{E} \log m_0$, donc la condition surcritique peut s'écrire $\mathbb{E}\log m_0 > 0$. La martingale fondamentale (W_n) du processus (Z_n) est relativement simple à définir, car elle correspond à la suite de la population normalisée par la moyenne :

$$W_0 := 1, \quad W_n := \frac{Z_n}{\mathbb{E}_\xi Z_n} = \frac{Z_n}{m_0 \cdots m_{n-1}}. \tag{1.11}$$

On montre que (W_n) est effectivement une martingale positive sous la mesure $\mathbb{P}$ ou $\mathbb{P}_\xi$ par rapport à la filtration

$$\mathcal{F}_0 = \sigma(\xi) \quad \text{et} \quad \mathcal{F}_n = \sigma(\xi, N^1_{l,k},\ 0 \le k \le n-1, l \ge 1), \quad n \ge 1.$$

On obtient ainsi la convergence

$$W_n = \frac{Z_n}{m_0 \cdots m_{n-1}} \to W \quad \mathbb{P}\text{-p.s.}, \tag{1.12}$$

où W est une variable aléatoire positive finie presque sûrement. Pour en déduire la taille asymptotique de Z_n, il est nécessaire d'étudier la non-dégénérescence de W. Athreya et Karlin [6] ont établi que la condition

$$\mathbb{E}\Big(\frac{Z_1}{m_0} \log^+ \frac{Z_1}{m_0}\Big) < +\infty \tag{1.13}$$

est suffisante pour que W soit non-dégénérée, et que sous cette condition on a $\mathbb{P}$-p.s.

$$\mathbb{E}_\xi W = 1 \quad \text{et} \quad \mathbb{P}_\xi(W = 0) = \mathbb{P}_\xi(Z_n \to 0). \tag{1.14}$$

Plus tard, Tanny [69] a montré que la condition (1.13) était nécessaire et suffisante pour la non-dégénérescence de W dans le cas où la suite des environnements (ξ_n) est i.i.d. En outre, on peut montrer que la non-dégénérescence de W est équivalente à la convergence dans L^1 de la martingale (W_n). Ainsi, le résultat de Tanny [69] implique que (1.13) est une condition nécessaire et suffisante pour avoir $W_n \to W$ dans L^1.

Par la suite, il a été intéressant d'étudier la convergence $W_n \to W$ dans L^p, pour $p > 1$, dans le cas d'un environnement (ξ_n) i.i.d. Guivarc'h et Liu dans [38, Théorème 1.3] ont établi une condition nécessaire et suffisante pour cette convergence. Ils ont montré que

$W_n = \frac{Z_n}{m_0 \cdots m_{n-1}}$ converge vers W dans L^p si et seulement si

$$\mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < +\infty \quad \text{et} \quad \mathbb{E}m_0^{1-p} < 1. \tag{1.15}$$

Ensuite, Huang et Liu [44, Théorème 1.5] ont renforcé ce résultat en prouvant que, sous la condition (1.15), $W_n \to W$ dans L^p avec une vitese exponentielle qu'ils ont explicitée :

$$\lim_{n\to+\infty} \delta^{-n}\left(\mathbb{E}\left|\frac{Z_n}{m_{0,n-1}} - W\right|^p\right)^{1/p} = 0 \quad \forall\delta > \delta_c(p), \tag{1.16}$$

avec

$$\delta_c(p) = \begin{cases} (\mathbb{E}m_0^{1-p})^{1/p} & \text{si } p \in (1,2), \\ \max\left\{(\mathbb{E}m_0^{1-p})^{1/p}, (\mathbb{E}m_0^{-p/2})^{1/p}\right\} & \text{si } p \geq 2. \end{cases} \tag{1.17}$$

Sous la condition de non-dégénérescence (1.13) de la limite W, on obtient le même type de comportement du processus (Z_n) que dans le cas de Galton-Watson : le processus s'éteind sur l'évènement $\{W = 0\}$, et $Z_n \to +\infty$ quand $W > 0$. Plus précisément, on observe la décomposition suivante :

$$\log Z_n = S_n + \log W_n, \tag{1.18}$$

où $S_n = \sum_{k=1}^d \log m_k$ est la marche aléatoire associée au processus (Z_n), et $(\log m_n)$ est une suite de variables aléatoires stationnaires ergodiques. On sait par le théorème ergodique de Birkhoff que $\frac{S_n}{n} \to \mathbb{E}\log m_0$ p.s. quand $n \to +\infty$. Par conséquent, sous la condition (1.13) et lorque $W > 0$, le terme $\log W_n$ dans la décomposition (1.18) sera négligeable devant S_n quand $n \to +\infty$. Suivant cette logique, il est possible de transférer certaines propriétés asymptotiques de la suite (S_n) à $(\log Z_n)$. Notamment, $\log Z_n$ satisfait la loi des grands nombres suivante : sur l'évènement $\{Z_n \to +\infty\}$, p.s.,

$$\frac{1}{n}\log Z_n \underset{n\to+\infty}{\to} \mathbb{E}\log m_0. \tag{1.19}$$

Par (1.19), on retrouve la vitesse exponentielle du processus (Z_n) que l'on avait dans le cas de Galton-Watson.

D'autres théorèmes limites ont été démontrés grâce à la décoposition (1.18). A partir de maintenant, l'environnement $\xi = (\xi_n)$ est i.i.d. On se place aussi sous la condition que $p_0 := \mathbb{P}(Z_1 = 0) = 0$, c'est à dire que chaque particule de chaque génération du processus produit au moins une autre particule. Huang et Liu [43] ont établi un théorème central limite (TCL) pour $\log Z_n$: sous la condition supplémentaire que la variance $\sigma^2 = \mathrm{var}(\log m_0)$ est finie, on a

$$\frac{\log Z_n - n\mathbb{E}\log m_0}{\sqrt{n}} \underset{n\to+\infty}{\to} \mathcal{N}(0,\sigma^2) \quad \text{en loi}, \tag{1.20}$$

où $\mathcal{N}(0,\sigma^2)$ désigne la loi normale de moyenne 0 et de variance σ^2.

Recemment, Grama, Liu et Miqueu dans [31] ont travaillé sur un théorème de type Berry-Esseen pour $\log Z_n$, c'est à dire un théorème limite donnant une vitesse de convergence pour le TCL (1.20) : sous certaines conditions de moments sur la loi de reproduction du processus de branchement, il existe une constante $C > 0$ telle que pour tout $n \geq 1$,

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{\sigma\sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}}, \tag{1.21}$$

où $\Phi : x \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, dt$ est la fonction de répartition de loi normale centrée réduite, définie sur $\mathbb{R}$.

La preuve de (1.21) est basée sur la décomposition (1.18), sur le théorème de Berry-Esseen pour la somme de variable aléatoire i.i.d. S_n, et sur un bon contrôle du terme reste $\log W_n$ obtenu en démontrant l'existence des moments harmoniques $\mathbb{E}W^{-a}$ avec $a > 0$. Le point le plus compliquer (mais nécessaire) reste l'étude des moments harmoniques des limites W^i. On introduit la condition de bornitude suivante : il existe des constantes $p \in (1, 2]$ et $A > A_1 > 1$ telles que, $\mathbb{P}$-p.s.,

$$A_1 \leq m_0 \quad \text{et} \quad \mathbb{E}_\xi(Z_1^p) \leq A^p. \tag{1.22}$$

Sous la condition (1.22), Huang et Liu [43] ont montré que

$$\mathbb{E}W^{-a} < +\infty \quad \text{si et seulement si} \quad \mathbb{E}[p_1(\xi_0) m_0^a] < 1, \tag{1.23}$$

avec $p_1(\xi_0) := \mathbb{P}_\xi(Z_1 = 1)$. On peut reformuler (1.23) en disant que la quantité $a_0 > 0$ solution de l'équation $\mathbb{E}[p_1(\xi_0) m_0^{a_0}] = 1$ (et en fait unique solution) est l'exposant critique pour l'existence des moments harmoniques, au sens que $\mathbb{E}W^{-a} < +\infty$ si $a < a_0$, et $\mathbb{E}W^{-a} = +\infty$ si $a \geq a_0$. Le seul défaut de l'équivalence (1.23) est qu'elle n'est satisafaite que sous la très forte condition de bornitude (1.22). D'autres études ont été faites afin d'alléger cette condition, il y a par exemple les résultats de Grama, Liu et Miqueu [30, 31] qui ont montrés l'existence des moments harmoniques $\mathbb{E}W^{-a}$ pour $a > 0$ petit, sous des conditions de moments à la place de (1.22) ; en contrepartie l'exposant critique n'est pas identifié.

On présente enfin un théorème de déviations modérées de type Cramér pour le processus de branchement uni-type en environnement aléatoire (Z_n). On définit $\Lambda(s) := \log \mathbb{E} m_0^s$, $s \in I_\Lambda$ avec $I_\Lambda := \{s \in \mathbb{R} : \mathbb{E} m_0^s < +\infty\}$. Soit $\gamma_k := \Lambda^{(k)}(0)$ pour tout $k \geq 1$, avec en particulier $\gamma_1 = \mathbb{E} \log m_0$ et $\gamma_2 = \sigma^2$. On définit la série de Cramér ζ associée à la fonction Λ par

$$\zeta(t) := \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3} t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}} t^2 + \cdots , \tag{1.24}$$

une série qui converge lorque $|t|$ est suffisamment petit (voir [19] et [63]). Il a été établi dans [31] le résultat suivant concernant les déviations modérées de type Cramér pour $\log Z_n$: sous certaines conditions de moments, pour tout $0 \leq x \leq o(\sqrt{n})$, quand $n \to +\infty$,

$$\frac{\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{\sigma\sqrt{n}} > x \right)}{1 - \Phi(x)} = \mathrm{e}^{\frac{x^3}{\sqrt{n}} \zeta(\frac{x}{\sqrt{n}})} \left[1 + O\left(\frac{1+x}{\sqrt{n}} \right) \right], \tag{1.25}$$

et

$$\frac{\mathbb{P}\left(\frac{\log Z_n - n\mathbb{E}\log m_0}{\sigma\sqrt{n}} < -x \right)}{\Phi(-x)} = \mathrm{e}^{-\frac{x^3}{\sqrt{n}} \zeta(-\frac{x}{\sqrt{n}})} \left[1 + O\left(\frac{1+x}{\sqrt{n}} \right) \right]. \tag{1.26}$$

Le modèle multi-type en environnement aléatoire

Concernant les processus de branchements multi-type en environnement aléatoire, très peu de résultats sont à ce jour connus pour le cas surcritique, les recherches actuelles étant essentiellement focalisées sur les régimes critiques et sous-critiques. Le résultat fondamental que l'on cherche à établir pour le cas surcritique est un théorème de type Kesten-Stigum nous disant exactement quand chaque composante $Z_n^i(j)$ du processus (Z_n^i) tend vers l'infini avec vitesse exponentielle. Plus précisément, on voudrait obtenir une généralisation du Théorème de Kesten-Stigum du modèle de Galton-Watson décrit par les relations (1.7)-(1.9).

Dans la littérature, le résultat qui semble se rapprocher le plus d'un théorème de type Kesten-Stigum est celui de Cohn [17]. Cohn dans [17] se place sous la condition surcritique

$\gamma > 0$; il suppose qu'il existe des constantes C_1, C_2 et C_3 strictement positives telles que, $\mathbb{P}$-p.s.,

$$C_1 \leq M_0(i,j) \leq C_2 \quad \text{et} \quad 0 \leq B_0^i(r,j) \leq C_2 \quad \forall 1 \leq i,r,j \leq d, \tag{1.27}$$

où B_0^i est définie par (1.6) ; il suppose de plus la condition d'intégrabilité

$$\mathbb{E}|\log \sum_{i=1}^{d}(1 - \mathbb{P}(\|Z_1^i\| = 0))| < \infty. \tag{1.28}$$

Sous toutes ces hypothèses, Cohn annonce dans [17] que pour tout $1 \leq i,j \leq d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \to W^i \quad \text{dans } L^2, \tag{1.29}$$

où W^i est une variable aléatoire positive et non-dégénérée (c'est à dire $\mathbb{P}(W^i > 0) > 0$) avec $\mathbb{E}W^i = 1$. Bien que la convergence (1.29) soit cohérente avec celle observée pour le cas déterministe (voir (1.7)), le résultat annoncé par Cohn se révèle être faux. En effet, prenant le cas uni-type $d = 1$, par le théorème de Guivarc'h et Liu [38, Théorème 1.3] on sait que la condition (1.15) avec $p = 2$ est nécessaire et suffisante pour avoir (1.29) ; autrement dit, (1.29) est vraie si et seulement si $\mathbb{E}\left(\frac{Z_1}{m_0}\right)^2 < +\infty$ et $\mathbb{E}m_0^{-1} < 1$. On constate cependant que la condition nécessaire $\mathbb{E}m_0^{-1} < 1$ n'est pas impliquée par celles de Cohn (1.27) et (1.28). En conséquence le résultat de Cohn [17] est faux sous les conditions annoncées.

D'autres résultats sont tout de même établis. Notamment, Jones [45] a montré un théorème de convergence L^2 pour les processus de branchement multi-type en environnement variable (c'est à dire que l'environnement est fixé, non aléatoire). Son résultat peut totalement s'appliquer pour le modèle multi-type en environnement aléatoire : il donne une condition suffisante pour que $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$ converge dans L^2 sous la mesure quenched $\mathbb{P}_\xi$. Biggins, Cohn and Nerman dans [12] ont quant à eux étudié la convergence L^p pour le modèle en environnement variable, pour $p > 1$. De même, leur résultat permet d'obtenir pour les PBMEA un théorème de convergence L^p sous $\mathbb{P}_\xi$.

1.4 Objectifs et présentation des résultats de la thèse

Le principale objectif de cette thèse est d'étudier la taille du processus (Z_n^i) en régime surcritique lorsque $n \to +\infty$. Plus précisément, on cherche à étendre au PBMEA les résultats fondamentaux pour le cas surcritique du modèle de Galton-Watson et de celui uni-type en environnement aléatoire, présentés dans la section précédente. Notre étude est divisée en 4 chapitres.

Dans le chapitre 2, notre objectif est de démontrer un théorème de type Kesten-Stigum pour le processus (Z_n^i), qui est une extension pour le PBMEA des résultats de Kesten et Stigum [49] et Athreya et Karlin [6]. Pour cela, en utilisant les propriétés asymptotiques des produits de matrices aléatoires postives, on construit la martingale fondamentale (W_n^i), et on établit une condition nécéssaire et suffisante pour que la limite $W^i = \lim_{n\to+\infty} W_n^i$ soit non-dégénérée. En particulier, on en déduit une loi des grands nombres pour $\log \|Z_n^i\|$, ce qui implique une croissance exponentielle de la suite $(\|Z_n^i\|)$ lorsque le processus survit. De plus, on établi la convergence en loi de la direction $\frac{Z_n^i}{\|Z_n^i\|}$ conditionnellement à l'évènement d'explosion $\{\|Z_n^i\| \to +\infty\}$.

Dans le chapitre 3, on établit une condition nécessaire et suffisante pour la convergence dans L^p des composantes normalisées $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$ vers W^i. De plus, on démontre que la vitesse de ces convergences est exponentielle. Dans ce but, on étudie tout d'abord la convergence L^p de la martingale (W_n^i), pour laquelle on explicite la vitesse exponentielle lorsqu'elle converge.

Le principal objectif du chapitre 4 est d'établir un théorème de type Berry-Esseen pour $\log \|Z_n^i\|$, sous la condition que chaque particule produit au moins une autre particule à chaque génération. Plus précisément on démontre un théorème central limite pour $\log \|Z_n^i\|$ en donnant précisément la vitesse de convergence. La preuve de ce résultat est basée sur l'étude de l'existence des moments harmoniques de W^i (c'est à dire $\mathbb{E}(W^i)^{-a}$, $a \geq 0$). On détermine notamment, sous certaines conditions, la plus grande valeur de l'exposant a au deçà de laquelle $\mathbb{E}(W^i)^{-a}$ est fini pour tout $i = 1, \cdots, d$.

Le chapitre 5 est consacré à l'étude des déviations modérées pour $\log \|Z_n^i\|$. L'objectif est d'établir un théorème de type Cramer pour $\log \|Z_n^i\|$, c'est à dire un développement asymptotique plus poussé que la borne de Berry-Esseen pour $\log \|Z_n^i\|$. Pour cette étude, on définit une nouvelle mesure pour le processus (Z_n^i), et on démontre un théorème de type Berry-Esseen pour $\log \|Z_n^i\|$ sous cette nouvelle mesure.

1.4.1 Théorème de type Kesten-Stigum pour un processus de branchement multi-type en environnement aléatoire

Un des théorèmes fondamentaux les plus important dans l'étude d'un processus de branchement en régime surcritique est celui que l'on nomme dans la littérature "théorème de type Kesten-Stigum". Démontré depuis longtemps pour le processus de Galton-Watson dans [49] (cf. (1.7)-(1.9)) et pour le modèle uni-type en environnement aléatoire dans [6, 69] (cf. (1.13) et (1.14)), il n'existe aucun véritable équivalent à ces derniers pour le multi-type en environnement aléatoire. Les résulats de Cohn [17], Jones [45] et Biggins, Cohn et Nerman [12] donnent tout de même un début de réponse. L'objectif principal dans cette section est d'établir un théorème de type Kesten-Stigum pour le processus de branchement multi-type en environnement aléatoire. Notre approche est similaire à celle utilisée pour les modèles de Galton-Watson et uni-type en environnement aléatoire. Cependant la difficulté principale est la construction de la martingale fondamentale (W_n^i) du processus de branchement (Z_n^i), l'objet clé pour démontrer notre théorème de type Kesten-Stigum.

Martingale fondamentale associée au processus de branchement

La construction de la martingale (W_n^i) nécessite quelques propriétés et notations relatives aux produits des matrices moyennes $M_{n,n+k}$, pour $n, k \geq 0$. Plus précisément, nous allons utilisé une extension du théorème de Perron-Frobenius pour le produit de matrices aléatoires, un résultat établi par Hennion [40]. Soit $\mathcal{G}_+$ le semi-groupe des matrices de $\mathcal{M}_d(\mathbb{R})$ avec les entrées positives et qui sont admissibles (allowable), au sens où chaque ligne et chaque colonne contiennent au moins un élément strictement positif. On définit également $\mathcal{G}_+^0$ le sous espace de $\mathcal{G}_+$ des matrices strictement positives (matrices dont tous les coeffients sont strictement positifs). Hennion a établi son théorème sous la condition de positivité suivante :

H1. *La matrice M_0 prend ses valeurs dans le semi-groupe $\mathcal{G}_+$ $\mathbb{P}$-p.s. et*

$$\mathbb{P}\left(\bigcup_{n\geq 0} \left\{ M_{0,n} \in \mathcal{S}^0 \right\} \right) > 0.$$

Cette hypothèse signifie que, avec probabilité positive, il existe un entier n tel que le produit de matrices $M_{0,n}$ est strictement positif. Soit $\rho_{n,n+k}$ le rayon spectral de $M_{n,n+k}$. Sous la condition **H1**, par le théorème de Perron-Frobenius (voir e.g. [7]) $\rho_{n,n+k}$ est une valeur propre strictement positive de $M_{n,n+k}$, et il existe deux vecteurs propres positifs à droite et à gauche, respectivement $U_{n,n+k}$ et $V_{n,n+k}$, que l'on choisit avec les normalisations $\|U_{n,n+k}\| = 1$ et $\langle V_{n,n+k}, U_{n,n+k}\rangle = 1$. Par les résultats de Hennion [40, Lemme 3.3 et Théorème 1], sous la condition **H1**, pour tout $n \geq 0$ la limite

$$U_{n,\infty} := \lim_{k\to\infty} U_{n,n+k} \tag{1.30}$$

existe $\mathbb{P}$-p.s., avec $U_{n,\infty} > 0$ et $\|U_{n,\infty}\| = 1$; de plus la suite $(U_{n,\infty})$ satisfait la relation

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty}, \tag{1.31}$$

où (λ_n) est une suite de variables aléatoires strictement positives. La relation (1.31) ressemble à celle satisfaite par le rayon spectral ρ_n de M_n, à savoir

$$M_n U_{n,n} = \rho_n U_{n,n}, \quad n \geq 0. \tag{1.32}$$

En ce sens λ_n et ρ_n auront des comportements très similaires. La difficulté avec la relation (1.32) est qu'elle n'est pas stable par le produit des matrices M_n, à la différence de (1.31) qui l'est, grâce au décalage temporel de $U_{n,\infty}$ dans la relation. Cette stabilité par produit de (1.31) est la propriété permettant de construire la martingale (W_n^i). La suite (λ_n) sera centrale dans toute cette présente thèse ; ces λ_n seront par la suite appelés pseudo-rayon spectraux des matrices aléatoires (M_n). On notera que l'itération de (1.31) donne

$$M_{n,n+k} U_{n+k+1,\infty} = \lambda_{n,n+k} U_{n,\infty}, \quad n,k \geq 0, \tag{1.33}$$

où

$$\lambda_{n,n+k} := \prod_{r=n}^{n+k} \lambda_r.$$

Soit T l'opérateur de décalage des environnements :

$$T\xi = (\xi_1, \xi_2, \cdots) \quad \text{si} \quad \xi = (\xi_0, \xi_1, \cdots),$$

et soit T^n sa n-ième itération. Par leurs définitions (1.30) et (1.31), pour chaque $n \geq 0$ le vecteur $U_{n,\infty}$ et le scalaire λ_n ne dépendent que de $T^n\xi = (\xi_n, \xi_{n+1}, \cdots)$, et les suites $(U_{n,\infty})$ et (λ_n) sont stationnaires ergodiques, puisque l'environnement aléatoire $\xi = (\xi_n)$ l'est.

On définit la suite (W_n^i) de la façon suivante : pour tout $1 \leq i \leq d$,

$$W_0^i := 1, \quad W_n^i := \frac{\langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geq 1. \tag{1.34}$$

Par (1.3) et (1.33), on calcule que

$$\mathbb{E}_\xi \langle Z_n^i, U_{n,\infty} \rangle = \langle M_{0,n-1}(i, \cdot), U_{n,\infty} \rangle = \lambda_{0,n-1} U_{0,\infty}(i),$$

ce qui implique que la suite (W_n^i) est normalisée de sorte que $\mathbb{E}_\xi W_n^i = 1$ $\mathbb{P}$-p.s. Notre premier résultat établit que (W_n^i) est une martingale par rapport à la filtration

$$\mathcal{F}_0 = \sigma(\xi), \quad \mathcal{F}_n = \sigma(\xi, N_{l,k}^r,\ 0 \leq k \leq n-1, 1 \leq r \leq d, l \geq 1), \quad n \geq 1.$$

Theorem 1.4.1. *Supposons la condition* ***H1****. Alors pour chaque $1 \leq i \leq d$ la suite (W_n^i) est une martingale positive par rapport à la filtration $(\mathcal{F}_n)$ sous les mesures $\mathbb{P}_\xi$ et $\mathbb{P}$, et donc elle converge $\mathbb{P}$-p.s. vers une variable aléatoire positive W^i qui satisfait $\mathbb{E}_\xi W^i \leq 1$ $\mathbb{P}$-p.s.*

Pour le cas d'un environnement ξ déterministe, les matrices M_n, $n \geq 0$, sont toutes identiques à la même matrice M (déterministe) ; de plus, $U_{n,\infty} = U_{n,n+k} = u$, où u est l'unique vecteur propre à droite de M associé avec le rayon spectral ρ, et $\lambda_{0,n-1} = \rho^n$, ce qui donne $W_n^i = \frac{\langle Z_n^i, u \rangle}{\rho^n u(i)}$. Donc la martingale (1.34) coïncide avec celle (1.10) du modèle de Galton-Watson. D'autre part, pour le cas uni-type en environnement aléatoire, il est clair que $U_{n,\infty} = 1$ $\mathbb{P}$-p.s., et $\lambda_{0,n-1} = m_0 \cdots m_{n-1}$, donc (1.34) coïncide avec (1.11).

Par la suite, on va chercher à étudier la limite W^i, et il sera intéressant de considérer sa transformée de Laplace quenched

$$\phi_\xi^i(t) = \mathbb{E}_\xi \, \mathrm{e}^{-tW^i}, \quad t \geq 0, \quad 1 \leq i \leq d. \tag{1.35}$$

Le résultat suivant montre que ϕ_ξ^i satisfait une certaine équation fonctionnelle, comme pour le modèle de Galton-Watson [7, Théorème 2].

Theorem 1.4.2. *Supposons la condition* ***H1****. Alors pour chaque $1 \leq i \leq d$, la transformée de Laplace quenched ϕ_ξ^i de W^i satisfait*

$$\phi_\xi^i(t) = f_0^i \left(\phi_{T\xi}^1 \left(t \frac{U_{1,\infty}(1)}{\lambda_0 U_{0,\infty}(i)} \right), \cdots, \phi_{T\xi}^d \left(t \frac{U_{1,\infty}(d)}{\lambda_0 U_{0,\infty}(i)} \right) \right), \quad t \geq 0. \tag{1.36}$$

Notons que par (1.34), la martingale (W_n^i) est une projection normalisée du vecteur population (Z_n^i) du processus de branchement sur une droite aléatoire donnée par la suite de vecteurs strictement positifs $(U_{n,\infty})$. En ce sens, W_n^i est la population totale normalisée du processus de branchement, pour une certaine norme aléatoire et dépendant de n ; ainsi la convergence $W_n^i \to W^i$ $\mathbb{P}$-p.s. va permettre un contrôle de la population normalisée lorsque la limite W^i n'est pas nulle, correspondant au cas non-dégénéré.

La principale difficultée dans l'étude de la martingale (W_n^i) est que pour chaque $n \geq 0$ fixé, W_n^i dépend de tout l'environnement ξ, y compris le futur $T^n\xi = (\xi_n, \xi_{n+1}, \cdots)$. C'est un phénomène nouveau qu'on ne rencontrait pas dans les modèles de Galton-Watson et uni-type en environnement aléatoire, ce qui rend l'étude à la fois plus intéressante et plus compliquée.

Non-dégénérescence de la limite W^i

La convergence p.s. $W_n^i \to W^i$ de la martingale est une propriété importante concernant le processus de branchement (Z_n^i), mais elle ne permet pas à elle seule d'étudier le comportement asymptotique de (Z_n^i), et plus précisément de le comparer avec celui du produit de matrices $M_{0,n-1}$. Pour arriver à cette comparaison, il faut que la limite W^i soit non-dégénérée, c'est à dire que $\mathbb{P}_\xi\Big(W^i > 0\Big) > 0$ $\mathbb{P}$-p.s., c'est pourquoi on cherche une condition suffisante (dans le meilleur des cas nécessaire et suffisante) pour la non-dégénérescence de W^i, $1 \leq i \leq d$.

On rappelle que l'exposant de Lyapunov γ définie par (1.5) existe sous la condition suivante :

H2. *La matrice aléatoire M_0 satisfait la condition de moment*

$$\mathbb{E}\log^+ \|M_0\| < +\infty.$$

À partir de maintenant on se placera toujours en régime surcritique, c'est à dire que $\gamma > 0$. On définit $q^i(\xi)$ la probabilité d'extinction du processus de branchement (Z_n^i) conditionnellement à l'environnement ξ, c'est à dire

$$q^i(\xi) := \mathbb{P}_\xi\Big(\lim_{n\to+\infty}\|Z_n^i\| = 0\Big).$$

Notre premier résultat donne une condition suffisante pour la non-dégénérescence de toutes les limites W^i, $1 \leq i \leq d$, sous la condition **H1**. Cette condition suffisante est la suivante :

H3. *Il existe une constante $C > 1$ telle que, pour tout $1 \leq i \leq d$, $\mathbb{P}$-p.s.*

$$\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_{1,n}^i, U_{n+1,\infty} \rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\{\langle N_{1,n}^i, U_{n+1,\infty} \rangle \geq C^n\}} \right) < +\infty.$$

Theorem 1.4.3. *Supposons les conditions* ***H1***, ***H2*** *et $\gamma > 0$. Alors* ***H3*** *est une condition suffisante pour W^i, $1 \leq i \leq d$ d'être non-dégénérée, c'est à dire*

$$\mathbb{P}_\xi\Big(W^i > 0\Big) > 0, \quad \mathbb{P}\text{-}p.s., \quad 1 \leq i \leq d. \tag{1.37}$$

De plus, quand W^i, $1 \leq i \leq d$ sont non-dégénérées, alors

$$\mathbb{E}_\xi W^i = 1 \quad \mathbb{P}\text{-}p.s., \tag{1.38}$$

et

$$\mathbb{P}_\xi\Big(W^i = 0\Big) = q^i(\xi) \quad \mathbb{P}\text{-}p.s. \tag{1.39}$$

Par le théorème de Sheffé, l'égalité (1.38) est équivalente à la convergence dans L^1 de W_n^i vers W^i. Par conséquent on peut voir **H3** comme une condition suffisante de la convergence dans L^1 de la martingale (W_n^i), pour tout $1 \leq i \leq d$. La propriété (1.39) indique que, lorsque tous les W^i sont non-dégénérés, alors W^i est nulle uniquement sur l'évènement d'extinction du processus de branchement $\{\|Z_n^i\| \to 0\}$.

Un des arguments clés pour démontrer ce résultat est la proposition suivante donnant une loi des grands nombres pour le produit $\lambda_{0,n-1}$ des pseudo-rayon spectraux.

Proposition 1.4.4. *Supposons les conditions* ***H1*** *et* ***H2***. *Alors l'espérance $\mathbb{E} \log \lambda_0$ est bien définie, à valeur dans $\mathbb{R} \cup \{-\infty\}$, et*

$$\lim_{n \to +\infty} \frac{1}{n} \log \lambda_{0,n-1} = \mathbb{E} \log \lambda_0 = \gamma \quad \mathbb{P}\text{-}p.s.$$

On notera que ce résultat permet de reformuler la classification des processus de branchement multi-type en environnements aléatoires : sous les conditions **H1** et **H2**, (Z_n^i) est surcritique si $\mathbb{E} \log \lambda_0 > 0$, critique si $\mathbb{E} \log \lambda_0 = 0$, et sous-critique si $\mathbb{E} \log \lambda_0 < 0$.

La condition suffisante **H3** est malheureusement complexe et difficile à vérifier. Cela étant, nous allons voir qu'il est possible de la remplacer par d'autres conditions, certes plus fortes, mais meilleures en pratique. Tout particulièrement nous allons établir une condition suffisante de non-dégénérescence du type $EX \log^+ X < \infty$, condition qui s'avérera être nécessaire et suffisante sous des hypothèses plus fortes. Les 3 conditions introduites ci-dessous sont plus fortes que **H3**, et sont toutes par conséquent des conditions suffisantes pour la non-dégénérescence des W^i.

H4. *Il existe une constante $C > 1$ telle que, pour tout $1 \leq i, j \leq d$, $\mathbb{P}$-p.s.,*

$$\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{N^i_{1,n}(j)}{M_n(i,j)} \mathbb{1}_{\left\{ \frac{N^i_{1,n}(j)}{M_n(i,j)} \geq C^n \right\}} \right) < +\infty.$$

H5. *Pour tout $1 \leq i \leq d$,*

$$\mathbb{E} \left(\frac{\langle Z^i_1, U_{1,\infty} \rangle}{\lambda_0 U_{0,\infty}(i)} \log^+ \langle Z^i_1, U_{1,\infty} \rangle \right) < +\infty.$$

H6. *Pour tout $1 \leq i, j \leq d$,*

$$\mathbb{E} \left(\frac{Z^i_1(j)}{M_0(i,j)} \log^+ \frac{Z^i_1(j)}{M_0(i,j)} \right) < +\infty.$$

On peut montrer que, sous les conditions **H1** et **H2**, on a les implications

$$\textbf{H6} \Rightarrow \textbf{H4} \Rightarrow \textbf{H3} \quad \text{et} \quad \textbf{H6} \Rightarrow \textbf{H5} \Rightarrow \textbf{H3}. \tag{1.40}$$

En conséquence, les conditions **H4**, **H5** et **H6** peuvent remplacer **H3** dans le Théorème 1.4.3. La condition **H6** est celle du type $EX \log^+ X < \infty$; elle généralise la condition classique du cas unitype (1.13) et celle du modèle de Galton-Watson (1.8) ; c'est la plus facile à vérifier, et elle sera toujours une condition suffisante de non-dégénérescence des W^i.

Le prochain résultat est une conséquence directe du Théorème 1.4.3 et met en avant la puissance de la propriété de non-dégénérescence des W^i. On définit l'évènement d'explosion du processus de branchement (Z^i_n), $1 \leq i \leq d$, par

$$E^i := \{ \lim_{n \to +\infty} \|Z^i_n\| = +\infty \}.$$

Corollary 1.4.5. *Supposons les conditions **H1**, **H2** et $\gamma > 0$. Supposons de plus que l'une des conditions suivantes **H3**, **H4**, **H5** ou **H6** est satisfaite. Alors pour tout $1 \leq i \leq d$ on* *a* $q^i(\xi) < 1$ $\mathbb{P}$-*p.s. et*

$$\mathbb{P}_\xi(E^i) = 1 - q^i(\xi) \quad \mathbb{P}\text{-}p.s. \tag{1.41}$$

De plus, sur l'événement d'explosion E^i *on a*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Z_n^i\| = \gamma \quad \mathbb{P}\text{-}p.s. \tag{1.42}$$

L'égalité (1.41) est en fait une réécriture de (1.39), en remarquant que $\mathbb{P}_\xi(W^i > 0) \leq \mathbb{P}_\xi(E^i)$ p.s. La relation (1.42) est une loi des grands nombres satisfaite pour le logarithme $\log \|Z_n^i\|$ de la population totale $\|Z_n^i\| = Z_n^i(1) + \cdots + Z_n^i(d)$. On notera que la relation (1.41) est une extension du résultat de Kaplan [46, Théorème 1], qui a démontré ce résultat dans le cas où $c \leq M_0(i,j) \leq C$ p.s. avec $c > 0$ et $C > 0$ des constantes, et où les moments d'ordre 2 conditionnellement à ξ des lois de reproductions sont bornés p.s. par une constante strictement positive. On fait aussi référence à Tanny [68, Théorème 1] qui a montrer (1.41) et (1.42) sous la condition que $q_i(\xi) = 1$ p.s. pour tout i ou $q_i(\xi) < 1$ p.s. pour tout i.

On introduit maintenant la condition suivante de Furstenberg et Kesten [26] :

H7. *Il existe une constante* $D > 1$ *telle que* $\mathbb{P}$-*p.s.,*

$$1 \leq \frac{\max\limits_{1\leq i,j\leq d} M_0(i,j)}{\min\limits_{1\leq i,j\leq d} M_0(i,j)} \leq D.$$

Il est évident que $M_0 > 0$ p.s. sous la condition **H7**, ainsi **H7** implique **H1**. La condition **H7** simplifie énormément l'étude du processus de branchement Z_n^i. Notamment, combinée à l'équation (1.31), elle permet d'obtenir la minoration $U_{n,\infty}(i) \geq C$ p.s. pour tout $1 \leq i \leq d$, où $C > 0$ est une constante. Avec cette minoration on peut contrôler les termes qui dépendent du futur $(\xi_n, \xi_{n+1}, \cdots)$ et qui compliquent de façon importante l'étude. Par exemple, on montre que, sous les conditions **H7** et **H2**, les implications (1.40) sont renforcées par les équivalences suivantes :

$$\textbf{H4} \Leftrightarrow \textbf{H3} \quad \text{et} \quad \textbf{H6} \Leftrightarrow \textbf{H5}; \tag{1.43}$$

si de plus la suite des environnements (ξ_n) est i.i.d., alors on a

$$\textbf{H3} \Leftrightarrow \textbf{H4} \Leftrightarrow \textbf{H5} \Leftrightarrow \textbf{H6}. \tag{1.44}$$

Les équivalences (1.43) et (1.44) permettent de donner une condition nécessaire et suffisante pour la non-dégénérescence de tous les W^i, sous la condition **H7**. Le résultat est le suivant :

Theorem 1.4.6. *Supposons les conditions **H2**, **H7** et $\gamma > 0$. Alors la condition **H4** est nécessaire et suffisante pour que W^i, $1 \leq i \leq d$, soient non-dégénérées (au sens de (1.37)) ; cette condition est équivalente à **H6** quand l'environnement (ξ_n) est i.i.d. De plus, lorsque W^i, $1 \leq i \leq d$, sont non-dégénérées, alors (1.38) et (1.39) sont satisfaites.*

Comportement asymptotique du processus de branchement (Z_n^i)

Le but de cette section est d'étudier le comportement asymptotique de (Z_n^i). Sous les hypothèses du corollaire 1.4.5, on sait déjà que $\|Z_n^i\| \to +\infty$ avec une vitesse exponentielle, dans le cas où les W^i sont non-dégénérés. On s'intéresse maintenant aux coordonnées $Z_n^i(j)$, $1 \leq i,j \leq d$. Pour cette étude, on se placera toujours sous la condition de Furstenberg-Kesten **H7** et pour un environnement (ξ_n) i.i.d. Notre objectif principal est d'établir un théorème de type Kesten-Stigum pour le processus de branchment (Z_n^i), généralisant le théorème de Kesten-Stigum [49] pour le modèle de Galton-Watson.

On considère tout d'abord la direction du vecteur Z_n^i. On a besoin ici d'un autre résultat de Hennion [40, Théorème 1] : sous la condition **H1**, pour tout $n \geq 0$ la suite $(V_{n,n+k}/\|V_{n,n+k}\|)_{k\geq 0}$ converge en loi vers un vecteur aléatoire de norme 1, noté $\overline{V}_{0,\infty} > 0$:

$$\frac{V_{n,n+k}}{\|V_{n,n+k}\|} \xrightarrow[k\to+\infty]{d(\mathbb{P})} \overline{V}_{0,\infty}, \tag{1.45}$$

où $\xrightarrow{d(\mathbb{P})}$ désigne la convergence en loi sous la mesure $\mathbb{P}$. Quand $\mathbb{P}(E^i) > 0$, on définit $\mathbb{P}_{E^i} := \mathbb{P}(\cdot|E^i)$ la mesure de probabilité conditionnellement à E^i. Notre premier résultat donne une comparaison de la direction du vecteur Z_n^i avec celle du vecteur propre à gauche $V_{0,n-1}$ du produit de matrices $M_{0,n-1}$. On établit de plus la convergence en loi du vecteur direction $Z_n^i/\|Z_n^i\|$.

Theorem 1.4.7. *Supposons les conditions **H2**, **H7** et $\gamma > 0$. On suppose de plus que la suite des environnements $\xi = (\xi_0, \xi_1, \cdots)$ est i.i.d. Alors, pour tout $1 \leq i \leq d$ tel que*

$\mathbb{P}(E^i) > 0$, *on a*

$$\left\| \frac{Z_n^i}{\|Z_n^i\|} - \frac{V_{0,n-1}}{\|V_{0,n-1}\|} \right\| \xrightarrow[n\to+\infty]{\mathbb{P}_{E^i}} 0; \tag{1.46}$$

de plus, conditionnellement à l'événement E^i*, la suite* $(Z_n^i/\|Z_n^i\|)_{n\geq 0}$ *converge en loi vers* $\overline{V}_{0,\infty}$ *:*

$$\frac{Z_n^i}{\|Z_n^i\|} \xrightarrow[n\to+\infty]{d(\mathbb{P}_{E^i})} \overline{V}_{0,\infty}. \tag{1.47}$$

Ce théorème généralise le résultat de Kurtz, Lyons, Pemantle et Perez [54] établi pour le modèle de Galton-Watson, où dans ce cas la convergence (1.47) est presque sûre et le vecteur $\overline{V}_{0,\infty}$ est déterministe.

C'est la combinaison du Théorème 1.4.7 et de la convergence presque sûre $W_n^i \to W^i$ de la martingale qui permet de trouver une bonne normalisation des coordonnées $Z_n^i(j)$ pour que celles-ci convergent. Plus précisément, le théorème suivant établit une convergence en probabilité de la composante $Z_n^i(j)$ sous deux différentes normalisations : $\mathbb{E}_\xi Z_n^i(j) = M_{0,n-1}(i,j)$ et $\rho_{0,n-1}V_{0,n-1}(j)$. On donne aussi une condition nécessaire et suffisante pour la non-dégénérescence de leurs limites.

Theorem 1.4.8. *Supposons les conditions du Théorème 1.4.7. Alors, pour* $1 \leq i,j \leq d$*,*

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i \tag{1.48}$$

et

$$\frac{Z_n^i(j)}{\rho_{0,n-1}V_{0,n-1}(j)} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i U_{0,\infty}(i). \tag{1.49}$$

De plus, les variables limites W^i*,* $1 \leq i \leq d$*, sont non-dégénérées (au sens de* (1.37)*) si et seulement si* ***H6*** *est satisfaite ; quand* ***H6*** *est vraie, on a* (1.38) *et* (1.39).

Notons que les deux normalisations $M_{0,n-1}(i,j)$ et $\rho_{0,n-1}V_{0,n-1}(j)$ sont en réalité équivalentes quand $n \to +\infty$ modulo une variable aléatoire strictement positive : c'est le résultat de Hennion [40, Théorème 1], qui annonce que sous la condition **H1** on a, quand $n \to +\infty$,

$$M_{0,n-1}(i,j) \sim U_{0,\infty}(i)\rho_{0,n-1}V_{0,n-1}(j), \quad \text{avec} \quad U_{0,\infty}(i) > 0 \ \mathbb{P}\text{-p.s.} \tag{1.50}$$

Ainsi les convergences (1.48) et (1.49) sont totalement équivalentes.

La première convergence (1.48) peut être comparé à celle de Cohn [17], qui a annoncée une convergence L^2 de $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$ sous certaines conditions. Même si on a montré que son résultat était faux (voir section 1.3), il avait tout de même trouvé la bonne normalisation. La convergence (1.49) correspond à celle du théorème de Kesten-Stigum (1.7) établie pour le modèle de Galton-Watson. La seule différence vient dans le mode de convergence (en probabilité pour (1.49) et presque sûre pour (1.7)).

Sous les conditions **H2** et **H7**, Furstenberg et Kesten [26] ont établi une loi des grands nombres pour les composantes $M_{0,n-1}(i,j)$ du produit de matrices $M_{0,n-1}$: pour tout $1 \leq i,j \leq d$,

$$\lim_{n\to+\infty} \frac{1}{n} \log M_{0,n-1}(i,j) = \gamma \quad \mathbb{P}\text{-p.s.} \tag{1.51}$$

En combinant les convergences (1.48) et (1.51) avec la relation (1.41) donnant $E^i = \{W^i > 0\}$ sous la condition de non-dégénérescence **H6**, on obtient la propriété suivante : sous **H6** et conditionnellement à l'événement d'explosion E^i, toutes les coordonées $Z_n^i(j)$ du processus de branchement tendent en probabilité vers $+\infty$ avec une vitesse exponentielle.

En conséquence du Théorème 1.4.8, on peut obtenir le comportement asymptotique de la norme $\|Z_n^i\|$, c'est à dire le nombre total de particules du processus Z_n^i à la génération n.

Corollary 1.4.9. *Supposons les conditions du Théorème 1.4.7. Alors pour tout* $1 \leq i \leq d$,

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} = \frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i$$

et

$$\frac{\|Z_n^i\|}{\rho_{0,n-1}\|V_{0,n-1}\|} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i U_{0,\infty}(i).$$

Sous des conditions de moments supplémentaires sur les lois de reproductions et sur les matrices moyennes, on montre que les convergences en probabilité des Théorèmes 1.4.7, 1.4.8 et 1.4.9 peuvent être renforcées en convergences presque sûres.

Theorem 1.4.10. *Supposons les conditions **H2**, **H7** et* $\gamma > 0$. *On suppose de plus la suite des environnements* $\xi = (\xi_0, \xi_1, \cdots)$ *est i.i.d. On fait aussi l'hypothèse que pour un* *certain* $p > 1$,

$$\max_{1\le i,j\le d} \mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)^p < +\infty \quad et \quad \mathbb{E}\|M_0\|^{1-p} < +\infty. \tag{1.52}$$

Alors les propositions suivantes sont satisfaites :

1. W^i, $1 \le i \le d$ *sont non-dégénérées, et* (1.38) *et* (1.39) *sont vérifiées.*
2. *Pour tout* $1 \le i \le d$, $\mathbb{P}$-*p.s. sur l'événement* E^i,

$$\left\| \frac{Z_n^i}{\|Z_n^i\|} - \frac{V_{0,n-1}}{\|V_{0,n-1}\|} \right\| \underset{n\to+\infty}{\longrightarrow} 0. \tag{1.53}$$

3. *Pour tout* $1 \le i,j \le d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \underset{n\to+\infty}{\longrightarrow} W^i \quad \mathbb{P}\text{-}p.s., \tag{1.54}$$

$$\frac{Z_n^i(j)}{\rho_{0,n-1} V_{0,n-1}(j)} \underset{n\to+\infty}{\longrightarrow} W^i U_{0,\infty}(i) \quad \mathbb{P}\text{-}p.s. \tag{1.55}$$

4. *Pour tout* $1 \le i \le d$,

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} = \frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \underset{n\to+\infty}{\longrightarrow} W^i \quad \mathbb{P}\text{-}p.s., \tag{1.56}$$

$$\frac{\|Z_n^i\|}{\rho_{0,n-1}\|V_{0,n-1}\|} \underset{n\to+\infty}{\longrightarrow} W^i U_{0,\infty}(i) \quad \mathbb{P}\text{-}p.s. \tag{1.57}$$

1.4.2 Convergence dans L^p pour un processus de branchement multi-type surcritique en environnement aléatoire

Dans cette section, on cherche à étudier la convergence dans L^p de la martingale fondamentale (W_n^i) et des composantes normalisées $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$ pour tout $1 \le i, j \le d$. Jusqu'à présent, on sait que (W_n^i) converge p.s. vers W^i (voir Théorème 1.4.1), et les suites $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$ convergent en probabilité vers W^i sous les conditions du Théorème 1.4.8. On a de plus établi des conditions suffisantes pour la non-dégénérescence des limites

W^i (voir Théorèmes 1.4.3 et 1.4.6), et par le théorème de Sheffé la non-dégénérescence de W^i est équivalente aux convergences $W_n^i \to W^i$ et $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j) \to W^i$ dans L^1. Notre objectif est maintenant de trouver une condition suffisante, ou mieux nécessaire et suffisante, pour la convergence dans L^p de ces suites, et de montrer que toutes ces convergences surviennent avec une vitesse exponentielle que l'on explicitera dans certains cas. On cherche en fait à étendre au modèle multi-type en environnement aléatoire les résultats établis par Guivarc'h et Liu [38] et Huang et Liu [44] pour le cas uni-type. Notons que Jones [45] et Biggins, Cohn et Nerman [12] ont respectivement étudier les convergences L^2 et L^p pour les processus de branchement en environnement variables ; leurs résultats peuvent s'appliquer pour les PBMEA, et donne ainsi des conditions suffisantes pour la convergence L^p conditionnellement à l'environnement ξ. On mentionne aussi le travail de Cohn [17] dans lequel il a anoncé une condition suffisante de convergence L^2 pour un PBMEA.

Convergence de la martingale (W_n^i) dans L^p

On commence par l'étude de la convergence L^p de la martingale (W_n^i), pour tout $1 \le i \le d$. Pour établir une condition suffisante pour la convergence L^p de (W_n^i), on a besoin de notations supplémentaires. On définit

$$I := \Big\{s \le 0 : \ \mathbb{E}M_0(i,j)^s < +\infty \quad \forall i, j = 1, \cdots, d\big\}.$$

On remarque que I est un intervalle, en conséquence de l'inégalité de Hölder. Il est clair que s'il existe $s \in I$ avec $s < 0$, alors $M_0 > 0$ p.s., et donc la condition de positivité **H1** est vérifiée. Soit $\mathcal{S} = \{x \in \mathbb{R}^d : x \ge 0, \|x\| = 1\}$ l'intersection de la sphère unité de $\mathbb{R}^d$ avec le quadrant positif. On définit pour $M \in \mathcal{G}_+^0$ l'action projective de M sur $\mathcal{S}$ par $M \cdot x := \frac{Mx}{\|Mx\|}$, $x \in \mathcal{S}$. On considère $\mathcal{C}(\mathcal{S})$ l'espace des fonctions continues sur $\mathcal{S}$ à valeurs réelles, que l'on munit de la norme

$$\|\varphi\|_\infty = \sup_{x\in\mathcal{S}} \|\varphi x\|, \quad \varphi \in \mathcal{C}(\mathcal{S}). \tag{1.58}$$

Pour tout $s \in I$, on peut définir l'opérateur de transfert P_s associé à la matrice M_0 : pour tout $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s\varphi(x) := \mathbb{E}\Big[\|M_0x\|^s \varphi(M_0 \cdot x)\Big], \quad x \in \mathcal{S}. \tag{1.59}$$

On montre que pour tout $s \in I$, la limite

$$\kappa(s) := \lim_{n\to+\infty} \Big(\mathbb{E}\|M_{0,n-1}\|^s\Big)^{1/n} \tag{1.60}$$

existe avec $\kappa(s) < +\infty$; de plus $\kappa(s)$ est le rayon spectral de l'opérateur P_s. En dimension 1, la fonction κ correspond à la transformée de Laplace de la variable aléatoire $m_0 = M_0$. On notera que κ est log-convexe sur I.

Le premier théorème que l'on présente donne une condition suffisante pour la convergence dans L^p des martingales (W_n^i), $1 \leq i \leq d$. On montre aussi que cette condition est nécessaire quand la condition de Furstenberg-Kesten **H7** est satisfaite.

Theorem 1.4.11. *Soit $p > 1$ tel que $1 - p \in I$. Si*

$$\max_{1\leq i,j\leq d} \mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)^p < +\infty \quad et \quad \kappa(1-p) < 1, \tag{1.61}$$

alors $W_n^i \underset{n\to+\infty}{\longrightarrow} W^i$ dans L^p pour tout $1 \leq i \leq d$. La réciproque est aussi vraie sous la condition de Furstenberg-Kesten ***H7****.*

Ce résultat généralise celui de Guivarc'h et Liu [38, Théorème 1.3] du modèle uni-type en environnement aléatoire ($d = 1$). Notons que la condition surcritique $\gamma > 0$ n'apparait pas dans le Théorème 1.4.11 ; en fait, sous la condition **H2**, on a $\log \kappa(1-p) \geq (1-p)\gamma$ par application de l'inégalité de Jensen, et donc (1.61) implique que $\gamma > 0$.

Le résultat suivant établit que, sous la condition (1.61), la convergence $W_n^i \to W^i$ dans L^p est avec une vitesse exponentielle.

Theorem 1.4.12. *Soit $p > 1$ tel que $1 - p \in I$. Supposons la condition (1.61).*

1. *Si $1 < p \leq 2$, alors en notant $\delta_c(p) := \kappa(1-p)^{1/p}$ on a*

$$\limsup_{n\to+\infty} \; \delta_c(p)^{-n} \Big(\mathbb{E}|W_n^i - W^i|^p\Big)^{1/p} < +\infty. \tag{1.62}$$

2. *Si $p > 2$, alors $\delta_c(p) := \max\Big\{\kappa(1-p)^{1/p}, \kappa(-p/2)^{1/p}\Big\} < 1$, et*

$$\lim_{n\to+\infty} \delta^{-n} \Big(\mathbb{E}|W_n^i - W^i|^p\Big)^{1/p} = 0 \quad \forall \delta > \delta_c(p). \tag{1.63}$$

Ce théorème étend le resultat de Huang et Liu [44] pour le cas $d = 1$ avec la vitesse $\delta_c(p)^n$ correspondante. On peut remarquer que pour $p \geq 2$, en utilisant l'inégalité de

Hölder's sur $\mathbb{E}\|M_{0,n-1}\|^{-p/2}$ puis en faisant $n \to +\infty$, on obtient que $\kappa(-p/2)^{2/p} \leq \kappa(1-p)^{1/(p-1)}$; par conséquent, la condition (1.61) implique que $\delta_c(p) < 1$. Il a de plus été montré dans [44] que $\delta_c(p)$ correspondait à la vitesse optimale pour $d = 1$ sous des conditions de moments additionnelles, ainsi cela devrait être pareille pour le cas multi-type.

La preuve des Théorèmes 1.4.11 et 1.4.12 se base sur celle du cas $d = 1$ (voir [44]). Toutefois notre étude est plus complexe, dûe au fait que W_n^i dépende de tous les environnements $(\xi_0, \xi_1, \cdots)$ et pas seulement des environnements passés $(\xi_0, \cdots, \xi_{n-1})$.

La première étape de notre démonstration est de montrer que (1.61) est une condition suffisante pour avoir les convergences $W_n^i \to W^i$ dans L^p. On prouve en même temps le Théorème 1.4.12 donnant la vitesse exponentielle. Pour $p \in (1, 2]$, le principe est de contrôler la norme L^p, conditionnellement à ξ, des accroissements de la martingale $W_{n+1}^i - W_n^i$ en termes de $(\lambda_{0,n-1} U_{0,\infty}(i))^{1-p}$ (on utilise pour cela l'inégalité de Marcinkiewicz-Zygmund [16, Théorème 1.5]). Ensuite, on prends l'espérance conditionnellement au futur $T^n \xi = (\xi_n, \xi_{n+1}, \cdots)$, donnant une borne en $\mathbb{E}_{T^n\xi}(\lambda_{0,n-1} U_{0,\infty}(i))^{1-p}$. On démontre enfin que $\mathbb{E}_{T^n\xi}(\lambda_{0,n-1} U_{0,\infty}(i))^{1-p} \leq C\kappa(1-p)^n$ p.s., où $C > 0$ est une constante, ce qui donne la bonne vitesse de convergence dans L^p de la martingale (W_n^i). Pour $p > 2$, on fait une preuve par récurrence sur la valeur de p.

La deuxième étape est la preuve que la condition (1.61) est nécessaire. Pour ce faire, on établit des propriétés spectrales de l'opérateur de transfert P_s avec $s \leq 0$. Plus précisément, on montre que pour tout $s \in I$, le rayon spectral $\kappa(s)$ de l'opérateur P_s est une valeur propre de P_s, et qu'il existe une fonction $r_s \in \mathcal{C}(\mathcal{S})$ strictement positive telle

$$P_s r_s = \kappa(s) r_s.$$

La fonction propre r_s permet de faire apparaitre dans les calculs la quantité $\kappa(s)$ sans passage à la limite quand $n \to +\infty$ (voir (1.60) pour la définition de $\kappa(s)$) ; avec ceci, on obtient la condition nécessaire $\kappa(1-p) < 1$ avec une inégalité stricte.

Convergence des composantes normalisées $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$ dans L^p

On s'intéresse maintenant à l'étude de la convergence L^p de $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, $1 \leq i, j \leq d$, les composantes du processus (Z_n^i) normalisées par leurs moyennes respectives conditionnellement à l'environnement ξ. Comme pour le Théorème 1.4.8, on se placera sous la condition de Furstenberg-Kesten **H7**. Le théorème suivant donne une condition nécessaire et suffisante pour la convergence dans L^p de toutes les composantes normalisées

$Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$.

Theorem 1.4.13. *Supposons la condition* ***H7***. *Soit* $p > 1$ *tel que* $1 - p \in I$. *Alors* $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \underset{n\to+\infty}{\longrightarrow} W^i$ *dans* L^p *pour tout* $1 \leq i, j \leq d$ *si et seulement si* (1.61) *est satisfaite.*

Pour finir, sous la condition (1.61) on montre la convergence dans L^p avec vitesse exponentielle de $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, $1 \leq i, j \leq d$.

Theorem 1.4.14. *Supposons la condition* ***H7***. *Soit* $p > 1$ *tel que* $1 - p \in I$ *et* (1.61) *est satisfaite. Alors il existe* $\delta \in (0,1)$ *tel que*

$$\lim_{n\to+\infty} \delta^{-n} \left(\mathbb{E} \left| \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} - W^i \right|^p \right)^{1/p} = 0. \tag{1.64}$$

Pour démontrer les Théorèmes 1.4.13 et 1.4.14, on se sert de nos résultats concernant la convergence L^p de la martingale W_n^i (Théorèmes 1.4.11 et 1.4.12), et on montre que pour tout $1 \leq i, j \leq d$, la différence $Z_n^i(j)/M_{0,n-1}(i,j) - W_n^i$ converge dans L^p vers 0 avec une vitesse exponentielle. La preuve de ces convergences est basée sur le résultat de Seneta [64, Théorème 4.19] qui établit sous des conditions convenables la convergence des produits de matrices stochastiques avec une vitesse exponentielle.

1.4.3 Borne de Berry-Esseen et moments harmoniques pour un processus de branchement multi-type surcritique en environnement aléatoire

On a précédemment établit une loi des grands nombres pour le logarithme de la population totale $\log \|Z_n^i\|$ (cf. Corollaire 1.4.5). Maintenant, notre principal objectif dans cette section est de démontrer un théorème central limite (TCL) pour $\log \|Z_n^i\|$, et d'établir un théorème de type Berry-Esseen pour $\log \|Z_n^i\|$ donnant une vitesse de convergence pour ce TCL. Pour cela, une étape importante de la preuve est l'étude de l'existence des moments harmoniques des limites presque sûres W^i des martingales (W_n^i).

Moments harmoniques de W^i

On cherche à établir l'existence des moments harmoniques $\mathbb{E}(W^i)^{-a}$ ($a > 0$), des variables aléatoires W^i. Pour le cas uni-type en environnement aléatoire, on peut citer le résultat de Huang et Liu [43] ainsi que ceux de Grama, Liu et Miqueu [30, 31]. Le cas multi-type ($d \geq 2$) n'avait pas été traité jusqu'à présent.

Pour tout $n \geq 0$, on définit le vecteur $p_0(\xi_n)$ et la matrice $P_1(\xi_n)$, dont les composentes sont :

$$p_0(\xi_n)(i) = f_n^i(\mathbf{0}) \quad \text{et} \quad P_1(\xi_n)(i,j) = \frac{\partial f_n^i}{\partial s_j}(\mathbf{0}), \quad 1 \leq i,j \leq d.$$

Autrement dit, pour tout $1 \leq i,j \leq d$, on a

$$p_0(\xi_n)(i) = \mathbb{P}_{T^n\xi}(\|Z_1^i\| = 0) \quad \text{et} \quad P_1(\xi_n)(i,j) = \mathbb{P}_{T^n\xi}(Z_1^i = e_j).$$

Remarquons qu'il est nécessaire d'avoir $W^i > 0$ p.s. pour que $\mathbb{E}(W^i)^{-a} < +\infty$ pour un certain $a > 0$. Le résultat du Théorème 1.4.3 nous assure, sous de bonnes conditions, de la non-dégénérescence des limites W^i, $1 \leq i \leq d$, et on a la relation (1.39), c'est à dire $\mathbb{P}_\xi\Big(W^i = 0\Big) = q^i(\xi)$ p.s. Ensuite, pour avoir $W^i > 0$ p.s., on introduit la condition suivante :

H8. *Le vecteur* $p_0(\xi_0) = (f_0^1(\mathbf{0}), \cdots, f_0^d(\mathbf{0}))$ *satisfait*

$$p_0(\xi_0) = \mathbf{0} \quad \mathbb{P}\text{-}p.s. \tag{1.65}$$

La condition **H8** signifie que chaque particule du processus de branchement donne naissance à au moins une autre particule ; ainsi, sous **H8**, le processus (Z_n^i) ne peut pas s'éteindre, donc $\mathbb{P}_\xi\Big(W^i = 0\Big) = q^i(\xi) = 0$ p.s. quand W^i est non-dégénérée.

Tout d'abord, on étudie l'existence des moments harmoniques $\mathbb{E}(W^i)^{-a}$ sous la condition suivante de bornitude de moments conditionnellement à l'environnement ξ.

H9. *Il existe des constantes* $p \in (1,2]$, $A > A_1 > 1$ *et* $A_2 > 0$ *telles que pour tout* $1 \leq i,j \leq d$, $\mathbb{P}$-*p.s.*

$$A_2 \leq M_0(i,j), \quad A_1 \leq \|M_0(i,\cdot)\| \quad \textit{et} \quad \mathbb{E}_\xi(Z_1^i(j)^p) \leq A^p.$$

La condition **H9** est très forte, en particulier elle implique que les entrées $M_0(i,j)$ de la matrice moyenne M_0 sont toutes bornées inférieurement et supérieurement par des constantes : on a $A_2 \leq M_0(i,j) \leq A$ p.s. pour tout $1 \leq i,j \leq d$. La condition $A_1 \leq \|M_0(i,\cdot)\|$ p.s. avec $A_1 > 1$ signifie qu'une particule de type i produit en moyenne plus qu'une seule particule à la génération suivante. De plus, **H9** implique les conditions **H1**, **H2**, **H6**, ainsi que la condition surcritique $\gamma > 0$.

L'intérêt d'introduire la condition **H9** est qu'elle va permettre d'établir une équation identifiant l'exposant critique des moments harmoniques des variables W^i. Pour écrire cette équation, on introduit la fonction κ_1 définie sur $\mathbb{R}_+$ par

$$\kappa_1(a) := \lim_{n\to+\infty}\Big(\mathbb{E}\|M_{0,n-1}\|^a\Big\|\prod_{k=0}^{n-1}P_1(\xi_k)\Big\|_\infty\Big)^{1/n}, \tag{1.66}$$

où $\|\cdot\|_\infty$ est la norme opérateur sur $\mathcal{M}_d(\mathbb{R})$ associée à la norme L^∞ usuelle de $\mathbb{R}^d$:

$$\|x\|_\infty := \max_{1\le i\le d}|x(i)|, \quad x\in\mathbb{R}^d;$$
$$\|M\|_\infty := \sup_{\|x\|_\infty=1}\|Mx\|_\infty, \quad M\in\mathcal{M}_d(\mathbb{R}).$$

On montre que, sous la condition **H9**, $\kappa_1(a)$ existe pour tout $a\ge 0$ avec $0\le\kappa_1(a)<+\infty$. De plus, κ_1 est une fonction continue et croissante sur $\mathbb{R}_+$, avec $\kappa_1(0)=\rho\Big(\mathbb{E}P_1(\xi_0)\Big)$, où $\rho\Big(\mathbb{E}P_1(\xi_0)\Big)$ est le rayon spectral de la matrice $\mathbb{E}P_1(\xi_0)$. Pour toute variable aléatoire X, on notera $\|X\|_{L^\infty} := \operatorname{ess\,sup}(\mathrm{X})$ le sup essentiel de X.

Theorem 1.4.15. *Supposons les conditions* ***H8****,* ***H9*** *et* $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty}<1$. *Pour* $a>0$ *fixé, les implications suivantes sont satisfaites :*

(1) si $\kappa_1(a)<1$ *alors* $\mathbb{E}(W^i)^{-a}<+\infty$ *pour tout* $1\le i\le d$ *;*

(2) si $\mathbb{E}(W^i)^{-a}<+\infty$ *pour tout* $1\le i\le d$*, alors* $\kappa_1(a)\le 1$.

Une conséquence directe du Théorème 1.4.15 est le résultat suivant.

Corollary 1.4.16. *Sous les conditions du Théorème 1.4.15, on a :*

(1) $\mathbb{E}(W^i)^{-a}<+\infty$ *pour tout* $1\le i\le d$ *et* $a>0$ *si et seulement si* $\mathbb{E}P_1(\xi_0)$ *est nilpotente ;*

(2) si $\mathbb{E}P_1(\xi_0)$ *n'est pas nilpotente, alors il existe une unique constante* $a_0>0$ *vérifiant*

$$\kappa_1(a_0)=1, \tag{1.67}$$

et

$$\max_{1\le i\le d}\mathbb{E}(W^i)^{-a}\begin{cases}<+\infty & si\ a\in[0,a_0),\\ =+\infty & si\ a\in(a_0,+\infty).\end{cases}$$

Ce résultat met en évidence deux situations possibles. La partie (1) donne une condition nécessaire et suffisante pour que tous les moments harmoniques $\mathbb{E}(W^i)^{-a}$ existent, à tous les ordres $a \geq 0$. Lorsque cette condition n'est pas satisfaite, la partie (2) nous dit que la quantité a_0, unique solution de l'équation (1.67), est la valeur critique pour l'existence des moments harmoniques des W^i, $1 \leq i \leq d$. Nos résultats du Théorème 1.4.15 et du Corollaire 1.4.16 étendent ceux établis par Huang et Liu [43] pour le cas uni-type ($d = 1$), à ceci près qu'ils ont montré dans leur cas que la condition $\kappa_1(a) < 1$ était nécessaire et suffisante pour l'existence des moments harmoniques $\mathbb{E}(W^1)^{-a}$. Malheureusement, nous ne savons pas si les moments harmoniques à l'ordre a_0 existent ou non pour le cas multi-type, mais nous intuitons que $\max_{1\leq i\leq d} \mathbb{E}(W^i)^{-a_0} = +\infty$, ce qui correspondrait au résultat pour $d = 1$.

On s'intéresse maintenant à démontrer l'existence des moments harmoniques $\mathbb{E}(W^i)^{-a}$ sous une condition plus faible que **H9**. Ce problème a été étudier par Grama, Liu et Miqueu dans [30, 31] pour $d = 1$. L'idée est de remplacer la condition de bornitude **H9** par une condition de moment plus faible, sous laquelle $\mathbb{E}(W^i)^{-a} < +\infty$ pour tout $1 \leq i \leq d$ et $a > 0$ suffisamment petit. En contrepartie, sous cette nouvelle condition nous n'aurons plus d'information sur l'exposant critique. Pour tout $n \geq 0$ et $p > 1$, on pose

$$\theta_n(p) := \max_{1\leq i,j\leq d} \mathbb{E}_\xi \left| \frac{N^i_{1,n}(j)}{M_n(i,j)} - 1 \right|^p.$$

La condition de moment est la suivante :

H10. *Il existe deux constantes $p \in (1, 2]$ et $\eta \in (0, 1)$ telles que*

$$\mathbb{E}\|M_0\|^\eta < +\infty, \quad \max_{1\leq i,j\leq d} \mathbb{E}M_0(i,j)^{-\eta} < +\infty \quad \textit{et} \quad \mathbb{E}\theta_0(p)^\eta < +\infty.$$

On remarquera que, tout comme **H9**, la condition **H10** implique aussi les conditions **H1**, **H2** et **H6**. Comme indiqué ci-dessus, le prochain résultat donne une condition suffisante pour l'existence des moments harmoniques des limites W^i pour un ordre $a > 0$ petit.

Theorem 1.4.17. *Supposons les conditions **H8**, **H10** et $\gamma > 0$. Alors il existe $a > 0$ tel que $\mathbb{E}(W^i)^{-a} < +\infty$ pour tout $1 \leq i \leq d$.*

Théorème central limite pour $\log \|Z^i_n\|$

Pour le processus de branchement uni-type en environnement aléatoire, il a été établi par Huang et Liu [43] un TCL pour $\log Z_n$, le logarithme de la population totale du processus. On voudrait maintenant une généralisation de ce résultat pour le multi-type, c'est à dire établir un TCL pour $\log \|Z_n^i\|$, pour chaque $1 \leq i \leq d$. Pour cela, en utilisant la définition de la martingale W_n^i (cf. (1.34)), on obtient deux importantes inégalités faisant le lien entre $\log \|Z_n^i\|$ et $\log \|M_{0,n-1}(i,\cdot)\|$:

$$\log \|Z_n^i\| \leq \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i - \min_{1\leq j\leq d} \log U_{n,\infty}(j), \tag{1.68}$$

$$\log \|Z_n^i\| \geq \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i + \min_{1\leq j\leq d} \log U_{n,\infty}(j). \tag{1.69}$$

On se souvient que $(U_{n,\infty})$ est une suite stationnaire ergodique de vecteurs aléatoires strictement positifs. De plus, si la limite W^i de la martingale (W_n^i) est non-dégénérée (cf. Théorème 1.4.3), alors on aura $\log W_n^i \to \log W^i$ p.s. quand $n \to +\infty$, où $\log W^i$ est une variable aléatoire finie p.s. De ces deux propriétés, on en déduit que les deux termes $\log W_n^i$ et $\min_{1\leq j\leq d} \log U_{n,\infty}(j)$ normalisés par $\sqrt{n}$ vont tendre vers 0 en probabilité. Ainsi, à partir du TCL pour $\log \|M_{0,n-1}(i,\cdot)\|$ établi par Hennion dans [40] et en utilisant les inégalités (1.68) et (1.69), on obtient un TCL pour $\log \|Z_n^i\|$. Pour appliquer le résultat de Hennion, on a besoin de la condition suivante :

H11. *La matrice M_0 satisfait*

$$\mathbb{E}(\log \|M_0\|)^2 < +\infty.$$

On notera que cette condition implique **H2** (la condition sous laquelle l'exposant de Lyapunov γ existe). On présente maintenant notre TCL pour $\log \|Z_n^i\|$:

Theorem 1.4.18. *Supposons les conditions **H1** et **H11**. On suppose de plus les conditions **H8**, **H6** et $\gamma > 0$. Alors il existe $\sigma \geq 0$ tel que pour tout $1 \leq i \leq d$, quand $n \to \infty$,*

$$\frac{\log \|Z_n^i\| - n\gamma}{\sqrt{n}} \to \mathcal{N}(0,\sigma^2) \quad \textit{en loi,}$$

où $\mathcal{N}(0,\sigma^2)$ est la loi normale de moyenne 0 et de variance σ^2.

Ce résultat est une généralisation du TCL établi par Huang et Liu [43] pour le cas uni-type. On peut remarquer deux jeux de conditions dans le Théorème 1.4.18. Le premier contient **H1** et **H11**, ce sont les conditions d'application du TCL pour $\log \|M_{0,n-1}(i,\cdot)\|$

dans [40] ; le deuxième , correspondant à **H8**, **H6** et $\gamma > 0$, est l'ensemble des conditions pour que les limites W^i soient non-dégénérées (Théorème 1.4.3).

Bornes de type Berry-Esseen pour $\log \|Z_n^i\|$

Le but de cette section est d'établir un théorème de type Berry-Esseen pour $\log \|Z_n^i\|$, le logarithme de la population totale $\|Z_n^i\|$ du processus de branchement. Plus précisément, on démontre une convergence uniforme dans le TCL de $\log \|Z_n^i\|$ (cf. Théorème 1.4.18), avec une vitesse en $\frac{1}{\sqrt{n}}$. Pour le cas uni-type $d = 1$, ce problème a déjà été étudier par Grama, Liu et Miqueu dans [31].

On remarque que la variance asymptotique σ^2 définie dans le Théorème 1.4.18 peut être nulle, un cas dit "dégénéré" qu'on ne cherchera pas à étudier. C'est pourquoi nous feront par la suite l'hypothèse suivante :

H12. *La variance asymptotique σ^2 satisfait*

$$\sigma^2 > 0.$$

Notre preuve du théorème de type Berry-Esseen pour $\log \|Z_n^i\|$ est essentiellement basée sur l'application du Théorème 1.4.17 donnant l'existence des moments harminiques $\mathbb{E}(W^i)^{-a}$ pour $a > 0$ suffisamment petit. En particulier, on supposera la condition de moment **H10** sous laquelle, par un résultat de [75], la variance σ^2 satisfait

$$\sigma^2 = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}[(\log \|M_{0,n-1}x\| - n\gamma)^2],$$

uniformément en $x \in \mathcal{S}$. On rappelle que $\Phi : x \mapsto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2} \, dt$ est la fonction de répartition de loi normale centrée réduite, définie sur $\mathbb{R}$. Le théorème suivant donne une borne de type Berry-Esseen pour $\log \|Z_n^i\|$.

Theorem 1.4.19. *Supposons les conditions **H8**, **H10**, **H12** et $\gamma > 0$. Alors il existe une constante $C > 0$ telle que pour tout $n \geq 1$, $x \in \mathbb{R}$ et $1 \leq i \leq d$,*

$$\left| \mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}}.$$

Notons que le Théorème 1.4.19 peut en particulier s'appliquer pour le cas uni-type (un résultat déjà existant sous d'autres hypothèses dans [31]). Dans ce cas, on a $\gamma = \mathbb{E} \log m_0$ et $\sigma^2 = \mathbb{E}(\log m_0 - \gamma)^2$, où $m_0 = \mathbb{E}_\xi Z_1$; de plus, il est intéressant de remarquer que la condition de moment **H10** peut se simplifier en la suivante : il existe deux constantes $p \in (1,2]$ et $\eta \in (0,1)$ telles que

$$\mathbb{E} m_0^\eta < +\infty \quad \text{et} \quad \mathbb{E}\theta_0(p)^\eta < +\infty, \text{ où } \theta_0(p) = \mathbb{E}_\xi \left| \frac{Z_1}{m_0} - 1 \right|^p. \tag{1.70}$$

Pour démontrer le Théorème 1.4.19, on revient aux inégalités (1.68) et (1.69) qui permettent une comparaison de $\log \|Z_n^i\|$ avec $\log \|M_{0,n-1}(i,\cdot)\|$. La première étape est d'établir un bon contrôle des termes restes $\log W_n^i$ et $\log U_{n,\infty}(j)$. Le plus difficile est $\log W_n^i$. Par le Théorème 1.4.17, on a l'existence des moments harmoniques $\mathbb{E}(W^i)^{-a}$ pour $a > 0$ suffisamment petit, ce qui aura pour conséquence la convergence dans L^1 de $\log W_n^i$ vers $\log W^i$ avec une vitesse exponentielle. Dans la seconde étape, on donne une vitesse en $\frac{1}{\sqrt{n}}$ de la convergence de la loi jointe $\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}}, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}}\right)$. Pour cela, on utilise les inégalités (1.68) et (1.69) avec les controles des restes $\log W_n^i$ et $\log U_{n,\infty}(j)$ précédemment obtenus, puis on applique le théorème de type Berry-Esseen pour $\log \|M_{0,n-1}(i,\cdot)\|$ établi par Xiao, Grama et Liu [75].

1.4.4 Déviations modérées de type Cramér pour un processus de branchement multi-type surcritique en environnement aléatoire

Toute cette section est consacrée à l'étude des déviations modérées de type Cramér pour $\log \|Z_n^i\|$ (c'est à dire le logarithme de la population totale $\|Z_n^i\| = Z_n^i(1) + \cdots Z_n^i(d)$), et cela pour tout $1 \leq i \leq d$. Ce problème a été résolu pour le cas uni-type dans [31] par Grama, Liu et Miqueu.

On rappelle que $\mathcal{C}(\mathcal{S})$ est l'espace des fonctions continues sur $\mathcal{S}$ à valeurs réelles, et qu'il est muni de la norme L^∞ définie par (1.58). On avait défini l'opérateur de transfert P_s pour $s \in I$ (cf. (1.59)). En fait, sous la condition de moment **H10**, on peut définir P_s pour tout $s \in [-\eta, \eta]$, ceci de la façon suivante : pour tout $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s\varphi(x) := \mathbb{E}\Big[\|M_0 x\|^s \varphi(M_0 \cdot x)\Big], \quad x \in \mathcal{S}. \tag{1.71}$$

De même, sous **H10**, pour tout $s \in [-\eta, \eta]$ la limite

$$\kappa(s) := \lim_{n\to+\infty} \Big(\mathbb{E}\|M_{0,n-1}\|^s\Big)^{1/n} \tag{1.72}$$

existe avec $\kappa(s) < +\infty$, et $\kappa(s)$ est le rayon spectral de l'opérateur P_s (cf. [13, Proposition 3.1] et [33, Proposition 3.1]). Soit $\Lambda(s) := \log \kappa(s)$, $s \in (-\eta, \eta)$. Il est connu que, sous la condition de moment **H10**, la fonction $s \mapsto \Lambda(s)$ est analytique sur l'intervalle $(-\eta, \eta)$ lorque $\eta > 0$ est suffisamment petit (voir par exemple [11, Lemme 10.17]). On peut alors définir $\gamma_k := \Lambda^{(k)}(0)$ pour tout $k \geq 1$, avec en particulier $\gamma_1 = \gamma$ et $\gamma_2 = \sigma^2$. On écrit la série de Cramér ζ associée à la fonction Λ comme suit :

$$\zeta(t) := \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3}t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}}t^2 + \cdots, \tag{1.73}$$

qui converge lorque $|t|$ est suffisamment petit.

Le théorème suivant donnent les déviations modérées de type Cramér pour $\log \|Z_n^i\|$.

Theorem 1.4.20. *Supposons les conditions **H8**, **H10**, **H12** et $\gamma > 0$. Alors, pour tout $0 \leq x \leq o(\sqrt{n})$ et $1 \leq i \leq d$, quand $n \to +\infty$,*

$$\frac{\mathbb{P}\Big(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\Big)}{1 - \Phi(x)} = e^{\frac{x^3}{\sqrt{n}}\zeta(\frac{x}{\sqrt{n}})}\left[1 + O\Big(\frac{1+x}{\sqrt{n}}\Big)\right], \tag{1.74}$$

et

$$\frac{\mathbb{P}\Big(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} < -x\Big)}{\Phi(-x)} = e^{-\frac{x^3}{\sqrt{n}}\zeta(-\frac{x}{\sqrt{n}})}\left[1 + O\Big(\frac{1+x}{\sqrt{n}}\Big)\right]. \tag{1.75}$$

Comme indiqué dans la section précédente, dans le cas uni-type ($d = 1$) la condition **H10** se simplifie en (1.70). On rappelle que l'on peut trouver se résultat pour $d = 1$ dans [31] sous un autre jeu d'hypothèses, mais il est totalement nouveau pour $d \geq 2$. Par le Théorème 1.4.20, on en déduit les déviations modérées pour $\log \|Z_n^i\|$ avec $x = o(n^{1/6})$ quand $n \to +\infty$.

Corollary 1.4.21. *Supposons les conditions du Théorème 1.4.20. Alors, pour tout $0 \leq x \leq o(n^{1/6})$ et $1 \leq i \leq d$, quand $n \to +\infty$,*

$$\frac{\mathbb{P}\Big(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\Big)}{1 - \Phi(x)} = 1 + O\Big(\frac{1+x}{\sqrt{n}}\Big), \tag{1.76}$$

et

$$\frac{\mathbb{P}\Big(\frac{\log\|Z_n^i\|-n\gamma}{\sigma\sqrt{n}} < -x\Big)}{\Phi(-x)} = 1 + O\Big(\frac{1+x}{\sqrt{n}}\Big). \tag{1.77}$$

Pour la preuve du Théorème 1.4.20, l'idée est de reprendre le même raisonnement que pour celle du Théorème 1.4.19 (théorème de type Berry-Esseen pour le MBPRE), mais en remplaçant la mesure $\mathbb{P}$ par une toute nouvelle mesure de probabilité notée $\mathbb{P}_s^{e_i}$, avec $s \in (-\eta, \eta)$ et $1 \leq i \leq d$. La définition de $\mathbb{P}_s^{e_i}$ s'appuie sur la théorie spectrale de l'opérateur de transfert P_s (voir Buraczewski, Damek, Guivarc'h et Mentemeier [13], Guivarc'h et Le Page [36], Xiao, Grama et Liu [75]). Notamment, sous **H10**, pour tout $s \in (-\eta, \eta)$ avec $\eta > 0$ suffisamment petit, il existe une unique fonction strictement positive $r_s \in \mathcal{C}(\mathcal{S})$ de norme $\|r_s\|_\infty = 1$ telle que

$$P_s r_s = \kappa(s) r_s; \tag{1.78}$$

autrement dit, $\kappa(s)$ est une valeur propre de l'opérateur P_s, et r_s est une fonction propre associée. Pour tout $1 \leq i \leq d$, on considère le processus

$$X_0^{e_i} = e_i, \quad \text{et} \quad X_n^{e_i} = M_{0,n-1}^T \cdot e_i, \quad n \geq 1,$$

qui forme une chaîne de Markov sur $\mathcal{S}$. Grâce à l'égalité (1.78), on définit la mesure de probabilité $\mathbb{P}_s^{e_i}$ et l'espérance associée $\mathbb{E}_s^{e_i}$ de la façon suivante : pour tout $n \geq 1$ et toute fonction mesurable bornée h sur $\mathbb{X}^n$,

$$\mathbb{E}\left[\frac{\|M_{0,n-1}^T e_i\|^s r_s(X_n^{e_i})}{\kappa(s)^n r_s(e_i)} h(\xi_0, \cdots, \xi_{n-1})\right] = \mathbb{E}_s^{e_i}\Big[h(\xi_0, \cdots, \xi_{n-1})\Big]. \tag{1.79}$$

Pour démontrer le Théorème 1.4.20, l'objectif est d'étendre le Théorème 1.4.19 pour la mesure changée $\mathbb{P}_s^{e_i}$, et cela uniformément en $s \in (-\eta, \eta)$. Plus précisément on montre que, sous les conditions du Théorème 1.4.20 et pour $\eta > 0$ suffisamment petit, il existe une constante $C > 0$ telle que pour tout $n \geq 1$ et $x \in \mathbb{R}$,

$$\sup_{s\in(-\eta,\eta)} \left|\mathbb{P}_s^{e_i}\Big(\frac{\log\|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leq x\Big) - \Phi(x)\right| \leq \frac{C}{\sqrt{n}}, \tag{1.80}$$

où $\sigma_s^2 = \Lambda''(s)$. Une fois que (1.80) est prouvé, on obtient un bon contrôle de la loi jointe

$\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}}, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}}\right)$ uniformément en $s \in (-\eta, \eta)$, et on conclut la preuve du Théorème 1.4.20 en utilisant les arguments de Petrov [63].

Il reste à voir comment on démontre (1.80), c'est à dire une borne de type Berry-Esseen pour la mesure changée $P_s^{e_i}$ uniforme en $s \in (-\eta, \eta)$. L'idée est de reprendre les arguments de la preuve du Théorème 1.4.19 en considérant la mesure $P_s^{e_i}$ à la place de $\mathbb{P}$. On repart des inégalités (1.68) et (1.69). Le principale défi est de donner un bon contrôle des termes restes $\log W_n^i$ et $\log U_{n,\infty}(j)$ sous la mesure changée $P_s^{e_i}$, et cela uniformément en $s \in (-\eta, \eta)$. Pour cela, on établit une condition suffisante pour l'existence des moments harmoniques des limites W^i sous $\mathbb{P}_s^{e_i}$ uniformément en $s \in (-\eta, \eta)$: sous les conditions du Théorème 1.4.20 et pour $\eta > 0$ suffisamment petit, on montre qu'il existe $a > 0$ tel que pour tout $1 \leq i \leq d$,

$$\sup_{s \in (-\eta,\eta)} \mathbb{E}_s^{e_i}(W^i)^{-a} < +\infty. \tag{1.81}$$

Par la relation (1.81), on en déduit la convergence dans L^1 de $\log W_n^i$ vers $\log W^i$ sous la mesure $\mathbb{P}_s^{e_i}$, uniformément en $s \in (-\eta, \eta)$, avec une vitesse exponentielle. On trouve plus facilement une borne pour le deuxième terme $\log U_{n,\infty}(j)$. On contrôle enfin le terme principal $\log \|M_{0,n-1}(i,\cdot)\|$ des inégalités (1.68) et (1.69) en utilisant la borne de type Berry-Esseen sous la mesure changée $\mathbb{P}_s^{e_i}$ (uniforme en $s \in (-\eta, \eta)$) établie par Xiao, Grama et Liu [75].

Chapter 2

A Kesten-Stigum type theorem for a supercritical multi-type branching process in a random environment

Résumé. On considère un processus de branchement d-type et surcritique $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d))$, $n \geqslant 0$, dans un environnement aléatoire $\xi = (\xi_0, \xi_1, \ldots)$ i.i.d., en commençant avec une particule de type i, pour qui la loi de reproduction à la génération n dépend de l'environnement ξ_n au temps n. Dans le cas d'un environnement déterministe, le célèbre théorème de Kesten-Stigum (1966) donne essentiellement que, si la matrice des moyennes des lois de reproduction a un rayon spectral $\rho > 1$, alors pour tout $i, j = 1, \ldots, d$, presque sûrement $\lim_{n\to\infty} \frac{Z_n^i(j)}{\rho^n}$ existe et est finie; de plus, les variables limites sont non-dégénérées si et seulement si $\mathbb{E}\left(Z_1^i(j)\log^+ Z_1^i(j)\right) < +\infty$ pour tout i et j. L'extension au modèle en environnement aléatoire avec $d = 1$ a été faite par Athreya et Karlin (1971) et Tanny (1988). Étendre le théorème de Kesten-Stigum au cas environnement aléatoire avec $d > 1$ est un problème de longue date. Le principal objectif de cet article est de résoudre ce problème épineux. En particulier, sous de simples conditions, on montre que pour tout $1 \leqslant i, j \leqslant d$, quand $n \to +\infty$, $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j) \to W^i$ en probabilité, où W^i est une variable aléatoire positive, $\mathbb{E}_\xi Z_n^i(j)$ est l'espérance conditionnelle de $Z_n^i(j)$ sachant l'environnement ξ, qui diverge vers ∞ avec une vitesse géométrique au sens que $\frac{1}{n}\log \mathbb{E}_\xi Z_n^i(j) \to \gamma > 0$ presque sûrement, γ étant l'exposant de Lyapunov des matrices des moyennes de productions; de plus les W^i sont non-dégénérées pour tout i si et seulement si $\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\log^+ \frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty$ pour tout i et j, où $M_0(i,j)$ est la moyenne conditionnelle du nombre d'enfants de type j produits par une particule de type i au temps 0, sachant l'environnement ξ. L'idée clé de la démonstration est l'introduction d'une martingale positive (W_n^i) qui converge p.s. vers W^i, et qui se réduit à la très connue martingale du cas déterministe. En outre, on prouve que la direction $Z_n^i/\|Z_n^i\|$ converge en loi conditionnellement à l'évènement d'explosion $\{\|Z_n^i\| \to +\infty\}$. Le cas d'un environnement stationnaire ergodique est aussi considéré. Nos résultas ouvrent la voie pour montrer d'importantes propriétés telles que des théorèmes central limit avec une vitesse de convergence et des théorèmes de grandes déviations.

Abstract. Consider a supercritical d-type branching process $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d))$, $n \geqslant 0$, in an i.i.d. environment $\xi = (\xi_0, \xi_1, \ldots)$, starting with one particle of type i, whose offspring distributions of generation n depend on the environment ξ_n at time n.

In the deterministic environment case, the famous Kesten-Stigum (1966) theorem states essentially that, if the mean matrix of the offspring distribution has spectral radius $\rho > 1$, then for all $i, j = 1, \ldots, d$, almost surely $\lim_{n\to\infty} \frac{Z_n^i(j)}{\rho^n}$ exists and is finite; moreover, the limit variables are non-degenerate if and only if $\mathbb{E}\left(Z_1^i(j)\log^+ Z_1^i(j)\right) < +\infty$ for all i and j. The extension to the random environment case with $d = 1$ has been done by Athreya and Karlin (1971) and Tanny (1988). Extending the Kesten-Stigum theorem to the random environment case with $d > 1$ is a long-standing problem. The main objective of this paper is to resolve this delicate problem. In particular, under simple conditions, we prove that for any $1 \leqslant i, j \leqslant d$, as $n \to +\infty$, $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j) \to W^i$ in probability, where W^i is a non-negative random variable, $\mathbb{E}_\xi Z_n^i(j)$ is the conditional expectation of $Z_n^i(j)$ given the environment ξ, which diverges to ∞ with geometric rate in the sense that $\frac{1}{n}\log \mathbb{E}_\xi Z_n^i(j) \to \gamma > 0$ almost surely, γ being the Lyapunov exponent of the mean matrices of the offspring distributions; moreover W^i are non-degenerate for all i if and only if $\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\log^+ \frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty$ for all i and j, where $M_0(i,j)$ is the conditioned mean of the number of children of type j produced by a particle of type i at time 0, given the environment ξ. The key idea of the proof is the introduction of a non-negative martingale (W_n^i) which converges a.s. to W^i, and which reduces to the well-known fundamental martingale in the deterministic environment case. In addition, we prove that the direction $Z_n^i/\|Z_n^i\|$ converges in law conditioned on the explosion event $\{\|Z_n^i\| \to +\infty\}$. The case of stationary and ergodic environment is also considered. Our results open ways to prove important properties such as central limit theorems with convergence rate and large deviation asymptotics.

2.1 Introduction

Branching processes are rapidly developing areas of the theory of random processes. Their importance is mainly due to the large spectrum of applications in many fields including biology, chemistry, population dynamics, nuclear physics, etc. See for example the classical books by Harris [39] and Athreya and Ney [7]. The introduction of a random environment by Smith and Wilkinson [65] and Athreya and Karlin [5] brought an important advancement in the theory and applications of branching processes. The role of random environment has been by now well understood in the case of single type branching processes, for which a number of important properties have been established, see for example the recent book by Kersting and Vatutin [50]. For multi-type branching processes in random environments (MBPRE's), recent progress has been made for the critical and subcritical cases: see for example Peigné, Le Page and Pham [57] , Vatutin and Dyakonova [70], and Vatutin and Wachtel [73], who studied the convergence rate of the survival probability;

for the supercritical case, we have not found recent work in the literature, and we feel that too few results are known.

For a supercritical multi-type branching process (MBP), the fundamental problem is the description of the population size at time n. Let us recall the famous Kesten-Stigum's theorem [49] established in the deterministic environment case, which tells us exactly when the population size grows at an exponential rate. Consider a MBP $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, where $Z_n(j)$ denotes the number of particles of types j at time n, Z_0 represents the initial population. Denote by M the (non-random) matrix of means of the offspring distributions, which is assumed to be primitive in the sense that there exists $k \geqslant 1$ such that $M^k > 0$. Let ρ be the spectral radius of the mean matrix M, and let $u = (u(1), \cdots, u(d))$ and $v = (v(1), \cdots, v(d))$ be respectively associated positive right and left eigenvectors with the normalization $\|u\| = 1$ and $\langle v, u \rangle = 1$, where $\| \cdot \|$ denotes the L_1-norm and $\langle \cdot, \cdot \rangle$ the scalar product. Assume that $\rho > 1$, which means that the branching process is in the supercritical regime. Denote by $(Z_n^i)_{n \geqslant 0}$ the branching process $(Z_n)_{n \geqslant 0}$ starting with one initial particle of type i, that is when $Z_0 = e_i$, where e_i is the unit vector whose i-th component is 1. Kesten and Stigum [49] showed that, for any $1 \leqslant i, j \leqslant d$, as $n \to +\infty$,

$$\frac{Z_n^i(j)}{\rho^n v(j)} \to W^i u(i) \quad \text{a.s.}, \tag{2.1.1}$$

where W^i is a non-negative random variable which is non-degenerate for all i if and only if $\mathbb{E}\left(Z_1^i(j) \log^+ Z_1^i(j)\right) < +\infty$ for all i and j, and when it is non-degenerate, $\mathbb{E}W^i = 1$. The proof of (2.1.1) is based on the fundamental non-negative martingale

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, u \rangle}{\rho^n u(i)}, \quad n \geqslant 1, \tag{2.1.2}$$

which converges a.s. to W^i.

Due to the importance of the Kesten-Stigum theorem and of the fundamental martingale (W_n^i), a challenging problem is to find the corresponding results for the random environment case. For the single type process, this problem was considered at the very beginning of the study of the topic in the fundamental work of Athreya and Karlin [6] (1971). In [6] it was found that for a single type branching process $(Z_n)_{n \geqslant 0}$ in a stationary and ergodic random environment, the sequence

$$W_0 = 1, \quad W_n = \frac{Z_n}{m_0 \cdots m_{n-1}}, \quad n \geqslant 1, \tag{2.1.3}$$

with m_k denoting the conditioned mean of the offspring distribution at time k given the environment, constitutes a martingale, and that, in the supercritical case where $\mathbb{E} \log m_0 > 0$, the limit variable $W = \lim_{n\to} W_n$ is non-degenerate if

$$\mathbb{E}\left(\frac{Z_1}{m_0} \log^+ Z_1\right) < +\infty. \tag{2.1.4}$$

In case of an independent and identically distributed (i.i.d.) environment, this condition was proved to be also necessary for the non degeneracy of W by Tanny [69] (1988). Notice that when $\mathbb{E}|\log m_0| < \infty$, the moment condition (2.1.4) is equivalent to

$$\mathbb{E}\left(\frac{Z_1}{m_0} \log^+ \frac{Z_1}{m_0}\right) < +\infty.$$

For a multi-type branching process in random environment $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, the situation is much more delicate. In fact, extending the Kesten-Stigum theorem to this case is a long-standing problem. The only result that we found in the literature about the subject is a theorem by Cohn [17], which we briefly recall below. For $n \geqslant 0$, denote by M_n the matrix of the conditioned means of the offspring distribution of n-th generation given the environment: the (i, j)-th entry of M_n is

$$M_n(i, j) = \mathbb{E}_\xi[Z_{n+1}(j) \mid Z_n = e_i],$$

where $\mathbb{E}_\xi$ denotes the conditional expectation given the environment ξ. Let $M_{0,n} = M_0 \cdots M_n$ be the product matrix. Assume that each entry of M_n is bounded a.s. from below and above by two positive constants, and that all the conditional second moments of the offspring distributions given the environment are bounded a.s. by a constant. We suppose that the MBPRE is in the supercritical regime, which means that

$$\gamma := \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| > 0, \tag{2.1.5}$$

where $\|M_{0,n-1}\|$ is the L_1-norm of the matrix $M_{0,n-1}$. This definition of the supercriticality agrees with that in the deterministic environment case, since in this case $\log \rho = \gamma$.

Assume also the integrability condition $\mathbb{E}|\log\sum_{i=1}^{d}(1-\mathbb{P}(\|Z_1^i\|=0))|<\infty$. The result of Cohn ([17], 1989) states that, under these conditions, it holds that for each $j=1,\cdots,d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \to W^i \quad \text{in } L^2, \tag{2.1.6}$$

where W^i is a non degenerate random variable satisfying $\mathbb{E}W^i=1$.This result is already very interesting. However, it only gives sufficient conditions which are not necessary for $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$ to converge to a non degenerate random variable. Moreover, the sequence $\left(\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}\right)_{n\geq 0}$ in general is not a martingale; it turns out very useful to find the martingale which corresponds the fundamental martingale known in the constant environment case. For multi-type branching processes in varying environment, Jones [45] obtained conditions for L^2 convergence, while Biggins, Cohn and Nerman [12] got conditions (including a uniform integrability condition) for L^p ($p\geqslant 1$) convergences. Their result in [12] about L^1 convergence can be applied to study the non-degeneracy of W^i in the random environment case, but the conditions therein are relatively complicated, from which we fail to deduce a simple condition for the non-degeneracy for a MBPRE.

Our objective in this paper is to obtain a full extension of the Kesten-Stigum result (2.1.1) for a supercritical MBPRE $Z_n=(Z_n(1),\cdots,Z_n(d))$, $n\geqslant 0$. For simplicity, let us consider the case where the Furstenberg-Kesten condition **A4** (see Section 2.2) is satisfied, and where the environment is i.i.d. Assume the supercritical condition $\gamma>0$. For $n,k\geqslant 0$, let $\rho_{n,n+k}$ be the spectral radius of the product matrix $M_{n,n+k}=M_n\cdots M_{n+k}$, and let $U_{n,n+k}$ and $V_{n,n+k}$ be respectively the associated non-negative right and left eigenvectors with the normalization $\|U_{n,n+k}\|=1$ and $\langle V_{n,n+k},U_{n,n+k}\rangle=1$. Set $U_{n,\infty}=\lim_{k\to\infty}U_{n,n+k}$, where the limit exists a.s. according to a result of Hennion [40] . Then we have the following analog of Kesten-Stigum's result (2.1.1) which describes the asymptotic behaviour of the coordinate $Z_n^i(j)$: for any $1\leqslant i,j\leqslant d$, as $n\to+\infty$,

$$\frac{Z_n^i(j)}{\rho_{0,n-1}V_{0,n-1}(j)} \to W^iU_{0,\infty}(i) \quad \text{in probability}, \tag{2.1.7}$$

where $U_{0,\infty}(i)\in(0,1)$, W^i is a non-negative random variable such that W^i is non-degenerate for all i if and only if

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\log^+\frac{Z_1^i(j)}{M_0(i,j)}\right)<+\infty \tag{2.1.8}$$

for all i and j, and when it is non-degenerate, $\mathbb{E}W^i = 1$. A result similar to (2.1.6) is also proved : in Theorem 2.2.11 we establish (under conditions weaker than those supposed by Cohn [17]) that for all $1 \leqslant i, j \leqslant d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \to W^i \quad \text{in probability,} \tag{2.1.9}$$

where W^i is the same variable as in (2.1.7). As it has been just noted, the condition (2.1.8) is sufficient and necessary for W^i to be non-degenerate.

The asymptotic behavior of the direction of the vector Z_n^i is also of interest. We show that the unit vector $Z_n^i/\|Z_n^i\|$ converges in law conditioned on the explosion event $\{\|Z_n^i\| \to +\infty\}$. This extends the corresponding result of Kurtz, Lyons, Pemantle and Perez [54] established for the deterministic environment case.

The key idea of the proof is the introduction of a non-negative martingale (W_n^i) which converges a.s. to W^i, and which reduces to the well-known fundamental martingale in the deterministic environment case and in the single-type random environment case. Since this is the key difficulty let us explain our construction in details. The straightforward way for a generalization of (2.1.2) would be replacing ρ^n and the right eigenvector u by the eigenvalues $\rho_{0,n-1}$ and the corresponding right eigenvectors $U_{0,n-1}$ of the matrix $M_{0,n-1}$; unfortunately, this does not lead to a martingale. Our definition is based on the analog of the Perron-Frobenius theorem for products of random matrices which has been established in Hennion [40]. From the results of [40, Theorem 1], the sequence of unit vectors $(U_{n,\infty})$ satisfies

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty},$$

where $\lambda_n = \|M_n U_{n+1,\infty}\|, n \geqslant 0$, is a stationary and ergodic sequence. Iterating the last relation leads to the identity

$$M_{0,n-1} U_{n,\infty} = \lambda_{0,n-1} U_{0,\infty},$$

with $\lambda_{0,n} = \lambda_0 \cdots \lambda_n$. This allows us to associate with the branching process (Z_n^i) the positive martingale

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geqslant 1. \tag{2.1.10}$$

When the environment is deterministic, the matrices M_n, $n \geqslant 0$, are identical to a single deterministic matrix, say M. In this case we have $U_{n,\infty} = U_{n,n+k} = u$, where u is the unit right eigenvector of M associated with the spectral radius ρ, and $\lambda_{0,n-1} = \rho^n$, so that $W_n^i = \frac{\langle Z_n^i, u\rangle}{\rho^n u(i)}$, which shows that the martingale (2.1.10) coincides with the martingale (2.1.2). For a single-type branching process in random environment, we have $U_{n,\infty} = 1$, $\lambda_{0,n-1} = m_0 \cdots m_{n-1}$, so that (2.1.10) coincides with (2.1.3).

In fact, in the paper we will establish more general results for stationary and ergodic environment without assuming the Furstengerg-Kesten condition. We refer the reader to Section 2.2 for details.

We mention that the results of this paper open ways to prove important properties such as central limit theorems with convergence rate and large deviation asymptotics, similar to those obtained in [9, 31]. This will be considered in a forthcoming paper .

The outline is as follows. In Section 2.2, we introduce the necessary notation and present the main results. In Section 2.3, we give some preliminaries for products of positive random matrices. The fundamental martingale (W_n^i) is constructed in Section 2.4; the non-degeneracy of its limit is considered in Sections 2.5-2.7. Section 2.8 is devoted to the convergence of the direction of Z_n. In Section 2.9 we study the convergence in probability of the normalized component $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$; its a.s. convergence is considered in Section 2.10. Section 2.11 is an appendix in which we prove several implications among the conditions used in the statements of the main results.

2.2 Background and main results

2.2.1 Notation and preliminary statements

We start this section by fixing some notation. For an integer $d \geqslant 1$ let $\mathbb{R}^d$ be the d-dimensional space of vectors with real coordinates. For $1 \leqslant i \leqslant d$ denote by e_i the d-dimensional vector with 1 in the i-th place and 0 elsewhere. $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^d$ stands for the vector with all coordinates equal to 1. For any $x, y \in \mathbb{R}^d$, let

$$\langle x, y\rangle := \sum_{i=1}^d x(i)\, y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^d |x(i)|$$

be the scalar product and the L^1 norm in $\mathbb{R}^d$. The operator norm of a matrix $M = (M(i,j))_{1\leqslant i,j\leqslant d} \in \mathcal{M}_d(\mathbb{R})$ is given by

$$\|M\| := \sup_{\|x\|=1} \|Mx\|.$$

For a matrix or a vector X, we write $X > 0$ to mean that each entry of X is strictly positive. The set of non-negative integers is denoted $\mathbb{N} = \{0, 1, \cdots\}$. The symbol C denotes positive constants. The indicator of an event A is denoted by $\mathbb{1}_A$. The symbol $\xrightarrow{d(\mathbb{P})}$ denotes the convergence in distribution under $\mathbb{P}$, while $\xrightarrow{\mathbb{P}}$ means the convergence in probability $\mathbb{P}$.

Let $\xi = (\xi_n)_{n\geqslant 0}$ be a stationary and ergodic sequence of random variables with values in an abstract space $\mathbb{X}$. Each realization of ξ_n is associated with d probability distributions on $\mathbb{N}^d$ characterized by their probability generating functions

$$f_n^r(s) = \sum_{k_1,\cdots,k_d=0}^{\infty} p^r_{k_1,\cdots,k_d}(\xi_n) s_1^{k_1} \cdots s_d^{k_d}, \quad s = (s_1, \cdots, s_d) \in [0,1]^d,$$

$1 \leqslant r \leqslant d$. A d-type branching process $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, in the random environment ξ is a process with values in $\mathbb{N}^d$ such that $Z_0 \in \mathbb{N}^d$ is fixed, and for all $n \geqslant 0$,

$$Z_{n+1} = \sum_{r=1}^{d} \sum_{l=1}^{Z_n(r)} N^r_{l,n}, \tag{2.2.1}$$

where $Z_n(j)$ represents the number of particles of type j of some population in generation n; conditioned on the environment ξ, the random vectors $N^r_{l,n} = (N^r_{l,n}(1), \cdots, N^r_{l,n}(d))$, with $N^r_{l,n}(j)$ denoting the offspring of type j at time $n+1$ of the l-th particle of type r in the generation n, are independent for $l \geqslant 1, n \geqslant 0, 1 \leqslant r \leqslant d$; each $N^r_{l,n}$ has the same probability generating function f_n^r for $l \geqslant 1$. In the sequel, when the branching process $(Z_n)_{n\geqslant 0}$ starts with one initial particle of type i, i.e. when $Z_0 = e_i$, we will write $(Z_n^i)_{n\geqslant 0}$ instead of $(Z_n)_{n\geqslant 0}$.

Let $\mathbb{P}_\xi$ be the additional probability under which the process (Z_n) is defined given the environment ξ. The total probability $\mathbb{P}$ can be formulated as $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$, where τ denotes the law of the environment sequence ξ. The probability $\mathbb{P}_\xi$ is usually called quenched law, while the total probability $\mathbb{P}$ is called annealed law. The quenched law $\mathbb{P}_\xi$ can be considered as the conditional law of $\mathbb{P}$ given the environment ξ. The expectation with respect to $\mathbb{P}_\xi$ and $\mathbb{P}$ will be denoted respectively by $\mathbb{E}_\xi$ and $\mathbb{E}$.

According to the definition of the model, under $\mathbb{P}_\xi$, the random vectors $N_{l,n}^r = (N_{l,n}^r(1), \cdots, N_{l,n}^r(d))$ are independent and have the same probability generating function f_n^r:

$$f_n^r(s) = \mathbb{E}_\xi\left(\prod_{j=1}^d s_j^{N_{l,n}^r(j)}\right), \quad s = (s_1, \cdots, s_d) \in [0,1]^d.$$

Set for brevity $N_n^r := N_{1,n}^r$, and let $f_n = (f_n^1, \cdots, f_n^d)$. Then f_n^r is the generating function of $N_n^r = (N_n^r(1), \cdots, N_n^r(d))$ under $\mathbb{P}_\xi$.

We now introduce the sequence of matrices $(M_n)_{n\in\mathbb{N}}$ of conditional means given the environment, which will play a central role in our developments. For all $n \geqslant 0$, set

$$M_n = M_n(\xi_n) := \left(\frac{\partial f_n^i}{\partial s_j}(\mathbf{1})\right)_{1\leqslant i,j\leqslant d},$$

i.e., for any $1 \leqslant i, j \leqslant d$, the (i,j)-th entry of the matrix M_n is

$$M_n(i,j) = \frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi\left[Z_{n+1}(j)\middle| Z_n = e_i\right],$$

which represents the conditioned mean of the number of children of type j produced by a particle of type i at time n, and which are supposed to be finite. Here and hereafter, for a d-dimensional probability generating function f, $\frac{\partial f}{\partial s_j}(\mathbf{1})$ denotes the left derivative at $\mathbf{1}$ of f with respect to s_j.

The matrix M_n depends only on ξ_n and the sequence of the matrices $(M_n)_{n\geqslant 0}$ is stationary and ergodic. Let $0 \leqslant k \leqslant n$. For the product of the matrices $M_k, \cdots, M_n$ it is convenient to use the notation

$$M_{k,n} := M_k \cdots M_n = \left(\frac{\partial f_k^i \circ f_{k+1} \circ \cdots \circ f_n}{\partial s_j}(\mathbf{1})\right)_{1\leqslant i,j\leqslant d},$$

where

$$\frac{\partial f_k^i \circ f_{k+1} \circ \cdots \circ f_n}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi\left[Z_{n+1}(j)\middle| Z_k = e_i\right].$$

In particular, with $k = 0$, we have for all $1 \leqslant i, j \leqslant d$,

$$\mathbb{E}_\xi Z_{n+1}^i(j) = M_{0,n}(i,j). \tag{2.2.2}$$

Denote by $\mathcal{S}$ the semigroup of matrices of $\mathcal{M}_d(\mathbb{R})$ with positive entries which are allowable in the sense that every row and column contains a strictly positive element, and by $\mathcal{S}^0$ the subset of the matrices with strictly positive entries. Following Hennion [40], we shall assume that the matrices M_n satisfy the condition

A1. The matrix M_0 belongs to the semigroup $\mathcal{S}$ $\mathbb{P}$-a.s. and

$$\mathbb{P}\left(\bigcup_{n\geqslant 0} \left\{ M_{0,n} \in \mathcal{S}^0 \right\} \right) > 0.$$

This means that with positive probability, there is n such that the product matrice $M_{0,n}$ is strictly positive.

Obviously if $G \in \mathcal{S}$ and $G^0 \in \mathcal{S}^0$ then $G^0 G \in \mathcal{S}^0$. Let θ_n be the least k such that $M_{n,n+k} \in S^0$:

$$\theta_n := \inf \left\{ k \geqslant 0 : M_{n,n+k} \in S^0 \right\},$$

with the convention that $\inf \emptyset = +\infty$. According to Lemma 3.1 in [40], under condition **A1**, we have $\theta_n < +\infty$ $\mathbb{P}$-a.s. for all $n \geqslant 0$.

We shall relate the branching process $(Z_n^i)_{n\geqslant 0}$ to a martingale which is the key point in our study. Our construction is based on the extension of the Perron-Frobenius theorem of Hennion [40] for products of random matrices. Recall that, under condition **A1**, for any $n, k \geqslant 0$, the product $M_{n,n+k}$ belongs to $\mathcal{S}$ $\mathbb{P}$-a.s. Let $\rho_{n,n+k}$ be the spectral radius of $M_{n,n+k}$. By the classical Perron-Frobenius theorem (see e.g. [7]), $\rho_{n,n+k}$ is a strictly positive eigenvalue of $M_{n,n+k}$, associated to positive right and left eigenvectors $U_{n,n+k}$ and $V_{n,n+k}$, respectively, with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k} \rangle = 1$.

The following propositions collect some results established by Hennion in [40, Lemma 3.3 and Theorem 1], which provide an analog of the Perron-Frobenius theorem for products of random matrices.

Proposition 2.2.1. *Assume condition **A1**. For all $n \geqslant 0$, the following assertions hold :*

1. *for all $1 \leqslant i, j \leqslant d$:*

$$\lim_{k\to+\infty} \frac{M_{n,n+k}(i,j)}{\rho_{n,n+k} U_{n,n+k}(i) V_{n,n+k}(j)} \mathbb{1}_{\{\theta_n \leqslant k\}} = 1 \quad \mathbb{P}\text{-}a.s., \tag{2.2.3}$$

or equivalently

$$\lim_{k\to+\infty}\left(\frac{M_{n,n+k-1}}{\rho_{n,n+k-1}}-U_{n,n+k-1}V_{n,n+k-1}^T\right)=0\quad \mathbb{P}\text{-}a.s.;$$

2. *the sequence* $(U_{n,n+k})_{k\geqslant 0}$ *converges* $\mathbb{P}$*-a.s. to a random unit vector, say* $U_{n,\infty}>0$*:*

$$U_{n,n+k}\underset{k\to+\infty}{\longrightarrow}U_{n,\infty};$$

3. *the sequence* $(V_{n,n+k}/\|V_{n,n+k}\|)_{k\geqslant 0}$ *converges in law to a random unit vector, say* $\overline{V}_{0,\infty}>0$*:*

$$\frac{V_{n,n+k}}{\|V_{n,n+k}\|}\overset{d(\mathbb{P})}{\underset{k\to+\infty}{\longrightarrow}}\overline{V}_{0,\infty};$$

4. *the scalars*

$$\lambda_n:=\lambda_n(\xi)=\|M_nU_{n+1,\infty}\| \tag{2.2.4}$$

are strictly positive and satisfy the relation

$$M_nU_{n+1,\infty}=\lambda_nU_{n,\infty}. \tag{2.2.5}$$

The sequence (λ_n) will play an important role in the following. The numbers λ_n will be called pseudo spectral radii of the products of random matrices. Notice that λ_n behaves as the spectral radius ρ_n which satisfies

$$M_nU_{n,n}=\rho_nU_{n,n}; \tag{2.2.6}$$

the point is that in (2.2.5), there is a shift of time in the vector $U_{n+1,\infty}$ appearing on the left side, which permits to iterate the formula leading to

$$M_{n,n+k}U_{n+k+1,\infty}=\lambda_{n,n+k}U_{n,\infty}, \tag{2.2.7}$$

where

$$\lambda_{n,n+k} = \prod_{r=n}^{n+k} \lambda_r, \qquad \text{for } n, k \geqslant 0.$$

This shows that the relation (2.2.5) is stable for products of random matrices, while the corresponding relation (2.2.6) for the spectral radius does not have this stability. Notice that by (2.2.7)

$$\lambda_{n,n+k} = \|M_{n,n+k} U_{n+k+1,\infty}\|. \tag{2.2.8}$$

Let T be the shift operator of the environment sequence:

$$T\xi = (\xi_1, \xi_2, \cdots) \quad \text{ if } \quad \xi = (\xi_0, \xi_1, \cdots),$$

and let T^n be its n-fold iteration. Note that the vector $U_{n,\infty}$ and the scalar λ_n depend on the whole sequence $T^n\xi = (\xi_n, \xi_{n+1}, \cdots)$. Since the random environment $\xi = (\xi_n)_{n\geqslant 0}$ is a stationary ergodic sequence, from (2.2.4) it follows that $(\lambda_n)_{n\geqslant 0}$ is also a stationary ergodic sequence.

We complement Proposition 2.2.1 by establishing a relation between the product sequence $\lambda_{0,n-1}$ and the spectral radius $\rho_{0,n-1}$, which will be useful in the proof of the main results of the paper. For its proof, see Section 2.3.

Proposition 2.2.2. *Assume condition **A1**. For all $n \geqslant 0$ and $1 \leqslant j \leqslant d$,*

$$\lambda_n = \lim_{k\to+\infty} \frac{\rho_{n,n+k} V_{n,n+k}(j)}{\rho_{n+1,n+k} V_{n+1,n+k}(j)} \mathbb{1}_{\{\theta_{n+1}\leqslant k\}} \quad \mathbb{P}\text{-}a.s. \tag{2.2.9}$$

and

$$\lim_{n\to+\infty} \frac{\lambda_{0,n-1}}{\rho_{0,n-1}\langle V_{0,n-1}, U_{n,\infty}\rangle} = 1 \quad \mathbb{P}\text{-}a.s. \tag{2.2.10}$$

2.2.2 Main results

We first introduce the martingale related to MBPRE. Our definition is quite different from the one for a MBP with deterministic environment. However, we shall see below that in the case of deterministic environment it comes to the same. Consider the following filtration: $\mathcal{F}_0 = \sigma(\xi)$ and, for $n \geqslant 1$,

$$\mathcal{F}_n = \sigma\left(\xi, N_{l,k}^r,\ 0 \leqslant k \leqslant n-1, 1 \leqslant r \leqslant d, l \geqslant 1\right).$$

Define the process $(W_n^i)_{n\geqslant 0}$: for all $1 \leqslant i \leqslant d$, set

$$W_0^i := 1, \quad W_n^i := \frac{\langle Z_n^i, U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geqslant 1. \tag{2.2.11}$$

Our first theorem states that $(W_n^i)_{n\geqslant 0}$ is a non-negative martingale.

Theorem 2.2.3. *Assume condition* ***A1****. For all* $1 \leqslant i \leqslant d$ *the sequence* $(W_n^i)_{n\geqslant 0}$ *is a non-negative martingale w.r.t.* $(\mathcal{F}_n)_{n\geqslant 0}$ *under the laws* $\mathbb{P}_\xi$ *and* $\mathbb{P}$*, and hence converges* $\mathbb{P}$*-a.s. to a random variable* $W^i \geqslant 0$ *which satisfies* $\mathbb{E}_\xi W^i \leqslant 1$ $\mathbb{P}$*-a.s.*

We next give a functional equation satisfied by the quenched Laplace transform $\phi_\xi^i(t) = \mathbb{E}_\xi\, \mathrm{e}^{-tW^i}$, $t \geqslant 0$, $1 \leqslant i \leqslant d$. For a similar result in the deterministic environment we refer to Theorem 2, p.192 in [7].

Theorem 2.2.4. *Assume condition* ***A1****. Then for each* $1 \leqslant i \leqslant d$*, the quenched Laplace transform* ϕ_ξ^i *of* W^i *satisfies*

$$\phi_\xi^i(t) = f_0^i\left(\phi_{T\xi}^1\left(t\frac{U_{1,\infty}(1)}{\lambda_0 U_{0,\infty}(i)}\right), \cdots, \phi_{T\xi}^d\left(t\frac{U_{1,\infty}(d)}{\lambda_0 U_{0,\infty}(i)}\right)\right), \quad t \geqslant 0. \tag{2.2.12}$$

We now introduce a condition under which we can define the Lyapunov exponent γ of the sequence of random matrices $(M_n)_{n\geqslant 0}$.

A2. The random matrix M_0 satisfies the moment condition

$$\mathbb{E} \log^+ \|M_0\| < +\infty.$$

By the sub-additivity lemma, under **A2**, the limit

$$\gamma := \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|$$

exists and is equal to the quantity $\inf_{k\geqslant 1} \frac{1}{k} \mathbb{E} \log \|M_{0,k-1}\|$. Moreover, the following strong law of large numbers has been established [26] :

$$\lim_{n\to+\infty}\frac{1}{n}\log\|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.} \tag{2.2.13}$$

The Lyapunov exponent γ allows to introduce the following classification of MBPRE's. We say that a MBPRE is subcritical if $\gamma < 0$, critical if $\gamma = 0$, and supercritical if $\gamma > 0$. It is easy to see that our classification coincides with the standard classification of a MBP in a deterministic environment and with that of the uni-type BPRE.

All over the rest of the paper we shall focus only on the supercritical regime where $\gamma > 0$, which by (2.2.13) implies that

$$\lim_{n\to+\infty}\|M_{0,n}\| = +\infty \quad \mathbb{P}\text{-a.s.}$$

Using an extension of Birkhoff's theorem and (2.2.13), we obtain the following strong law of large numbers for the product sequence $\lambda_{0,n-1}$, and a new expression of γ (see Section 2.3.2).

Proposition 2.2.5. *Assume conditions **A1** and **A2**. Then the expectation $\mathbb{E}\log\lambda_0$ is well defined with value in $\mathbb{R}\cup\{-\infty\}$, and*

$$\lim_{n\to+\infty}\frac{1}{n}\log\lambda_{0,n-1} = \mathbb{E}\log\lambda_0 = \gamma \quad \mathbb{P}\text{-}a.s.$$

From Proposition 2.2.5 it is clear that, under the conditions **A1** and **A2**, the classification stated above can be reformulated in terms of the quantity $\mathbb{E}\log\lambda_0$.

We then investigate the non-degeneracy of the limits W^i, $1\leqslant i\leqslant d$. Our first result gives a sufficient condition for non-degeneracy of W^i, $1\leqslant i\leqslant d$ under condition **A1**. To state the result we need to introduce the following condition.

A3. There exists a constant $C > 1$ such that, for all $1\leqslant i\leqslant d$, $\mathbb{P}$-a.s.

$$\sum_{n=0}^{+\infty}\mathbb{E}_\xi\left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)}\mathbb{1}_{\{\langle N_n^i, U_{n+1,\infty}\rangle\geqslant C^n\}}\right) < +\infty.$$

Let $q^i(\xi)$ be the probability of extinction of the process $(Z_n^i)_{n\geqslant 0}$:

$$q^i(\xi) := \mathbb{P}_\xi\Big(\lim_{n\to+\infty}\|Z_n^i\| = 0\Big).$$

Theorem 2.2.6. *Assume conditions **A1**, **A2** and $\gamma > 0$. Then **A3** is a sufficient condition for W^i, $1 \leqslant i \leqslant d$ to be non-degenerate, that is,*

$$\mathbb{P}_\xi\Big(W^i > 0\Big) > 0, \quad \mathbb{P}\text{-}a.s., \quad 1 \leqslant i \leqslant d. \tag{2.2.14}$$

Furthermore, when W^i, $1 \leqslant i \leqslant d$ are non-degenerate, then

$$\mathbb{E}_\xi W^i = 1 \quad \mathbb{P}\text{-}a.s., \tag{2.2.15}$$

and

$$\mathbb{P}_\xi\Big(W^i = 0\Big) = q^i(\xi) \quad \mathbb{P}\text{-}a.s. \tag{2.2.16}$$

We will see that the sufficient condition **A3** can be replaced by a condition of type $EX \log^+ X < \infty$: see Remark 2.2.7. To obtain a necessary and sufficient condition for the non-degeneracy of W^i, $1 \leqslant i \leqslant d$, we need the following condition introduced by Furstenberg and Kesten [26]:

A4. There exists a constant $D > 1$ such that $\mathbb{P}$-a.s.,

$$1 \leqslant \frac{\max_{1 \leqslant i,j \leqslant d} M_0(i,j)}{\min_{1 \leqslant i,j \leqslant d} M_0(i,j)} \leqslant D.$$

Under condition **A4** we have $M_0 \in \mathcal{S}^0$ $\mathbb{P}$-a.s., so that condition **A1** is satisfied, and $\theta_n = 1$ $\mathbb{P}$-a.s. for any $n \geqslant 0$.

The following conditions, which are stronger than **A3**, will also be used.

A5. There exists a constant $C > 1$ such that, for all $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-a.s.,

$$\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbb{1}_{\left\{ \frac{N_n^i(j)}{M_n(i,j)} \geqslant C^n \right\}} \right) < +\infty.$$

A6. For all $1 \leqslant i \leqslant d$,

$$\mathbb{E}\left(\frac{\langle Z_1^i, U_{1,\infty} \rangle}{\lambda_0 U_{0,\infty}(i)} \log^+ \langle Z_1^i, U_{1,\infty} \rangle \right) < +\infty.$$

A7. For all $1 \leqslant i, j \leqslant d$,

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty.$$

Remark 2.2.7. *In Theorem 2.2.6, condition **A3** can be replaced by each of the conditions **A5**, **A6** and **A7**. This can be seen by the following implications which will be proved in Appendix 2.11:*

1. *under **A1**,**A2**, we have: **A7** $\Rightarrow$ **A5** $\Rightarrow$ **A3**, **A7** $\Rightarrow$ **A6** $\Rightarrow$ **A3**;*
2. *under **A4**, **A2**, we have: **A5** $\Leftrightarrow$ **A3**; **A7** $\Leftrightarrow$ **A6**;*
3. *under **A4**, **A2** and when the environment is i.i.d., we have:*
   ***A3** $\Leftrightarrow$ **A5** $\Leftrightarrow$ **A6** $\Leftrightarrow$ **A7**.*

The following assertion is a consequence of Theorem 2.2.6. Let

$$E^i = \{\lim_{n\to+\infty} \|Z_n^i\| = +\infty\}$$

be the explosion event on which the branching process explodes, starting with one particle of type i, $1 \leqslant i \leqslant d$.

Corollary 2.2.8. *Assume conditions **A1**, **A2** and $\gamma > 0$. Assume also that one of the conditions **A3**, **A5**, **A6** or **A7** holds. Then for all $1 \leqslant i \leqslant d$ we have $q^i(\xi) < 1$ $\mathbb{P}$-a.s. and*

$$\mathbb{P}_\xi(E^i) = 1 - q^i(\xi) \quad \mathbb{P}\text{-}a.s. \tag{2.2.17}$$

Moreover, on the explosion event E^i we have

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Z_n^i\| = \gamma \quad \mathbb{P}\text{-}a.s. \tag{2.2.18}$$

Kaplan [46, Theorem 1] proved (2.2.17) under stronger conditions: he assumed that all the conditional means (given the environment) are bounded a.s. from below and above by two positive constants, and that all the conditional second moments of the offspring distributions are bounded a.s. by a constant. Tanny [68, Theorem 9.6] obtained (2.2.17) and (2.2.18) when $\gamma > 0$, under the condition that $q^i(\xi) = 1$ a.s. for all $1 \leqslant i \leqslant d$ or $q^i(\xi) < 1$ a.s. for all $1 \leqslant i \leqslant d$.

The following theorem gives a necessary and sufficient condition for the non-degeneracy of W^i, $1 \leqslant i \leqslant d$, under the Furstenberg - Kesten condition **A4**. Notice that when the environment is i.i.d., the condition that we obtain is of the form $\mathbb{E}X \log^+ X < \infty$ as in the classic Kesten-Stigum theorem [49] established for the deterministic environment case. In this case our result coincides with the corresponding one of Kesten-Stigum [49].

Theorem 2.2.9. *Assume conditions* ***A2****,* ***A4*** *and* $\gamma > 0$*. Then condition* ***A5*** *is necessary and sufficient for* W^i*,* $1 \leqslant i \leqslant d$*, to be non-degenerate (in the sense of* (2.2.14)*); this condition is equivalent to* ***A7*** *when the environment* $(\xi_n)_{n \geqslant 0}$ *is i.i.d. Furthermore, when* W^i*,* $1 \leqslant i \leqslant d$*, are non-degenerate, then* (2.2.15) *and* (2.2.16) *hold.*

We finally present our results about the asymptotic behavior of the branching process (Z_n). All these results will be stated for an i.i.d. environment under the Furstenberg-Kesten condition **A4**.

Under conditions **A2** and **A4**, Furstenberg and Kesten established in [26] a strong law of large numbers for all the components of the product of random matrices $M_{0,n-1}$: for all $1 \leqslant i, j \leqslant d$,

$$\lim_{n \to +\infty} \frac{1}{n} \log M_{0,n-1}(i,j) = \gamma \quad \mathbb{P}\text{-a.s.} \tag{2.2.19}$$

Let $\mathbb{P}_{E^i} := \mathbb{P}(\cdot|E^i)$ be the probability conditioned on E^i, when $\mathbb{P}(E^i) > 0$. The next result compares the direction of the vector Z_n with that of the left eigenvector $V_{0,n-1}$ of the matrix $M_{0,n-1}$ and provides its limit law.

Theorem 2.2.10. *Assume conditions* ***A2****,* ***A4*** *and* $\gamma > 0$*. Assume additionally that the random environment sequence* $\xi = (\xi_0, \xi_1, \cdots)$ *is i.i.d. Then, for all* $1 \leqslant i \leqslant d$ *such that* $\mathbb{P}(E^i) > 0$*, we have*

$$\left\| \frac{Z_n^i}{\|Z_n^i\|} - \frac{V_{0,n-1}}{\|V_{0,n-1}\|} \right\| \xrightarrow[n \to +\infty]{\mathbb{P}_{E^i}} 0; \tag{2.2.20}$$

moreover, conditional on the event E^i*, the sequence* $(Z_n^i / \|Z_n^i\|)_{n \geqslant 0}$ *converges in law to* $\overline{V}_{0,\infty}$*:*

$$\frac{Z_n^i}{\|Z_n^i\|} \xrightarrow[n \to +\infty]{d(\mathbb{P}_{E^i})} \overline{V}_{0,\infty}. \tag{2.2.21}$$

From Theorem 2.2.10 and from the convergence of the martingale $(W_n^i)_{n \geqslant 0}$ we deduce the asymptotic behaviour of the components $Z_n^i(j)$, under two different normalizations $\mathbb{E}_\xi Z_n^i(j)$ and $\rho_{0,n-1}V_{0,n-1}(j)$. Recall that by (2.2.2), $\mathbb{E}_\xi Z_n^i(j) = M_{0,n-1}(i,j)$, and by (2.2.3) and **A4**, it holds that $\rho_{0,n-1}V_{0,n-1}(j) \sim \frac{M_{0,n-1}(i,j)}{U_{0,\infty}(i)}$ with $U_{0,\infty}(i) > 0$, as $n \to +\infty$.

Theorem 2.2.11. *Assume the conditions of Theorem 2.2.10. Then, for all* $1 \leqslant i, j \leqslant d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i \tag{2.2.22}$$

and

$$\frac{Z_n^i(j)}{\rho_{0,n-1}V_{0,n-1}(j)} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i U_{0,\infty}(i). \tag{2.2.23}$$

Moreover, the limit variables W^i, $1 \leqslant i \leqslant d$, *are non-degenerate (in the sense of* (2.2.14)*) if and only if* ***A7*** *holds; when* ***A7*** *holds, we have* (2.2.15) *and* (2.2.16)*.*

Under stronger assumptions than those used in Theorem 2.2.11, namely that the entries of the mean matrices M_n and those of the corresponding Hessian matrices are bounded, Cohn [17] proved that the convergence in (2.2.22) can be reinforced to the L^2-convergence. Our result (2.2.23) can be compared to the well-known Kesten-Stigum theorem [49, Theorem 1] established in the deterministic environment case. In fact, when the environment is determinist, (2.2.23) reduces to Kesten-Stigum's result (2.1.1), but with the a.s. convergence therein replaced by the convergence in probability. We will give below a sufficient condition to have the a.s. convergence in (2.2.23) .

From Theorem 2.2.11 we get the following corollary which gives the asymptotic behaviour of the norm $\|Z_n^i\|$ of Z_n^i.

Corollary 2.2.12. *Assume the conditions of Theorem 2.2.10. Then for all* $1 \leqslant i \leqslant d$,

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} = \frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i$$

and

$$\frac{\|Z_n^i\|}{\rho_{0,n-1}\|V_{0,n-1}\|} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i U_{0,\infty}(i).$$

Our last result states a sufficient condition to get the a.s. convergence instead of the convergence in probability in Theorems 2.2.10, 2.2.11 and Corollary 2.2.12.

Theorem 2.2.13. *Assume conditions **A2**, **A4** and $\gamma > 0$. Assume additionally that the random environment sequence $\xi = (\xi_0, \xi_1, \cdots)$ is i.i.d. Assume also that for some $p > 1$,*

$$\max_{1 \leqslant i,j \leqslant d} \mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)^p < +\infty \quad \text{and} \quad \mathbb{E}\|M_0\|^{1-p} < +\infty. \tag{2.2.24}$$

Then the following assertions hold :

1. *W^i, $1 \leqslant i \leqslant d$ are non-degenerate, and (2.2.15) and (2.2.16) hold.*

2. *For all $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s. on the event E^i,*

$$\left\| \frac{Z_n^i}{\|Z_n^i\|} - \frac{V_{0,n-1}}{\|V_{0,n-1}\|} \right\| \underset{n \to +\infty}{\longrightarrow} 0. \tag{2.2.25}$$

3. *For all $1 \leqslant i, j \leqslant d$,*

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \underset{n \to +\infty}{\longrightarrow} W^i \quad \mathbb{P}\text{-a.s.}, \tag{2.2.26}$$

$$\frac{Z_n^i(j)}{\rho_{0,n-1} V_{0,n-1}(j)} \underset{n \to +\infty}{\longrightarrow} W^i U_{0,\infty}(i) \quad \mathbb{P}\text{-a.s.} \tag{2.2.27}$$

4. *For all $1 \leqslant i \leqslant d$,*

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} = \frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \underset{n \to +\infty}{\longrightarrow} W^i \quad \mathbb{P}\text{-a.s.}, \tag{2.2.28}$$

$$\frac{\|Z_n^i\|}{\rho_{0,n-1}\|V_{0,n-1}\|} \underset{n \to +\infty}{\longrightarrow} W^i U_{0,\infty}(i) \quad \mathbb{P}\text{-a.s.} \tag{2.2.29}$$

Under assumptions stronger than those of Theorem 2.2.13, one can show the L^p-convergence (with $p > 1$) instead of the a.s. convergence stated above, with an exponential rate. However this task is outside the scope of the present paper and will be done in a forthcoming work.

2.3 Asymptotic properties of the pseudo spectral radii for products of positive random matrices

In this section we prove Propositions 2.2.2 and 2.2.5 on the asymptotic properties of the sequence of pseudo spectral radii (λ_n) related to the products of positive random matrices (M_n).

2.3.1 Proof of Proposition 2.2.2

From (2.2.3), it holds that for all $n \geqslant 1$ and $1 \leqslant j \leqslant d$,

$$\begin{aligned} M_n U_{n+1,\infty} &= \lim_{k \to +\infty} M_n U_{n+1,n+k} \\ &= \lim_{k \to +\infty} M_n \frac{M_{n+1,n+k}(\cdot, j)}{\rho_{n+1,n+k} V_{n+1,n+k}(j)} \mathbb{1}_{\{\theta_{n+1} \leqslant k\}} \\ &= \lim_{k \to +\infty} \frac{\rho_{n,n+k} V_{n,n+k}(j)}{\rho_{n+1,n+k} V_{n+1,n+k}(j)} \mathbb{1}_{\{\theta_{n+1} \leqslant k\}} \\ &\quad \times \lim_{k \to +\infty} \frac{M_{n,n+k}(\cdot, j)}{\rho_{n,n+k} V_{n,n+k}(j)} \mathbb{1}_{\{\theta_n \leqslant k\}} \\ &= \lim_{k \to +\infty} \frac{\rho_{n,n+k} V_{n,n+k}(j)}{\rho_{n+1,n+k} V_{n+1,n+k}(j)} \mathbb{1}_{\{\theta_{n+1} \leqslant k\}} U_{n,\infty}. \end{aligned}$$

Combining this with (2.2.5), the relation (2.2.9) follows. Now we prove (2.2.10). From (2.2.7) we get that for all $n \geqslant 1$,

$$V_{0,n-1}^T M_{0,n-1} U_{n,\infty} = \lambda_{0,n-1} V_{0,n-1}^T U_{0,\infty},$$

with $V_{0,n-1}^T M_{0,n-1} = \rho_{0,n-1} V_{0,n-1}^T$, so

$$\frac{\lambda_{0,n-1}}{\rho_{0,n-1} \langle V_{0,n-1}, U_{n,\infty} \rangle} = \frac{1}{\langle V_{0,n-1}, U_{0,\infty} \rangle}. \tag{2.3.1}$$

Moreover, by Proposition 2.2.1 we know that $U_{0,n-1} \underset{n \to +\infty}{\longrightarrow} U_{0,\infty} > 0$ $\mathbb{P}$-a.s., so there exist two random variables $A > 0$ and $N_0 \geqslant 1$ such that $\mathbb{P}$-a.s. for all $n \geqslant N_0$,

$$0 < A \leqslant \min_{1 \leqslant i \leqslant d} U_{0,n-1}(i) \leqslant 1.$$

Since for all $n \geqslant 1$ we have $\langle U_{0,n-1}, V_{0,n-1}\rangle = 1$, it follows that for all $n \geqslant N_0$,

$$1 \leqslant \|V_{0,n-1}\| \leqslant \frac{1}{A}. \tag{2.3.2}$$

Consequently, for all $n \geqslant N_0$,

$$|\langle V_{0,n-1}, U_{0,\infty}\rangle - 1| = |\langle V_{0,n-1}, U_{0,\infty} - U_{0,n-1}\rangle| \leqslant \frac{1}{A}\,\|U_{0,\infty} - U_{0,n-1}\|,$$

so that

$$\langle V_{0,n-1}, U_{0,\infty}\rangle \underset{n\to+\infty}{\longrightarrow} 1 \quad \mathbb{P}\text{-a.s.} \tag{2.3.3}$$

Combining (2.3.1) and (2.3.3) gives (2.2.10), which ends the proof of Proposition 2.2.2.

2.3.2 Proof of Proposition 2.2.5

By (2.2.4) we have

$$\lambda_0 = \|M_0 U_{1,\infty}\| \leqslant \|M_0\| \quad \mathbb{P}\text{-a.s.}.$$

Using condition **A2**, it follows that $\mathbb{E}\log^+ \lambda_0 < +\infty$, so that $\mathbb{E}\log\lambda_0$ is well defined with value in $\mathbb{R}\cup\{-\infty\}$. Recall that $(\lambda_n)_{n\geqslant 0}$ is a stationary ergodic sequence of random variables. Applying an extension of the Birkhoff Ergodic Theorem [41, Theorem 1] we deduce that, $\mathbb{P}$-a.s.,

$$\lim_{n\to+\infty} \frac{1}{n}\log\lambda_{0,n-1} = \mathbb{E}\log\lambda_0. \tag{2.3.4}$$

Moreover, from (2.2.8) we see that for all $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$\frac{1}{n}\log\min_{1\leqslant j\leqslant d} U_{n,\infty}(j) + \frac{1}{n}\log\|M_{0,n-1}\| \leqslant \frac{1}{n}\log\lambda_{0,n-1} \leqslant \frac{1}{n}\log\|M_{0,n-1}\|. \tag{2.3.5}$$

Since $\left(\log\min_{1\leqslant j\leqslant d} U_{n,\infty}(j)\right)_{n\geqslant 0}$ is a stationary sequence of random variables, by Slutsky's lemma it follows that

$$\frac{1}{n}\log\min_{1\leqslant j\leqslant d} U_{n,\infty}(j) \underset{n\to+\infty}{\overset{\mathbb{P}}{\longrightarrow}} 0.$$

Combining this with the law of large numbers (2.2.13) and letting $n \to +\infty$ in (2.3.5), we obtain that

$$\frac{1}{n} \log \lambda_{0,n-1} \xrightarrow[n\to+\infty]{\mathbb{P}} \gamma. \tag{2.3.6}$$

By (2.3.4), (2.3.6) and the uniqueness of the limit in probability, it holds that $\mathbb{E} \log \lambda_0 = \gamma$. This concludes the proof of Proposition 2.2.5.

2.4 The fundamental martingale (W_n^i)

In this section we prove that (W_n^i) is a martingale, and that the quenched Laplace transform of its limit variable satisfies a functional equation that we make precise.

2.4.1 Proof of Theorem 2.2.3

Clearly $(W_n^i)_{n\geqslant 0}$ is adapted to $(\mathcal{F}_n)_{n\geqslant 0}$, and using (2.2.7) we have for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}_\xi W_n^i = \frac{\mathbb{E}_\xi \langle Z_n^i, U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} = \frac{\langle M_{0,n-1}(i,\cdot), U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} = 1 \quad \mathbb{P}\text{-a.s.}$$

Moreover, we know that $\mathbb{E}_\xi \left[Z_{n+1}^i \middle| \mathcal{F}_n \right] = M_n^T Z_n^i$, so we obtain that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}_\xi \left[W_{n+1}^i \middle| \mathcal{F}_n \right] = \frac{\langle M_n^T Z_n^i, U_{n+1,\infty}\rangle}{\lambda_{0,n} U_{0,\infty}(i)} = \frac{\langle Z_n^i, M_n U_{n+1,\infty}\rangle}{\lambda_{0,n} U_{0,\infty}(i)}.$$

Then applying (2.2.5) we get that for all $n \geqslant 0$,

$$\mathbb{E}_\xi \left[W_{n+1}^i \middle| \mathcal{F}_n \right] = \frac{\langle Z_n^i, \lambda_n U_{n,\infty}\rangle}{\lambda_{0,n} U_{0,\infty}(i)} = \frac{\langle Z_n^i, U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} = W_n^i.$$

This proves that the sequence $(W_n^i)_{n\geqslant 0}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\geqslant 0}$ under the law $\mathbb{P}_\xi$. The argument is similar under the law $\mathbb{P}$. By Fatou's Lemma we have $\mathbb{E}_\xi W^i \leqslant 1$ $\mathbb{P}$-a.s. This ends the proof of Theorem 2.2.3.

2.4.2 Proof of Theorem 2.2.4

Conditioned on the environment ξ, the random vectors $Z^r_{l,n,k} = (Z^r_{l,n,k}(1), \cdots, Z^r_{l,n,k}(d))$, with $Z^r_{l,n,k}(j)$ denoting the offspring of type j at time $n+k$ of the l-th particle of type r in the generation n, are independent and have the same probability generating function $f^r_n \circ f_{n+1} \circ \cdot f_{n+k-1}$. By iterating (2.2.1), it is easy to see that the process $(Z_n)_{n\geqslant 0}$ satisfies the relation

$$Z_{n+k} = \sum_{r=1}^{d} \sum_{l=1}^{Z_n(r)} Z^r_{l,n,k}, \quad n \geqslant 0, k \geqslant 1. \tag{2.4.1}$$

From (2.4.1) and (2.2.11) we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} W^i_{n+1} &= \sum_{r=1}^{d} \sum_{l=1}^{Z^i_1(r)} \frac{\langle Z^r_{l,1,n}, U_{n+1,\infty}\rangle}{\lambda_{0,n} U_{0,\infty}(i)} \\ &= \sum_{r=1}^{d} \frac{U_{1,\infty}(r)}{\lambda_0 U_{0,\infty}(i)} \sum_{l=1}^{Z^i_1(r)} W^r_{l,1,n}, \end{aligned} \tag{2.4.2}$$

where

$$W^r_{l,1,n} := \frac{\langle Z^r_{l,1,n}, U_{n+1,\infty}\rangle}{\lambda_{1,n} U_{1,\infty}(r)}.$$

Clearly $\left(Z^r_{l,1,n}\right)_{n\geqslant 0}$ is a supercritical MBPRE in the random environment $T\xi$, and $\left(W^r_{l,1,n}\right)_{n\geqslant 0}$ is its associated martingale which converges to a random variable denoted $W^r_{l,1}$. Moreover, when ξ is given, the random variables $W^r_{l,1}$, $l \geqslant 0$, are independent of each other and independent of Z^i_1 under $\mathbb{P}_\xi$, with a common distribution such that

$$\mathbb{P}_\xi\left(W^r_{l,1} \in \cdot\right) = \mathbb{P}_{T\xi}\left(W^r \in \cdot\right).$$

Letting $n \to +\infty$ in (2.4.2) it follows that for all $1 \leqslant i \leqslant d$,

$$W^i = \sum_{r=1}^{d} \frac{U_{1,\infty}(r)}{\lambda_0 U_{0,\infty}(i)} \sum_{l=1}^{Z^i_1(r)} W^r_{l,1}.$$

Taking the Laplace transform and using the independence under $\mathbb{P}_\xi$ of the random variables $W^r_{l,1}$ and $Z^i_1(r)$ for $l \geqslant 0$ and $1 \leqslant r \leqslant d$, we get that for all $1 \leqslant i \leqslant d$ and $t \geqslant 0$,

$$\begin{aligned}\phi^i_\xi(t) &= \mathbb{E}_\xi\left[\prod_{r=1}^{d}\prod_{l=1}^{Z^i_1(r)}\mathbb{E}_\xi\left[\mathrm{e}^{\frac{-tU_{1,\infty}(r)}{\lambda_0 U_{0,\infty}(i)}W^r_{l,1}}\right]\right]\\ &= \mathbb{E}_\xi\left[\prod_{r=1}^{d}\left(\phi_{T\xi}\left(t\frac{U_{1,\infty}(r)}{\lambda_0 U_{0,\infty}(i)}\right)\right)^{Z^i_1(r)}\right]\\ &= f^i_0\left(\phi^1_{T\xi}\left(t\frac{U_{1,\infty}(1)}{\lambda_0 U_{0,\infty}(i)}\right),\cdots,\phi^d_{T\xi}\left(t\frac{U_{1,\infty}(d)}{\lambda_0 U_{0,\infty}(i)}\right)\right),\end{aligned}$$

which is the desired equation.

2.5 Proof of Theorem 2.2.6

In this section we prove Theorem 2.2.6 about the non degeneracy of the limit variables W^i. We shall adapt the proof of Lyons, Permantle and Peres in [62], which first consists to interpret a branching process as a random tree. Let $\mathcal{T}$ be the set of (colored) trees and denote by $\mathbb{T}^i \in \mathcal{T}$ the random tree associated to the MBRE $(Z^i_n)_{n\geqslant 0}$. In fact a multi-type branching process can be identified naturally as a random colored-tree (type i is considered as color i), which is a subset of $\cup_{n=1}^\infty\{1,\cdots,d\}^n \times \cup_{n=0}^\infty \mathbb{N}^{*n}$ with $\mathbb{N}^{*0}=\{\emptyset\}$. The initial particle $\emptyset$ of type i is denoted $(i,\emptyset)$; a particle of type i of generation n is denoted by (i,u) with $u \in \mathbb{N}^{*n}$ a sequence of length n; its k-th child of type j is denoted (ij,uk), which is linked with its ancestor (i,u).

We write $t \overset{n}{=} t'$ for $n \geqslant 0$ and $t,t' \in \mathcal{T}$, if t and t' coincide up to height n. It is known that this defines a relation of equivalence. The associated equivalence classes generate the σ-fields $\mathcal{G}_n$, which form a filtration on $\mathcal{T}$. For any $s \in t$ and $t \in \mathcal{T}$, denote by $y(s) \in \mathbb{N}^d$ the number of children, by $\mathrm{gen}(s)$ the generation and by $\mathrm{type}(s)$ the type of the particle s. The distribution of $\mathbb{T}^i$ is characterized by

$$\mathbb{P}_\xi\left(\mathbb{T}^i \overset{n}{=} t\right) = \prod_{s\in t,\mathrm{gen}(s)<n}\mathbb{P}_\xi\left(N^{\mathrm{type}(s)}_{\mathrm{gen}(s)} = y(s)\right), \tag{2.5.1}$$

for any $n \geqslant 0$ and $t \in \mathcal{T}$; it is well defined by the Kolmogorov extension theorem.

We shall construct an auxiliary random tree $\mathbb{T}^i_*$ called "size-biased tree", for all $1 \leqslant i \leqslant d$. At time 0, we start with one initial particle of type i, labeled $(D_0, L_0) := (i,1)$, which forms the generation 0 of the tree $\mathbb{T}_*^i$. In the following, the environment environment ξ is fixed, and the notion of independence is conditioned on ξ. We generate d independent random vectors $Y_0^j \in \mathbb{N}^d$, $1 \leqslant j \leqslant d$, such that

$$\mathbb{P}_\xi\left(Y_0^j = y\right) = \frac{\langle y, U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(j)}\, \mathbb{P}_\xi\left(N_0^j = y\right), \quad y \in \mathbb{N}^d.$$

The above formula defines a probability due to the fact that $\mathbb{E}_\xi W_1^i = 1$. Let $Y_0 := Y_0^{D_0} = Y_0^i$ be the number of children of the initial particle $(D_0, L_0) = (i, 1)$. They constitue the particles of the first generation of the tree $\mathbb{T}_*^i$. At time $n = 1$, we pick at random one particle of type D_1 of the first generation, labeled (D_1, L_1), with the distribution

$$\mathbb{P}_\xi\left((D_1, L_1) = (j, l)\Big| Y_0\right) = \frac{U_{1,\infty}(j)}{\langle Y_0, U_{1,\infty}\rangle}, \quad 1 \leqslant l \leqslant Y_0(j), 1 \leqslant j \leqslant d.$$

The l-th particle of type j of the first generation, except for the particle (D_1, L_1), produces its descendants of the next generations according to a random tree $\mathbb{T}_1^j(l)$ (which forms the subtree of $\mathbb{T}_*^i$ starting from this particle), with distribution

$$\mathbb{P}_\xi\left(\mathbb{T}_1^j(l) \in \cdot\right) = \mathbb{P}_{T\xi}\left(\mathbb{T}^j \in \cdot\right), \quad (j, l) \neq (D_1, L_1),$$

$1 \leqslant l \leqslant Y_0(j)$, $1 \leqslant j \leqslant d$; the random trees $\mathbb{T}_1^j(l)$, $1 \leqslant j \leqslant d, l \geqslant 1$, are independent of each other. Moreover, we generate independent random vectors $Y_1^j \in \mathbb{N}^d$, $1 \leqslant j \leqslant d$, which are also independent of (D_1, L_1) and independent of the trees $\mathbb{T}_1^j(l)$, with distributions

$$\mathbb{P}_\xi\left(Y_1^j \in \cdot\right) = \mathbb{P}_{T\xi}\left(Y_0^j \in \cdot\right).$$

The particle (D_1, L_1) of the first generation produces its children of the next generation according to

$$Y_1 := Y_1^{D_1} = \sum_{j=1}^{d} Y_1^j \mathbb{1}_{\{D_1 = j\}},$$

namely, $Y_1(j)$ is the number of children of type j generated by the particle (D_1, L_1). We then proceed in the same way. Assume that at time $n \geqslant 2$, we have defined all the particles of generation n, and all the genealogical trees of the particles of generation n except for the direct children of (D_{n-1}, L_{n-1}). We pick at random one particle of type D_n

of the generation n, labeled (D_n, L_n), with the distribution

$$\mathbb{P}_\xi\left((D_n, L_n) = (j, l) \middle| Y_{n-1}\right) = \frac{U_{n,\infty}(j)}{\langle Y_{n-1}, U_{n,\infty}\rangle}, \quad 1 \leqslant l \leqslant Y_{n-1}(j), 1 \leqslant j \leqslant d,$$

where $Y_{n-1} = (Y_{n-1}(1), \cdots, Y_{n-1}(d))$, with $Y_{n-1}(j)$ denoting the number of children of type j of the particle (D_{n-1}, L_{n-1}). The l-th particle of type j of the children of (D_{n-1}, L_{n-1}), except for the particle (D_n, L_n), produces its descendants of the next generations according to a random tree $\mathbb{T}_n^j(l)$ (which forms the subtree of $\mathbb{T}_*^i$ starting from this particle), with distribution

$$\mathbb{P}_\xi\left(\mathbb{T}_n^j(l) \in \cdot\right) = \mathbb{P}_{T^n\xi}\left(\mathbb{T}^j \in \cdot\right), \quad (j, l) \neq (D_n, L_n),$$

$1 \leqslant l \leqslant Y_{n-1}(j)$, $1 \leqslant j \leqslant d$; these trees $\mathbb{T}_n^j(l)$, $1 \leqslant j \leqslant d, l \geqslant 1$, are independent of each other. Moreover, we generate independent random vectors $Y_n^j \in \mathbb{N}^d$, $1 \leqslant j \leqslant d$, which are independent of the past, also independent of (D_n, L_n) and independent of the trees $\mathbb{T}_n^j(l)$, with distributions

$$\mathbb{P}_\xi\left(Y_n^j \in \cdot\right) = \mathbb{P}_{T^n\xi}\left(Y_0^j \in \cdot\right).$$

The particle (D_n, L_n) of the generation n produces its children of the next generation according to

$$Y_n := Y_n^{D_n} = \sum_{j=1}^d Y_n^j \mathbb{1}_{\{D_n = j\}},$$

namely, $Y_n(j)$ is the number of children of type j generated by the particle (D_n, L_n). Therefore, by recurrence on n, we have defined the random tree $\mathbb{T}_*^i$.

For all $n \geqslant 0$, denote by Δ_n^i the distinguished path in $\mathbb{T}_*^i$ formed by the particles (D_k, L_k), $k \leqslant n$, which is identified to the last particle of the path. We show by induction that

$$\mathbb{P}_\xi\left(\mathbb{T}_*^i \overset{n}{=} t, \Delta_n^i = \sigma_j\right) = \frac{U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \, \mathbb{P}_\xi\left(\mathbb{T}^i \overset{n}{=} t\right), \tag{2.5.2}$$

for all $n \geqslant 1$, $t \in \mathcal{T}$ a tree of height at least n, and $\sigma_j \in t$ a particle of type j in generation

n. For $n=1$ we have

$$\begin{aligned}\mathbb{P}_\xi\left(\mathbb{T}^i_* \stackrel{1}{=} t, \Delta^i_1 = \sigma_j\right) &= \frac{U_{1,\infty}(j)}{\langle y, U_{1,\infty}\rangle}\ \frac{\langle y, U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(i)}\ \mathbb{P}_\xi\left(N^i_0 = y\right)\\ &= \frac{U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)}\ \mathbb{P}_\xi\left(\mathbb{T}^i \stackrel{1}{=} t\right),\end{aligned}$$

where y is the number of children of the initial particle in generation 0 in t. Now assume that (2.5.2) is true for some $n \geqslant 1$. Let $t \in \mathcal{T}$ be a tree of height at least $n+1$, $\sigma_j \in t$ a particle of type j in generation $n+1$, $\tilde{\sigma}_r \in t$ his ancestor of type r in generation n. Then using (2.5.1) and the notation introduced before (2.5.1), we have

$$\begin{aligned}&\mathbb{P}_\xi\left(\mathbb{T}^i_* \stackrel{n+1}{=} t, \Delta^i_{n+1} = \sigma_j\right)\\ &= \mathbb{P}_\xi\left(\mathbb{T}^i_* \stackrel{n}{=} t, \Delta^i_n = \tilde{\sigma}_r\right)\ \mathbb{P}_\xi\left((D_{n+1}, L_{n+1}) = \sigma_j\right) \times\\ &\quad \mathbb{P}_\xi\left(Y^r_n = y(\tilde{\sigma}_r)\right) \prod_{s\in t, s\neq \tilde{\sigma}_r, \mathrm{gen(s)=n}} \mathbb{P}_\xi\left(N^{\mathrm{type(s)}}_n = y(s)\right)\\ &= \mathbb{P}_\xi\left(\mathbb{T}^i_* \stackrel{n}{=} t, \Delta^i_n = \tilde{\sigma}_r\right) \frac{U_{n+1,\infty}(j)}{\langle y(\tilde{\sigma}_r), U_{n+1,\infty}\rangle} \times\\ &\quad \frac{\langle y(\tilde{\sigma}_r), U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(r)} \mathbb{P}_\xi\left(N^r_n = y(\tilde{\sigma}_r)\right) \prod_{s\in t, s\neq \tilde{\sigma}_r, \mathrm{gen(s)=n}} \mathbb{P}_\xi\left(N^{\mathrm{type(s)}}_n = y(s)\right)\\ &= \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)}\ \mathbb{P}_\xi\left(\mathbb{T}^i \stackrel{n}{=} t\right) \frac{U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(r)} \prod_{s\in t, \mathrm{gen(s)=n}} \mathbb{P}_\xi\left(N^{\mathrm{type(s)}}_n = y(s)\right)\\ &= \frac{U_{n+1,\infty}(j)}{\lambda_{0,n} U_{0,\infty}(i)}\ \mathbb{P}_\xi\left(\mathbb{T}^i \stackrel{n+1}{=} t\right).\end{aligned}$$

Hence (2.5.2) remains true for $n+1$. By induction (2.5.2) holds for all $n \geqslant 1$. Summing over σ_j in (2.5.2), we see that for any $n \geqslant 1$ and $t \in \mathcal{T}$,

$$\mathbb{P}_\xi\left(\mathbb{T}^i_* \stackrel{n}{=} t\right) = \frac{\langle z_n(t), U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}\ \mathbb{P}_\xi\left(\mathbb{T}^i \stackrel{n}{=} t\right), \tag{2.5.3}$$

where $z_n(t)$ is the vector counting the number of particles in t at generation n, of types $j = 1, \cdots, d$.

By abuse of notation we denote by $\mathbb{P}^i$ and $\mathbb{P}^i_\xi$ respectively the annealed and quenched distributions of the tree $\mathbb{T}^i \in \mathcal{T}$. The annealed and quenched laws of biased tree $\mathbb{T}^i \in \mathcal{T}$

are denoted by $\mathbb{P}^i_*$ and $\mathbb{P}^i_{*\xi}$, and defined according to

$$\mathbb{P}^i_*(\mathbb{T}^i \in \cdot) := \mathbb{P}^i(\mathbb{T}^i_* \in \cdot), \quad \mathbb{P}^i_{*\xi}(\mathbb{T}^i \in \cdot) := \mathbb{P}^i_\xi(\mathbb{T}^i_* \in \cdot). \tag{2.5.4}$$

By $\mathbb{P}^i_{*|\mathcal{G}_n}$ and $\mathbb{P}^i{}_{|\mathcal{G}_n}$ we denote the restrictions of the respective laws to the σ-field $\mathcal{G}_n$. Then by (2.5.3) we obtain that for all $n \geqslant 0$,

$$d\mathbb{P}^i_{*|\mathcal{G}_n} = W^i_n \, d\mathbb{P}^i{}_{|\mathcal{G}_n},$$

which means that $\mathbb{P}^i_{*|\mathcal{G}_n}$ has the density W^i_n with respect to $\mathbb{P}^i{}_{|\mathcal{G}_n}$. However, $\mathbb{P}^i_*$ is not necessarily absolutely continuous with respect to $\mathbb{P}^i$. Define

$$W^i := \limsup_{n\to+\infty} W^i_n.$$

Then according to Theorem 5.3.3 in [21] we have the following two equivalences :

$$\begin{cases} W^i = +\infty \quad \mathbb{P}^i_*\text{-a.s.} & \Leftrightarrow \quad W^i = 0 \quad \mathbb{P}^i\text{-a.s.}; \\ W^i < +\infty \quad \mathbb{P}^i_*\text{-a.s.} & \Leftrightarrow \quad \mathbb{E} W^i = 1. \end{cases} \tag{2.5.5}$$

Now we prove that the condition **A3** is sufficient for the random variable W^i to be finite $\mathbb{P}^i_*$-a.s., which will conclude the proof of Theorem 2.2.6 by (2.5.5). Assume **A3**. So there exists $C > 0$ such that for all $1 \leqslant j \leqslant d$, $\mathbb{P}$-a.s.,

$$\sum_{n=0}^{+\infty} \mathbb{P}_\xi \left(\frac{\log^+ \langle Y^j_n, U_{n+1,\infty}\rangle}{n} \geqslant C \right) = \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N^j_n, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(j)} \mathbb{1}_{\{\langle N^j_n, U_{n+1,\infty}\rangle \geqslant \mathrm{e}^{Cn}\}} \right) < \infty.$$

Since the random variables $\langle Y^j_n, U_{n+1,\infty}\rangle$, $n \geqslant 0$, are independent under $\mathbb{P}_\xi$, by the Borel-Cantelli lemma we deduce that for all $1 \leqslant j \leqslant d$, $\mathbb{P}_\xi$-a.s.,

$$\limsup_{n\to+\infty} \frac{\log^+ \langle Y^j_n, U_{n+1,\infty}\rangle}{n} \leqslant C. \tag{2.5.6}$$

Moreover $(\log^+ \langle Y^j_n, U_{n+1,\infty}\rangle)_{n\geqslant 0}$ is a non negative stationary and ergodic stochastic process, hence by a result of Tanny [66, Theorem 1] we know that $\limsup_{n\to+\infty} \log^+ \langle Y^j_n, U_{n+1,\infty}\rangle / n$

is either 0 $\mathbb{P}$-a.s. or $+\infty$ $\mathbb{P}$-a.s. Therefore, by (2.5.6) it follows that for all $1 \leqslant j \leqslant d$,

$$\lim_{n\to+\infty} \frac{\log^+\langle Y_n^j, U_{n+1,\infty}\rangle}{n} = 0 \quad \mathbb{P}^i\text{-a.s.} \tag{2.5.7}$$

Since $\log^+\langle Y_n, U_{n+1,\infty}\rangle = \sum_{j=1}^d \log^+\langle Y_n^j, U_{n+1,\infty}\rangle \mathbb{1}_{\{D_n=j\}}$, this implies that

$$\lim_{n\to+\infty} \frac{\log^+\langle Y_n, U_{n+1,\infty}\rangle}{n} = 0 \quad \mathbb{P}^i\text{-a.s.}$$

Furthermore, using Proposition 2.2.5, as $n \to +\infty$ we have

$$\log \lambda_{0,n-1} \sim \gamma n \quad \mathbb{P}\text{-a.s.}$$

The last two assertions imply that

$$\sum_{n=0}^{+\infty} \frac{\langle Y_n, U_{n+1,\infty}\rangle}{\lambda_{0,n-1}U_{0,\infty}(i)} < +\infty \quad \mathbb{P}^i\text{-a.s.} \tag{2.5.8}$$

For all $1 \leqslant i \leqslant d$ and $n \geqslant 0$, let $Z_{*n}^i \in \mathbb{N}^d$ be the vector whose j-th component is the number of particles in $\mathbb{T}_*^i$ at generation n, of type j. For all $0 \leqslant k \leqslant n$, let $Z_{*k,n}$ be the vector whose j-th component is the number of particles at generation n, of type j, which have as ancestor one of the children of (D_k, L_k), except (D_{k+1}, L_{k+1}). Then the processes $\{Z_{*k,n},\ n \geqslant k\}$, $k \geqslant 0$, are independent (under $\mathbb{P}_\xi$), with environment $T^{k+1}\xi$ and initial state $Z_{*k,k} = Y_k - e_{D_{k+1}}$. So, for all $n \geqslant 0$, we have the decomposition

$$Z_{*n}^i = e_{D_n} + \sum_{k=1}^n Z_{*k,n} \quad \mathbb{P}^i\text{-a.s.}$$

Set $\mathcal{Y} = \{Y_n^j, D_n : n \geqslant 0, 1 \leqslant j \leqslant d\}$. Then by Lemma 2.2.2, for $n \geqslant 1$, $\mathbb{P}^i$-a.s.,

$$\begin{aligned}
&\mathbb{E}_\xi\Big[\langle Z_{*n}^i - e_{D_n}, U_{n,\infty}\rangle \Big| \mathcal{Y}, Z_{*k,n-1}, k \leqslant n-1\Big] \\
&\quad = \sum_{k=1}^n \Big\langle \mathbb{E}_\xi\Big[Z_{*k,n}\Big|\mathcal{Y}, Z_{*k,n-1}, k \leqslant n-1\Big], U_{n,\infty}\Big\rangle \\
&\quad = \sum_{k=1}^{n-1} \langle M_{n-1}^T Z_{*k,n-1}, U_{n,\infty}\rangle + \langle Y_{n-1} - e_{D_n}, U_{n,\infty}\rangle \\
&\quad = \lambda_{n-1}\langle Z_{*n-1}^i - e_{D_{n-1}}, U_{n-1,\infty}\rangle + \langle Y_{n-1} - e_{D_n}, U_{n,\infty}\rangle.
\end{aligned}$$

Consequently, conditioned on $\mathcal{Y}$ and on the environment ξ, the process

$$A_n := \frac{\langle Z^i_{*n} - e_{D_n}, U_{n,\infty}\rangle}{\lambda_{0,n-1}U_{0,\infty}(i)} - \sum_{k=0}^{n-1} \frac{\langle Y_k - e_{D_{k+1}}, U_{k+1,\infty}\rangle}{\lambda_{0,k-1}U_{0,\infty}(i)}, \quad n \geqslant 0, \tag{2.5.9}$$

is a martingale w.r.t. $\sigma(\xi, \mathcal{Y}, \{Z_{*k,n}, k \leqslant n\})$, $n \geqslant 0$, under the law $\mathbb{P}^i_\xi$. Notice that A_n is bounded from below by the opposite of the series (2.5.8) which converges a.s., so this martingale converges $\mathbb{P}^i$-a.s. to a finite limit. From (2.5.9) and using the a.s. convergence of A_n and of the series (2.5.8), together with the fact that $\frac{\langle e_{D_n}, U_{n,\infty}\rangle}{\lambda_{0,n-1}U_{0,\infty}(i)} \to 0$ a.s., we deduce that

$$\lim_{n\to+\infty} \frac{\langle Z^i_{*n}, U_{n,\infty}\rangle}{\lambda_{0,n-1}U_{0,\infty}(i)} \quad \text{exists and is finite } \mathbb{P}^i\text{-a.s.}$$

Therefore, by the definition of $\mathbb{P}^i_*$ (see (2.5.4)),

$$W^i = \limsup_{n\to+\infty} \frac{\langle Z^i_n, U_{n,\infty}\rangle}{\lambda_{0,n-1}U_{0,\infty}(i)} < +\infty \quad \mathbb{P}^i_*\text{-a.s.}$$

So applying (2.5.5) we see that $\mathbb{E}W^i = 1$, or equivalently $\mathbb{E}_\xi W^i = 1$ $\mathbb{P}$-a.s., which implies that W^i is non-degenerate.

Finally, if we denote by

$$\overline{q}^i(\xi) := \mathbb{P}_\xi\left(W^i = 0\right), \quad 1 \leqslant i \leqslant d,$$

then by letting $t \to +\infty$ in (2.2.12) we see that

$$\overline{q}(\xi) = f_0\left(\overline{q}(T\xi)\right), \tag{2.5.10}$$

where $\overline{q}(\xi) = \left(\overline{q}^1(\xi), \cdots, \overline{q}^d(\xi)\right)$. Clearly,

$$\left\{\|Z^i_n\| \underset{n\to+\infty}{\longrightarrow} 0\right\} \subset \left\{W^i = 0\right\}. \tag{2.5.11}$$

So, if W^i is non-degenerate, then we have $q^i(\xi) \leqslant \overline{q}^i(\xi) < 1$ $\mathbb{P}$-a.s. Hence, using [46, Proposition 3.1] we deduce from (2.5.10) that $\overline{q}(\xi) = q(\xi)$ $\mathbb{P}$-a.s. This concludes the proof of Theorem 2.2.6.

2.6 Proof of Corollary 2.2.8

By Theorem 2.2.6, we know that

$$\mathbb{P}_\xi \left(W^i = 0 \right) = q^i(\xi) < 1.$$

So from (2.5.11) we conclude that for all $1 \leqslant i \leqslant d$,

$$\left\{ \|Z_n^i\| \underset{n\to+\infty}{\longrightarrow} 0 \right\} = \left\{ W^i = 0 \right\} \quad \mathbb{P}\text{-a.s.} \tag{2.6.1}$$

By the definition of W_n^i (cf. (2.2.11)), we obtain that for all $1 \leqslant i \leqslant d$ and $n \geqslant 0$,

$$W_n^i \leqslant \frac{\|Z_n^i\|}{\lambda_{0,n-1} U_{0,\infty}(i)}. \tag{2.6.2}$$

Using Proposition 2.2.5, it follows from (2.6.2) that, $\mathbb{P}$-a.s. on the event $\{W^i > 0\}$,

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|Z_n^i\| \geqslant \gamma. \tag{2.6.3}$$

For any $\varepsilon > 0$, all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, we have

$$\mathbb{P}\left(\|Z_n^i\| \geqslant \mathrm{e}^{\varepsilon n} \|M_{0,n-1}\| \right) \leqslant \mathbb{E}\left(\frac{\mathbb{E}_\xi \|Z_n^i\|}{\|M_{0,n-1}\|} \mathrm{e}^{-\varepsilon n} \right) \leqslant \mathrm{e}^{-\varepsilon n}.$$

It follows that $\sum_{n\geqslant 1} \mathbb{P}\left(\|Z_n^i\| \geqslant \mathrm{e}^{\varepsilon n} \|M_{0,n-1}\| \right) < +\infty$. Applying the Borel-Cantelli lemma, we deduce that

$$\mathbb{P}\left(\|Z_n^i\| \geqslant \mathrm{e}^{\varepsilon n} \|M_{0,n-1}\| \quad \text{i.o.} \right) = 0,$$

where i.o. means infinitely often. Combining this with (2.2.13) we get that for all $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|Z_n^i\| \leqslant \varepsilon + \lim_{n\to+\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \varepsilon + \gamma.$$

Letting $\varepsilon \to 0$ and using (2.6.1) and (2.6.3), we see that $\mathbb{P}$-a.s. on the explosion event E^i, $\lim_{n\to+\infty} \frac{1}{n} \log \|Z_n^i\| = \gamma$. This ends the proof of Corollary 2.2.8.

2.7 Proof of Theorem 2.2.9

2.7.1 Auxiliary results

We will need the following preliminary lemma.

Lemma 2.7.1. *Assume condition* ***A4***. *Then :*

1. *for all* $n, k \geqslant 0$ *and* $1 \leqslant i, j, r \leqslant d$, $\mathbb{P}$-*a.s.*,

$$\frac{1}{D} \leqslant \frac{M_{n,n+k}(i,j)}{M_{n,n+k}(i,r)} \leqslant D \quad \textit{and} \quad \frac{1}{D} \leqslant \frac{M_{n,n+k}(i,j)}{M_{n,n+k}(r,j)} \leqslant D; \tag{2.7.1}$$

2. *for all* $n \geqslant 0$ *and* $1 \leqslant i \leqslant d$, $\mathbb{P}$-*a.s.*,

$$\frac{1}{dD} \leqslant U_{n,\infty}(i) \leqslant 1; \tag{2.7.2}$$

3. *for all* $n, k \geqslant 0$ *and* $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-*a.s.*,

$$\frac{1}{dD^2} \leqslant \frac{M_{n,n+k}(i,j)U_{n+k+1,\infty}(j)}{\lambda_{n,n+k}U_{n,\infty}(i)} \leqslant 1. \tag{2.7.3}$$

Proof. For $k = 0$, (2.7.1) is a direct consequence of condition **A4** and the fact that the sequence $(M_n)_{n\geqslant 0}$ is stationary. Moreover, for all $n \geqslant 0$, $k \geqslant 1$ and $1 \leqslant i, j, r \leqslant d$, we have

$$\frac{M_{n,n+k}(i,j)}{M_{n,n+k}(i,r)} = \sum_{l=1}^{d} \frac{M_{n,n+k-1}(i,l)M_{n+k}(l,r)}{\sum_{s=1}^{d} M_{n,n+k-1}(i,s)M_{n+k}(s,r)} \frac{M_{n+k}(l,j)}{M_{n+k}(l,r)}.$$

We note that

$$\left(\frac{M_{n,n+k-1}(i,l)M_{n+k}(l,r)}{\sum_{s=1}^{d} M_{n,n+k-1}(i,s)M_{n+k}(s,r)}\right)_{1\leqslant i,l\leqslant d}$$

is a positive stochastic matrix. Therefore we get the first inequalities in (2.7.1) : for all $n \geqslant 0$, $k \geqslant 1$ and $1 \leqslant i, j, r \leqslant d$, $\mathbb{P}$-a.s.,

$$\frac{1}{D} \leqslant \min_{1\leqslant l\leqslant d} \frac{M_{n+k}(l,j)}{M_{n+k}(l,r)} \leqslant \frac{M_{n,n+k}(i,j)}{M_{n,n+k}(i,r)} \leqslant \max_{1\leqslant l\leqslant d} \frac{M_{n+k}(l,j)}{M_{n+k}(l,r)} \leqslant D.$$

A similar argument gives the second inequality in (2.7.1). So the proof of (2.7.1) is complete.

By (2.2.3) and (2.7.1) we get that for all $n \geqslant 0$ and $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-a.s.,

$$\frac{U_{n,\infty}(i)}{U_{n,\infty}(j)} = \lim_{k\to+\infty} \frac{M_{n,n+k}(i,i)}{M_{n,n+k}(j,i)} \geqslant \frac{1}{D}.$$

Since $\|U_{n,\infty}\| = 1$, this implies (2.7.2).

Using (2.2.7), it is clear that for all $n, k \geqslant 0$,

$$\left(\frac{M_{n,n+k}(i,j)U_{n+k+1,\infty}(j)}{\lambda_{n,n+k}U_{n,\infty}(i)}\right)_{1\leqslant i,j\leqslant d}$$

is a positive stochastic matrix. Then, applying (2.2.8), (2.7.2) and (2.7.1), it follows that for all $n, k \geqslant 0$ and $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\frac{M_{n,n+k}(i,j)U_{n+k+1,\infty}(j)}{\lambda_{n,n+k}U_{n,\infty}(i)} &= \frac{M_{n,n+k}(i,j)U_{n+k+1,\infty}(j)}{\sum_{r=1}^{d} M_{n,n+k}(i,r)U_{n+k+1,\infty}(r)} \\
&\geqslant \frac{1}{dD}\Big(\sum_{r=1}^{d} \frac{M_{n,n+k}(i,r)U_{n+k+1,\infty}(r)}{M_{n,n+k}(i,j)}\Big)^{-1} \\
&\geqslant \frac{1}{dD^2}\Big(\sum_{r=1}^{d} U_{n+k+1,\infty}(r)\Big)^{-1} = \frac{1}{dD^2}. \qquad (2.7.4)
\end{aligned}$$

This ends the proof of (2.7.3). □

2.7.2 Proof of Theorem 2.2.9

Notice that the conclusion of Theorem 2.2.9 for an i.i.d. environment follows from that for a stationary and ergodic environment and the fact that the conditions **A5** and **A7** are equivalent in the i.i.d. case (cf. Lemma 2.11.1). So we need only to prove Theorem 2.2.9 when the environment is stationary and ergodic.

By Theorem 2.2.6 and Lemma 2.11.1 we know that **A5** is sufficient for the non-degeneracy of all the W^i, $1 \leqslant i \leqslant d$.

We now prove that if **A5** fails, then each W^i is degenerate. Assume that **A5** fails. Then **A3** fails, since **A3** ⇔ **A5**, which means that for all $C > 0$,

$$\mathbb{P}\left(\max_{1\leqslant r\leqslant d} \sum_{n=0}^{+\infty} \mathbb{E}_{\xi}\left(\frac{\langle N_n^r, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(r)} \mathbb{1}_{\{\langle N_n^r, U_{n+1,\infty}\rangle \geqslant C^n\}}\right) = +\infty\right) > 0. \qquad (2.7.5)$$

We keep the notation of the proof of Theorem 2.2.6. By the definition of the tree $\mathbb{T}^i_*$, for $n \geqslant 0$,

$$Z^i_{*n} \geqslant Y_{n-1} \quad \mathbb{P}^i\text{-a.s.} \tag{2.7.6}$$

Let $(\mathcal{F}_{*n})_{n\geqslant 0}$ be the filtration defined by $\mathcal{F}_{*0} = \sigma(\xi)$, and for $n \geqslant 1$,

$$\mathcal{F}_{*n} = \sigma\Big(\xi, N^r_{l,k}, Y^r_k, D_k, 0 \leqslant k \leqslant n-1, 1 \leqslant r \leqslant d, l \geqslant 1\Big).$$

By the conditional Borel-Cantelli lemma [21, Theorem 5.3.2] we get that for all $C > 0$,

$$\Big\{\log^+\langle Y_n, U_{n+1,\infty}\rangle \geqslant Cn \quad \text{i.o.}\Big\} = \left\{\sum_{n=1}^{+\infty} \mathbb{P}\left(\log^+\langle Y_n, U_{n+1,\infty}\rangle \geqslant Cn \Big| \mathcal{F}_{*n}\right) = +\infty\right\}. \tag{2.7.7}$$

By the independence under $\mathbb{P}_\xi$ between $\{Y^r_n : 1 \leqslant r \leqslant d\}$ and $\mathcal{F}_{*n}$, we have

$$\begin{aligned}
&\sum_{n=1}^{+\infty} \mathbb{P}\left(\log^+\langle Y_n, U_{n+1,\infty}\rangle \geqslant Cn \Big| \mathcal{F}_{*n}\right) \\
&\quad = \sum_{n=1}^{+\infty} \mathbb{P}_\xi\left(\sum_{r=1}^{d} \log^+\langle Y^r_n, U_{n+1,\infty}\rangle \mathbb{1}_{\{D_n=r\}} \geqslant Cn \Big| \mathcal{F}_{*n}\right) \\
&\quad = \sum_{n=1}^{+\infty}\sum_{r=1}^{d} \mathbb{P}_\xi\Big(\log^+\langle Y^r_n, U_{n+1,\infty}\rangle \geqslant Cn\Big)\, \mathbb{P}_\xi\left(D_n = r \Big| \mathcal{F}_{*n}\right).
\end{aligned} \tag{2.7.8}$$

For any $n \geqslant 1$, under $\mathbb{P}_\xi$, D_n is independent of the family $\{N^r_{l,k}, Y^r_k\}$ with $0 \leqslant k \leqslant n-1, 1 \leqslant r \leqslant d$ and $l \geqslant 1$. Therefore, for $1 \leqslant r \leqslant d$ and $n \geqslant 1$,

$$\mathbb{P}_\xi\Big(D_n = r\Big|\mathcal{F}_{*n}\Big) = \mathbb{P}_\xi\Big(D_n = r\Big|D_k, 1 \leqslant k \leqslant n-1\Big)$$

Moreover by the construction of (D_n), for all $n \geqslant 1$ and $(j_1, \cdots, j_{n-1}, r) \in \{1, \cdots, d\}^n$,

$$\begin{aligned}
&\mathbb{P}_\xi\Big(D_n = r\Big|D_{n-1} = j_{n-1}, \cdots, D_1 = j_1, D_0 = i\Big)\\
&\quad = \mathbb{P}_\xi\Big(D_n = r\Big|D_{n-1} = j_{n-1}\Big)\\
&\quad = \sum_{y\in\mathbb{N}^d} \frac{y(r)U_{n,\infty}(r)}{\langle y, U_{n,\infty}\rangle}\mathbb{P}_\xi\left(Y_{n-1}^{j_{n-1}} = y\right)\\
&\quad = \sum_{y\in\mathbb{N}^d} \frac{y(r)U_{n,\infty}(r)}{\lambda_{n-1}U_{n-1,\infty}(j_{n-1})}\mathbb{P}_\xi\left(N_{n-1}^{j_{n-1}} = y\right)\\
&\quad = \frac{M_{n-1}(j_{n-1}, r)U_{n,\infty}(r)}{\lambda_{n-1}U_{n,\infty}(j_{n-1})}.
\end{aligned}$$

This implies that for all $1 \leqslant r \leqslant d$ and $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$\mathbb{P}_\xi\Big(D_n = r\Big|\mathcal{F}_{*n}\Big) = \frac{M_{n-1}(D_{n-1}, r)U_{n,\infty}(r)}{\lambda_{n-1}U_{n,\infty}(D_{n-1})}.$$

Then, using (2.7.3), it follows that for all $1 \leqslant r \leqslant d$ and $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$\mathbb{P}_\xi\Big(D_n = r\Big|\mathcal{F}_{*n}\Big) \geqslant \frac{1}{dD^2}. \tag{2.7.9}$$

Combining equality (2.7.8) with inequalities (2.7.9) and (2.7.3), we get that for all $C > 0$, $\mathbb{P}$-a.s.,

$$\sum_{n=1}^{+\infty} \mathbb{P}\left(\log^+\langle Y_n, U_{n+1,\infty}\rangle \geqslant Cn\Big|\mathcal{F}_{*n}\right) \geqslant \frac{1}{dD^2}\sum_{n=1}^{+\infty}\sum_{r=1}^{d}\mathbb{P}_\xi\left(\log^+\langle Y_n^r, U_{n+1,\infty}\rangle \geqslant Cn\right). \tag{2.7.10}$$

By the definition of Y_n^r, for all $C > 0$,

$$\mathbb{P}_\xi\left(\log^+\langle Y_n^r, U_{n+1,\infty}\rangle \geqslant Cn\right) = \mathbb{E}_\xi\left(\frac{\langle N_n^r, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(r)}\mathbb{1}_{\{\log^+\langle N_n^r, U_{n+1,\infty}\rangle \geqslant Cn\}}\right).$$

Using this together with (2.7.5) and (2.7.10), we deduce that for all $C > 0$,

$$\mathbb{P}\left(\sum_{n=1}^{+\infty}\sum_{r=1}^{d}\mathbb{P}_\xi\left(\log^+\langle Y_n^r, U_{n+1,\infty}\rangle \geqslant Cn\right) = +\infty\right) > 0. \tag{2.7.11}$$

Since $(\log^+\langle Y_n^r, U_{n+1,\infty}\rangle)_{n\geqslant 0}$ is a non negative stationary and ergodic sequence, by [66, Theorem 1] of Tanny we know that

$$\limsup_{n\to+\infty} \frac{\log^+\langle Y_n^r, U_{n+1,\infty}\rangle}{n} \text{ is either } 0 \ \mathbb{P}\text{-a.s. or } +\infty \ \mathbb{P}\text{-a.s.}$$

As $(Y_n^r, U_{n+1,\infty}), n \geqslant 1$, are independent under $\mathbb{P}_\xi$, by Borel-Cantelli lemma this implies that for all $C > 0$ and $1 \leqslant r \leqslant d$, either

$$\sum_{n=1}^{+\infty} \mathbb{P}_\xi \left(\log^+\langle Y_n^r, U_{n+1,\infty}\rangle \geqslant Cn\right) < +\infty \quad \mathbb{P}\text{-a.s.},$$

or

$$\sum_{n=1}^{+\infty} \mathbb{P}_\xi \left(\log^+\langle Y_n^r, U_{n+1,\infty}\rangle \geqslant Cn\right) = +\infty \quad \mathbb{P}\text{-a.s.}$$

This statement remains valid while $\mathbb{P}_\xi \left(\log^+\langle Y_n^r, U_{n+1,\infty}\rangle \geqslant Cn\right)$ is replaced by $\sum_{r=1}^d \mathbb{P}_\xi\Big(\log^+\langle Y_n^r, U_{n+1,\infty}\rangle \geqslant Cn\Big)$. Therefore from (2.7.11) we obtain that for all $C > 0$, $\mathbb{P}$-a.s.,

$$\sum_{n=1}^{+\infty} \sum_{r=1}^{d} \mathbb{P}_\xi \left(\log^+\langle Y_n^r, U_{n+1,\infty}\rangle \geqslant Cn\right) = +\infty. \tag{2.7.12}$$

Combining (2.7.7), (2.7.10) and (2.7.12), we deduce that

$$\limsup_{n\to+\infty} \frac{\log^+\langle Y_n, U_{n+1,\infty}\rangle}{n} = +\infty \quad \mathbb{P}\text{-a.s.}$$

It follows from (2.7.6) that, $\mathbb{P}^i$-a.s.,

$$\limsup_{n\to+\infty} \frac{\log^+\langle Z_{*n}^i, U_{n,\infty}\rangle}{n} \geqslant \limsup_{n\to+\infty} \frac{\log^+\langle Y_{n-1}, U_{n,\infty}\rangle}{n} = +\infty. \tag{2.7.13}$$

By Proposition 2.2.5 we have $\log \lambda_{0,n-1} \sim \gamma n$ $\mathbb{P}$-a.s. as $n \to +\infty$. So we get from (2.7.13) that

$$\limsup_{n\to+\infty} \frac{\langle Z_{*n}^i, U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} = +\infty \quad \mathbb{P}^i\text{-a.s.},$$

or equivalently

$$W^i = \limsup_{n\to+\infty} \frac{\langle Z_n^i, U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} = +\infty \quad \mathbb{P}_*^i\text{-a.s.}$$

By (2.5.5) we conclude that $W^i = 0$ $\mathbb{P}^i$-a.s. , for any $1 \leqslant i \leqslant d$. This ends the proof of Theorem 2.2.9.

2.8 Proof of Theorem 2.2.10

2.8.1 Auxiliary results

We need additional results on the products of the mean matrices (M_n). Set

$$\delta = \frac{D^2 - 1}{D^2 + 1} \in (0, 1).$$

The following Lemma was proved by Kesten and Spitzer in [48]. It gives a uniform convergence in (2.2.3) with an exponential rate, under the condition **A4** of Furstenberg and Kesten [26].

Lemma 2.8.1. *Assume condition* ***A4****. Then there exists a constant $C > 0$ such that for all $1 \leqslant i, j \leqslant d$ and $k \geqslant 0$,*

$$\sup_{n \geqslant 0} \left| \frac{M_{n,n+k}(i,j)}{\rho_{n,n+k} U_{n,n+k}(i) V_{n,n+k}(j)} - 1 \right| \leqslant C\delta^k \quad \mathbb{P}\text{-}a.s. \tag{2.8.1}$$

The next result establishes a uniform convergence with an exponential rate for the left and right eigenvectors $U_{n,n+k}$ and $V_{n,n+k}/\|V_{n,n+k}\|$, as $k \to \infty$.

Lemma 2.8.2. *Assume condition* ***A4****. Then there exists a constant $C > 0$ such that for all $k \geqslant 0$,*

$$\sup_{n \geqslant 0} \|U_{n,n+k} - U_{n,\infty}\| \leqslant C\delta^k \quad \mathbb{P}\text{-}a.s. \tag{2.8.2}$$

and

$$\sup_{n \geqslant 0} \left\| \frac{V_{0,n+k}}{\|V_{0,n+k}\|} - \frac{V_{n,n+k}}{\|V_{n,n+k}\|} \right\| \leqslant C\delta^k \quad \mathbb{P}\text{-}a.s. \tag{2.8.3}$$

Proof. We only prove (2.8.3), since one can obtain (2.8.2) by similar arguments. Let $C > 0$ be as in Lemma 2.8.1. Denote by $k_0 \geqslant 0$ such that $C\delta^{k_0} < 1$. By Lemma 2.8.1, for all $n \geqslant 0$, $k \geqslant k_0$ and $1 \leqslant i, j \leqslant d$ we have

$$\frac{1-C\delta^k}{1+C\delta^k}\frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \leqslant \frac{M_{n,n+k}(i,j)}{\|M_{n,n+k}(i,\cdot)\|} \leqslant \frac{1+C\delta^k}{1-C\delta^k}\frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \quad \mathbb{P}\text{-a.s.}$$

From this and the fact that $\frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \leqslant 1$, we deduce that for all $1 \leqslant i \leqslant d$ and $k \geqslant k_0$,

$$\sup_{n\geqslant 0}\left|\frac{M_{n,n+k}(i,j)}{\|M_{n,n+k}(i,\cdot)\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|}\right| \leqslant \frac{2C}{1-C\delta^{k_0}}\delta^k \quad \mathbb{P}\text{-a.s.}$$

Therefore we get that for all $1 \leqslant i \leqslant d$ and $k \geqslant k_0$,

$$\sup_{n\geqslant 0}\left|\frac{M_{n,n+k}(i,j)}{\|M_{n,n+k}(i,\cdot)\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|}\right| \leqslant C_1\delta^k \quad \mathbb{P}\text{-a.s.}, \tag{2.8.4}$$

with $C_1 = 2C/(1-C\delta^{k_0})$. From (2.8.4) it follows that for all $n \geqslant 0$, $k \geqslant k_0$ and $1 \leqslant j \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\left|\frac{V_{0,n+k}(j)}{\|V_{0,n+k}\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|}\right| \\
\leqslant\ & \left|\frac{M_{0,n+k}(j,j)}{\|M_{0,n+k}(j,\cdot)\|} - \frac{M_{n,n+k}(j,j)}{\|M_{n,n+k}(j,\cdot)\|}\right| + C_1\delta^{n+k} + C_1\delta^k \\
\leqslant\ & \left|\sum_{i=1}^d M_{0,n-1}(j,i)\frac{M_{n,n+k}(i,j)}{\|M_{0,n+k}(j,\cdot)\|} - \frac{M_{n,n+k}(j,j)}{\|M_{n,n+k}(j,\cdot)\|}\right| + 2C_1\delta^k \\
=\ & \left|\sum_{i=1}^d M_{0,n-1}(j,i)\frac{\|M_{n,n+k}(i,\cdot)\|}{\|M_{0,n+k}(j,\cdot)\|}\left(\frac{M_{n,n+k}(i,j)}{\|M_{n,n+k}(i,\cdot)\|} - \frac{M_{n,n+k}(j,j)}{\|M_{n,n+k}(j,\cdot)\|}\right)\right| \\
& +2C_1\delta^k,
\end{aligned}$$

So we obtain that for all $n \geqslant 0$, $k \geqslant k_0$ and $1 \leqslant j \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\left|\frac{V_{0,n+k}(j)}{\|V_{0,n+k}\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|}\right| \\
\leqslant\ & \sum_{i=1}^d M_{0,n-1}(j,i)\frac{\|M_{n,n+k}(i,\cdot)\|}{\|M_{0,n+k}(j,\cdot)\|}\left|\frac{M_{n,n+k}(i,j)}{\|M_{n,n+k}(i,\cdot)\|} - \frac{M_{n,n+k}(j,j)}{\|M_{n,n+k}(j,\cdot)\|}\right| \\
& +2C_1\delta^k \\
\leqslant\ & \max_{1\leqslant r\leqslant d}\left|\frac{M_{n,n+k}(r,j)}{\|M_{n,n+k}(r,\cdot)\|} - \frac{M_{n,n+k}(j,j)}{\|M_{n,n+k}(j,\cdot)\|}\right| + 2C_1\delta^k \\
\leqslant\ & 4C_1\delta^k,
\end{aligned}$$

where the last step holds by (2.8.4). Hence (2.8.3) holds for all $k \geqslant k_0$ and $C = 4C_1$. For $k < k_0$, since $\frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|} \leqslant 1$, we have $|\frac{V_{0,n+k}(j)}{\|V_{0,n+k}\|} - \frac{V_{n,n+k}(j)}{\|V_{n,n+k}\|}| \leqslant 1 \leqslant \delta^{-k_0}\delta^k$. Therefore (2.8.3) holds for all $k \geqslant 0$ and $C = \max(4C_1, \delta^{-k_0})$. □

The next assertion shows that conditioned on the explosion event $E^i = \{\|Z_n^i\| \to +\infty\}$, each component $Z_n^i(j)$ of Z_n^i tends to $+\infty$ in probability.

Proposition 2.8.3. *Assume conditions* ***A2****,* ***A4****, and $\gamma > 0$. Then, for all $1 \leqslant i, j \leqslant d$ such that $\mathbb{P}(E^i) > 0$, we have*

$$Z_n^i(j) \xrightarrow[n\to+\infty]{\mathbb{P}_{E^i}} +\infty.$$

Proof. Clearly, it suffices to prove that for all $1 \leqslant i, j \leqslant d$ and $K \geqslant 0$,

$$\mathbb{P}\Big(Z_n^i(j) \geqslant K, E^i\Big) \underset{n\to+\infty}{\longrightarrow} \mathbb{P}(E^i), \tag{2.8.5}$$

Set $K_1, K_2 \geqslant 0$. By (2.4.1), for $n, k \geqslant 1$ and $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-a.s., we have

$$\begin{aligned}
&\mathbb{P}_\xi\left(Z_{n+k}^i(j) \leqslant K_1, \|Z_n^i\| \geqslant K_2\right)\\
= \;& \mathbb{P}_\xi\left(\sum_{r=1}^{d}\sum_{l=1}^{Z_n^i(r)} Z_{l,n,k}^r(j) \leqslant K_1, \|Z_n^i\| \geqslant K_2\right)\\
\leqslant \;& \mathbb{P}_\xi\left(Z_{l,n,k}^r(j) \leqslant K_1, \|Z_n^i\| \geqslant K_2, 1 \leqslant r \leqslant d, 1 \leqslant l \leqslant Z_n^i(r)\right)\\
= \;& \mathbb{E}_\xi\left[\mathbb{P}_{T^n\xi}\left(Z_k^1(j) \leqslant K_1\right)^{Z_n^i(1)} \cdots\ \mathbb{P}_{T^n\xi}\left(Z_k^d(j) \leqslant K_1\right)^{Z_n^i(d)} \mathbb{1}_{\{\|Z_n^i\| \geqslant K_2\}}\right].
\end{aligned}$$

It follows that, $\mathbb{P}$-a.s.,

$$\mathbb{P}_\xi\left(Z_{n+k}^i(j) \leqslant K_1, \|Z_n^i\| \geqslant K_2\right) \leqslant \left(\max_{1\leqslant r\leqslant d} \mathbb{P}_{T^n\xi}\left(Z_k^r(j) \leqslant K_1\right)\right)^{K_2}.$$

This together with the fact that

$$\limsup_{n\to+\infty} \mathbb{P}\{E^i, \|Z_n^i\| < K_2\} \leqslant \mathbb{P}(\limsup_{n\to+\infty}\{E^i, \|Z_n^i\| < K_2\}) = 0,$$

implies that

$$\begin{aligned}\limsup_{n\to+\infty} \mathbb{P}(Z_n^i(j) \leqslant K_1, E^i) &\leqslant \limsup_{n\to+\infty} \mathbb{P}\left(Z_{n+k}^i(j) \leqslant K_1, \|Z_n^i\| \geqslant K_2\right) \\ &\leqslant \mathbb{E}\Big(\max_{1\leqslant r\leqslant d} \mathbb{P}_\xi\left(Z_k^r(j) \leqslant K_1\right)\Big)^{K_2}.\end{aligned}$$

Letting $K_2 \to +\infty$, it follows that

$$\begin{aligned}\limsup_{n\to+\infty} \mathbb{P}(Z_n^i(j) \leqslant K_1, E^i) &\leqslant \mathbb{P}\Big(\max_{1\leqslant r\leqslant d} \mathbb{P}_\xi(Z_k^r(j) \leqslant K_1) = 1\Big) \\ &\leqslant \sum_{r=1}^{d} \mathbb{P}\Big(\mathbb{P}_\xi(Z_k^r(j) \leqslant K_1) = 1\Big). \end{aligned} \tag{2.8.6}$$

By (2.2.19) we know that

$$\mathbb{E}_\xi Z_k^r(j) = M_{0,k-1}(r,j) \underset{k\to+\infty}{\longrightarrow} +\infty \quad \mathbb{P}\text{-a.s.}, \tag{2.8.7}$$

which implies that for all $K_1 \geqslant 0$,

$$\mathbb{P}\Big(\mathbb{P}_\xi(Z_k^r(j) \leqslant K_1) = 1\Big) \leqslant \mathbb{P}\Big(\mathbb{E}_\xi Z_k^r(j) \leqslant K_1\Big) \underset{k\to+\infty}{\longrightarrow} 0.$$

Therefore from (2.8.6), we conclude that for all $K_1 \geqslant 0$,

$$\mathbb{P}\Big(Z_n^i(j) \leqslant K_1, E^i\Big) \underset{n\to+\infty}{\longrightarrow} 0,$$

which implies (2.8.5) and ends the proof of Proposition 2.8.3. □

2.8.2 Proof of Theorem 2.2.10

Let $1 \leqslant i \leqslant d$. For all $n, k \geqslant 0$ set

$$\overline{Z}_n^i = \frac{Z_n^i}{\|Z_n^i\|} \quad \text{and} \quad \overline{V}_{n,n+k} = \frac{V_{n,n+k}}{\|V_{n,n+k}\|}.$$

From (2.4.1) and on the event E^i, for any $n, k \geqslant 1$ we have

$$\begin{aligned}
&\frac{1}{\rho_{n,n+k-1}\|V_{n,n+k-1}\|}\left\|\frac{Z^i_{n+k}}{\|Z^i_n\|} - M^T_{n,n+k-1}\overline{Z}^i_n\right\| \\
\leqslant\ & \frac{1}{\rho_{n,n+k-1}\|V_{n,n+k-1}\|}\frac{1}{\|Z^i_n\|}\sum_{j=1}^{d}\sum_{r=1}^{d}\left|\sum_{l=1}^{Z^i_n(r)}\Big(Z^r_{l,n,k}(j) - M_{n,n+k-1}(r,j)\Big)\right| \\
=\ & \sum_{j=1}^{d}\sum_{r=1}^{d}\frac{M_{n,n+k-1}(r,j)}{\rho_{n,n+k-1}\|V_{n,n+k-1}\|}\frac{1}{\|Z^i_n\|}\left|\sum_{l=1}^{Z^i_n(r)}\left(\frac{Z^r_{l,n,k}(j)}{M_{n,n+k-1}(r,j)} - 1\right)\right|. \qquad (2.8.8)
\end{aligned}$$

By the weak law of large numbers and Proposition 2.8.3 we get that for all $1 \leqslant r, j \leqslant d$,

$$\frac{1}{\|Z^i_n\|}\sum_{l=1}^{Z^i_n(r)}\left(\frac{Z^r_{l,n,k}(j)}{M_{n,n+k-1}(r,j)} - 1\right) \xrightarrow[n\to+\infty]{\mathbb{P}_{E^i}} 0. \qquad (2.8.9)$$

Let $C > 0$ be sufficiently large such that (2.8.1) and (2.8.3) hold. By (2.8.1), for any $1 \leqslant r, j \leqslant d$, $n \geqslant 0$ and $k \geqslant 1$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\frac{M_{n,n+k-1}(r,j)}{\rho_{n,n+k-1}\|V_{n,n+k-1}\|} &\leqslant (1 + C\delta^k)\frac{U_{n,n+k-1}(r)V_{n,n+k-1}(j)}{\|V_{n,n+k-1}\|} \\
&\leqslant 1 + C\delta^k.
\end{aligned}$$

Combining this with (2.8.8) and (2.8.9), we deduce that for all $k \geqslant 1$,

$$\frac{1}{\rho_{n,n+k-1}\|V_{n,n+k-1}\|}\left\|\frac{Z^i_{n+k}}{\|Z^i_n\|} - M^T_{n,n+k-1}\overline{Z}^i_n\right\| \xrightarrow[n\to+\infty]{\mathbb{P}_{E^i}} 0. \qquad (2.8.10)$$

Moreover, by Lemma 2.8.1 we get that that for any $n, k \geqslant 1$,

$$\begin{aligned}
&\left\| \frac{M^T_{n,n+k-1}}{\rho_{n,n+k-1}\|V_{n,n+k-1}\|} \overline{Z}^i_n - \langle \overline{Z}^i_n, U_{n,n+k-1}\rangle \overline{V}_{n,n+k-1} \right\| \\
\leqslant\ & \sum_{r=1}^{d}\sum_{j=1}^{d} \left| \frac{M_{n,n+k-1}(r,j)}{\rho_{n,n+k-1}\|V_{n,n+k-1}\|} \overline{Z}^i_n(r) - U_{n,n+k-1}(r)\overline{V}_{n,n+k-1}(j)\overline{Z}^i_n(r) \right| \\
\leqslant\ & \sum_{r=1}^{d}\sum_{j=1}^{d} U_{n,n+k-1}(r)\overline{V}_{n,n+k-1}(j)\overline{Z}^i_n(r) \times \\
& \quad \left| \frac{M_{n,n+k-1}(r,j)}{\rho_{n,n+k-1}U_{n,n+k-1}(r)V_{n,n+k-1}(j)} - 1 \right| \\
\leqslant\ & \max_{1\leqslant r,j\leqslant d} \left| \frac{M_{n,n+k-1}(r,j)}{\rho_{n,n+k-1}U_{n,n+k-1}(r)V_{n,n+k-1}(j)} - 1 \right| \\
\leqslant\ & C\delta^k. \qquad (2.8.11)
\end{aligned}$$

Let $k_0 \in \mathbb{N}$ be large enough such that $C\delta^{k_0-1} \leqslant 1/(dD)$. Then, combining (2.7.2) and (2.8.2), we see that for all $1 \leqslant r \leqslant d$, $n \geqslant 1$ and $k \geqslant k_0$,

$$U_{n,n+k-1}(r) \geqslant \frac{1}{2dD} \quad \mathbb{P}\text{-a.s.}$$

It follows that for all $n \geqslant 1$ and $k \geqslant k_0$,

$$\langle \overline{Z}^i_n, U_{n,n+k-1}\rangle \geqslant \frac{1}{2dD} \quad \mathbb{P}\text{-a.s.} \qquad (2.8.12)$$

Let $\varepsilon > 0$. Let $k_1 \in \mathbb{N}$ be such that $2dDC\delta^{k_1} \leqslant \varepsilon/8$ and $k_1 \geqslant k_0$. For all $n \geqslant 0$ and $k \geqslant k_1$, set

$$Y^i_{n,k} = \frac{\|Z^i_{n+k}\|}{\rho_{n,n+k-1}\|V_{n,n+k-1}\|\langle \overline{Z}^i_n, U_{n,n+k-1}\rangle \|Z^i_n\|},$$

which is well defined on the explosion event E^i. Notice that

$$\begin{aligned}
Y^i_{n,k}\overline{Z}^i_{n+k} - \overline{V}_{n,n+k-1} = & \frac{1}{\langle \overline{Z}^i_n, U_{n,n+k-1}\rangle} \times \\
& \left[\frac{Z^i_{n+k}}{\rho_{n,n+k-1}\|V_{n,n+k-1}\|\|Z^i_n\|} - \langle \overline{Z}^i_n, U_{n,n+k-1}\rangle \overline{V}_{n,n+k-1} \right].
\end{aligned}$$

Therefore, combining the relations (2.8.10) and (2.8.11), together with (2.8.12), we obtain that for all $k \geqslant k_1$,

$$\mathbb{P}_{E^i}\left(\left\|Y^i_{n,k}\overline{Z}^i_{n+k} - \overline{V}_{n,n+k-1}\right\| > \frac{\varepsilon}{4}\right) \underset{n\to+\infty}{\longrightarrow} 0. \tag{2.8.13}$$

Applying (2.8.13) and the triangle inequality we have that for all $k \geqslant k_1$,

$$\begin{aligned}
&\limsup_{n\to+\infty} \mathbb{P}_{E^i}\left(\left|Y^i_{n,k} - 1\right| > \frac{\varepsilon}{4}\right) \\
&\quad = \limsup_{n\to+\infty} \mathbb{P}_{E^i}\left(\left|\ \left\|Y^i_{n,k}\overline{Z}^i_{n+k}\right\| - \left\|\overline{V}_{n,n+k-1}\right\|\ \right| > \frac{\varepsilon}{4}\right) \\
&\quad \leqslant \limsup_{n\to+\infty} \mathbb{P}_{E^i}\left(\left\|Y^i_{n,k}\overline{Z}^i_{n+k} - \overline{V}_{n,n+k-1}\right\| > \frac{\varepsilon}{4}\right) \\
&\quad = 0.
\end{aligned} \tag{2.8.14}$$

Combining (2.8.13) with (2.8.14), we obtain that for all $k \geqslant k_1$,

$$\begin{aligned}
&\mathbb{P}_{E^i}\left(\left\|\overline{Z}^i_{n+k} - \overline{V}_{n,n+k-1}\right\| > \frac{\varepsilon}{2}\right) \\
\leqslant\ & \mathbb{P}_{E^i}\left(\left\|Y^i_{n,k}\overline{Z}^i_{n+k} - \overline{V}_{n,n+k-1}\right\| + \left|Y^i_{n,k} - 1\right| > \frac{\varepsilon}{2}\right) \\
\leqslant\ & \mathbb{P}_{E^i}\left(\left\|Y^i_{n,k}\overline{Z}^i_{n+k} - \overline{V}_{n,n+k-1}\right\| > \frac{\varepsilon}{4}\right) + \mathbb{P}_{E^i}\left(\left|Y^i_{n,k} - 1\right| > \frac{\varepsilon}{4}\right) \\
\underset{n\to+\infty}{\to}\ & 0.
\end{aligned} \tag{2.8.15}$$

Notice that for any $k_2 \geqslant 0$,

$$\begin{aligned}
&\limsup_{n\to+\infty} \mathbb{P}_{E^i}\left(\left\|\overline{Z}^i_n - \overline{V}_{0,n-1}\right\| > \varepsilon\right) \\
\leqslant\ & \limsup_{n\to+\infty} \mathbb{P}_{E^i}\left(\left\|\overline{Z}^i_{n+k_2} - \overline{V}_{n,n+k_2-1}\right\| > \frac{\varepsilon}{2}\right) \\
&+ \limsup_{n\to+\infty} \mathbb{P}_{E^i}\left(\left\|\overline{V}_{0,n+k_2-1} - \overline{V}_{n,n+k_2-1}\right\| > \frac{\varepsilon}{2}\right)
\end{aligned}$$

Let $k_2 \geqslant k_1$ be such that $C\delta^{k_2-1} \leqslant \varepsilon/2$. Then by (2.8.3), the second term in the right hand side is 0. The first one is also 0 by (2.8.15). Hence

$$\limsup_{n\to+\infty} \mathbb{P}_{E^i}\left(\left\|\overline{Z}^i_n - \overline{V}_{0,n-1}\right\| > \varepsilon\right) = 0.$$

This proves (2.2.20).

Since $(\overline{V}_{0,n})_{n\geqslant 0}$ convergences in law to $\overline{V}_{0,\infty}$ (see Proposition 2.2.1), from (2.2.20) we obtain directly the convergence in law of $(\overline{Z}^i_n)_{n\geqslant 0}$ to $\overline{V}_{0,\infty}$. This concludes the proof of Theorem 2.2.10.

2.9 Proof of Theorem 2.2.11 and Corollary 2.2.12

Proof of Theorem 2.2.11. By (2.7.3), for all $1\leqslant i\leqslant d$ and $n\geqslant 1$, $\mathbb{P}$-a.s., we have

$$\begin{aligned} W^i_n &= \sum_{j=1}^d \frac{M_{0,n-1}(i,j)U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \frac{Z^i_n(j)}{M_{0,n-1}(i,j)} \\ &\geqslant \frac{1}{dD^2}\sum_{j=1}^d \frac{Z^i_n(j)}{\mathbb{E}_\xi Z^i_n(j)}. \end{aligned}$$

Consequently we get that $Z^i_n(j)/\mathbb{E}_\xi Z^i_n(j) \to 0$ $\mathbb{P}$-a.s. on the event $\{W^i = 0\}$, for all $1\leqslant i,j\leqslant d$. Now we investigate on the event $\{W^i > 0\}$. By (2.6.2) and Proposition 2.2.5 it holds that $\|Z^i_n\|\to+\infty$ $\mathbb{P}$-a.s. on $\{W^i>0\}$. Moreover, using (2.2.3) and Proposition 2.2.10 we have that for all $1\leqslant i,j\leqslant d$, as $n\to+\infty$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \frac{1}{W^i_n}\frac{Z^i_n(j)}{\mathbb{E}_\xi Z^i_n(j)} &= \frac{U_{0,\infty}(i)\lambda_{0,n-1}}{M_{0,n-1}(i,j)}\frac{Z^i_n(j)}{\langle Z^i_n, U_{n,\infty}\rangle} \\ &\sim \frac{\langle V_{0,n-1}, U_{n,\infty}\rangle}{V_{0,n-1}(j)}\frac{Z^i_n(j)}{\langle Z^i_n, U_{n,\infty}\rangle} \\ &\sim \frac{\|V_{0,n-1}\|}{V_{0,n-1}(j)}\frac{Z^i_n(j)}{\|Z^i_n\|}\sum_{r=1}^d \frac{Z^i_n(r)U_{n,\infty}(r)}{\langle Z^i_n, U_{n,\infty}\rangle}\frac{V_{0,n-1}(r)}{\|V_{0,n-1}\|}\frac{\|Z^i_n\|}{Z^i_n(r)}. \end{aligned} \tag{2.9.1}$$

Applying Theorem 2.2.10 it follows that for all $1\leqslant i,j\leqslant d$,

$$\left|\frac{Z^i_n(j)}{\|Z^i_n\|} - \frac{V_{0,n-1}(j)}{\|V_{0,n-1}\|}\right| \xrightarrow[n\to+\infty]{\mathbb{P}_{E^i}} 0.$$

Since $(V_{0,n}/\|V_{0,n}\|)_{n\geqslant 0}$ converges in law to $\overline{V}_{0,\infty}$ with $\overline{V}_{0,\infty}>0$ $\mathbb{P}$-a.s., this implies that

$$\left|\frac{\|V_{0,n-1}\|}{V_{0,n-1}(j)}\frac{Z^i_n(j)}{\|Z^i_n\|} - 1\right| \xrightarrow[n\to+\infty]{\mathbb{P}_{E^i}} 0. \tag{2.9.2}$$

Combining (2.9.1) and (2.9.2), we deduce that for all $1 \leqslant i, j \leqslant d$,

$$\frac{1}{W_n^i} \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \xrightarrow[n\to+\infty]{\mathbb{P}_{E^i}} 1.$$

It follows that

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \xrightarrow[n\to+\infty]{\mathbb{P}_{E^i}} W^i,$$

which concludes the proof of (2.2.23). From (2.2.23) and (2.2.3), we deduce (2.2.22) : for all $1 \leqslant i, j \leqslant d$,

$$\begin{aligned}\frac{Z_n^i(j)}{\rho_{0,n-1}V_{0,n-1}(j)} &= \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} U_{0,n-1}(i) \frac{M_{0,n-1}(i,j)}{\rho_{0,n-1}U_{0,n-1}(i)V_{0,n-1}(j)} \\ &\xrightarrow[n\to+\infty]{\mathbb{P}} W^i U_{0,\infty}(i).\end{aligned}$$

This concludes the proof of Theorem 2.2.11. □

Proof of Corollary 2.2.12. Notice that, for all $1 \leqslant i \leqslant d$ and $n \geqslant 0$,

$$\frac{\|Z_n^i\|}{\|\mathbb{E}_\xi Z_n^i\|} - W^i = \sum_{j=1}^{d} \frac{M_{0,n-1}(i,j)}{\|M_{0,n-1}(i,\cdot)\|} \left(\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} - W^i \right).$$

Then, letting $n \to +\infty$ and using Theorem 2.2.11, we get the first convergence in Corollary 2.2.12. Combining this with (2.2.3), we get the second convergence, and we conclude the proof of Corollary 2.2.12. □

2.10 Proof of Theorem 2.2.13

We need an auxiliary result to prove Theorem 2.2.13.

Lemma 2.10.1. *Let $(X_k)_{k\in\mathbb{N}^*}$ be a sequence of i.i.d. random centered variables. Then for all $n \in \mathbb{N}^*$ and $p > 1$:*

$$\mathbb{E}\left|\sum_{k=1}^{n} X_k\right|^p \leqslant \begin{cases} (B_p)^p \, \mathbb{E}\,|X_k|^p \, n, & \text{if } 1 < p \leqslant 2, \\ (B_p)^p \, \mathbb{E}\,|X_k|^p \, n^{\frac{p}{2}}, & \text{if } p > 2, \end{cases}$$

where $B_p = 2 \min\left\{k^{1/2} : k \in \mathbb{N}, k \geqslant \frac{p}{2}\right\}$.

This result is a direct consequence of the Marcinkiewicz-Zygmund inequality, see [16, Theorem 1.5].

Proof of Theorem 2.2.13. Notice that the condition (2.2.24) implies **A7**. Therefore, using Theorem 2.2.9, we deduce that W^i, $1 \leqslant i \leqslant d$ are non-degenerate, and (2.2.15) and (2.2.16) hold. Now we shall prove the a.s. convergence (2.2.25)-(2.2.29). For that, it is sufficient to show that the convergence in probability in (2.8.9) can be reinforced to a.s. convergence. Indeed, if we prove that for all $1 \leqslant i, r, j \leqslant d$ and $k \geqslant 1$, $\mathbb{P}$-a.s. on the event E^i,

$$\frac{1}{\|Z_n^i\|} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \underset{n\to+\infty}{\longrightarrow} 0, \tag{2.10.1}$$

then all the convergences in probability in the proofs of Theorems 2.2.10, 2.2.11 and Corollary 2.2.12 can be reinforced to a.s. convergences. Now we shall prove (2.10.1), which is equivalent to the following statement: for all $1 \leqslant i, r, j \leqslant d$, $k \geqslant 1$ and $0 \leqslant b < k$, $\mathbb{P}$-a.s. on E^i,

$$\frac{1}{\|Z_{kn+b}^i\|} \sum_{l=1}^{Z_{kn+b}^i(r)} \left(\frac{Z_{l,kn+b,k}^r(j)}{M_{kn+b,k(n+1)+b-1}(r,j)} - 1 \right) \underset{n\to+\infty}{\longrightarrow} 0. \tag{2.10.2}$$

Let $\left(\tilde{\mathcal{F}}_n\right)_{n\geqslant 0}$ be the filtration defined by: $\tilde{\mathcal{F}}_0 = \{\emptyset\}$ and, for $n \geqslant 1$,

$$\tilde{\mathcal{F}}_n = \sigma\left(\xi_s, N_{l,s}^r, 0 \leqslant s \leqslant n-1, 1 \leqslant r \leqslant d, l \geqslant 1\right).$$

Applying the conditional Borel-Cantelli lemma [21, Theorem 5.3.2], we see that (2.10.2) holds if and only if for all $1 \leqslant i, r, j \leqslant d$, $k \geqslant 1$, $0 \leqslant b < k$, and $C > 0$, $\mathbb{P}$-a.s. on the event E^i,

$$\sum_{n=1}^{+\infty} \mathbb{P}\left(\left| \sum_{l=1}^{Z_{kn+b}^i(r)} \left(\frac{Z_{l,kn+b,k}^r(j)}{M_{kn+b,k(n+1)+b-1}(r,j)} - 1 \right) \right| \geqslant C \|Z_{kn+b}^i\| \middle| \tilde{\mathcal{F}}_{kn+b} \right) < +\infty. \tag{2.10.3}$$

We can always assume that condition (2.2.24) holds for some $1 < p \leqslant 2$. Since the environment sequence $(\xi_n)_{n\geqslant 0}$ is i.i.d., $Z_{l,n,k}^r(j)/M_{n,n+k-1}(r,j)$ is independent of $\tilde{\mathcal{F}}_n$ for all $1 \leqslant i, r, j \leqslant d$ and $n, k, l \geqslant 1$. Therefore, using Tchebychev's inequality and Lemma

2.10.1, the series in (2.10.3) can be bounded as follows :

$$\begin{aligned}&\sum_{n=1}^{+\infty}\mathbb{P}\left(\left|\sum_{l=1}^{Z^i_{kn+b}(r)}\left(\frac{Z^r_{l,kn+b,k}(j)}{M_{kn+b,k(n+1)+b-1}(r,j)}-1\right)\right|\geqslant C\|Z^i_{kn+b}\|\,\middle|\,\tilde{\mathcal{F}}_{kn+b}\right)\\ \leqslant\;&\sum_{n=1}^{+\infty}\frac{1}{C^p\|Z^i_{kn+b}\|^p}\mathbb{E}\left(\left|\sum_{l=1}^{Z^i_{kn+b}(r)}\left(\frac{Z^r_{l,kn+b,k}(j)}{M_{kn+b,k(n+1)+b-1}(r,j)}-1\right)\right|^p\,\middle|\,\tilde{\mathcal{F}}_{kn+b}\right)\\ \leqslant\;&\sum_{n=1}^{+\infty}\left(\frac{B_p^p\,Z^i_{kn+b}(r)}{C^p\|Z^i_{kn+b}\|^p}\mathbb{E}\left|\frac{Z^r_k(j)}{M_{0,k-1}(r,j)}-1\right|^p\right).\end{aligned}$$

The last series converges provided that $\mathbb{P}$-a.s. on E^i,

$$\mathbb{E}\left|\frac{Z^r_k(j)}{M_{0,k-1}(r,j)}-1\right|^p\sum_{n=1}^{+\infty}\|Z^i_{kn+b}\|^{1-p}<+\infty. \tag{2.10.4}$$

Therefore (2.10.3) holds if (2.10.4) is satisfied for all $1\leqslant i,r,j\leqslant d$, $k\geqslant 1$ and $0\leqslant b<k$.

It remains to prove (2.10.4), which is done below. By Corollary 2.2.8, we know that for all $1\leqslant i\leqslant d$, $\mathbb{P}$-a.s. on E^i,

$$\lim_{n\to+\infty}\frac{1}{n}\log\|Z^i_n\|=\gamma>0.$$

Therefore we deduce that, $\mathbb{P}$-a.s. on E^i,

$$\sum_{n=1}^{+\infty}\|Z^i_n\|^{1-p}<+\infty. \tag{2.10.5}$$

Now using (2.4.1) and the inequality $(x+y)^p\leqslant 2^{p-1}(x^p+y^p)$, $x,y\geqslant 0$, for all $1\leqslant i,j\leqslant d$

and $k \geqslant 1$, we have

$$\begin{aligned}
&\mathbb{E}\left(\frac{Z^i_{k+1}(j)}{M_{0,k}(i,j)}\right)^p \\
= \ &\mathbb{E}\left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} \sum_{l=1}^{Z_1^i(r)} \left|\frac{Z^r_{l,1,k}(j)}{M_{1,k}(r,j)} - 1 + 1\right|\right)^p \\
\leqslant \ &\mathbb{E}\left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} \sum_{l=1}^{Z_1^i(r)} \left|\frac{Z^r_{l,1,k}(j)}{M_{1,k}(r,j)} - 1\right| + \sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} Z_1^i(r)\right)^p \\
\leqslant \ &2^{p-1}\mathbb{E}\left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} \sum_{l=1}^{Z_1^i(r)} \left|\frac{Z^r_{l,1,k}(j)}{M_{1,k}(r,j)} - 1\right|\right)^p \\
&+ 2^{p-1}\mathbb{E}\left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} Z_1^i(r)\right)^p . \qquad (2.10.6)
\end{aligned}$$

By the convexity of the function $x \mapsto x^p$ on $\mathbb{R}_+$ we get that, for all $1 \leqslant i, j \leqslant d$ and $k \geqslant 1$,

$$\begin{aligned}
\mathbb{E}\left(\sum_{r=1}^d \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} Z_1^i(r)\right)^p &= \mathbb{E}\left(\sum_{r=1}^d \frac{M_0(i,r)M_{1,k}(r,j)}{M_{0,k}(i,j)} \frac{Z_1^i(r)}{M_0(i,r)}\right)^p \\
&\leqslant \mathbb{E}\left(\sum_{r=1}^d \frac{M_0(i,r)M_{1,k}(r,j)}{M_{0,k}(i,j)} \left(\frac{Z_1^i(r)}{M_0(i,r)}\right)^p\right) \\
&\leqslant \sum_{r=1}^d \mathbb{E}\left(\frac{Z_1^i(r)}{M_0(r,j)}\right)^p . \qquad (2.10.7)
\end{aligned}$$

Using again the convexity of $x \mapsto x^p$ on $\mathbb{R}_+$, together with Lemma 2.10.1, we obtain that for all $1 \leqslant i, j \leqslant d$ and $k \geqslant 1$,

$$\begin{aligned}
&\mathbb{E}\left(\sum_{r=1}^{d} \frac{M_{1,k}(r,j)}{M_{0,k}(i,j)} \sum_{l=1}^{Z_1^i(r)} \left|\frac{Z_{l,1,k}^r(j)}{M_{1,k}(r,j)} - 1\right|\right)^p \\
= \;& \mathbb{E}\left(\sum_{r=1}^{d} \frac{M_0(i,r)M_{1,k}(r,j)}{M_{0,k}(i,j)} \frac{1}{M_0(i,r)} \sum_{l=1}^{Z_1^i(r)} \left|\frac{Z_{l,1,k}^r(j)}{M_{1,k}(r,j)} - 1\right|\right)^p \\
\leqslant \;& \mathbb{E}\left(\sum_{r=1}^{d} \frac{M_0(i,r)M_{1,k}(r,j)}{M_{0,k}(i,j)M_0(i,r)^p} \mathbb{E}_\xi\left[\left(\sum_{l=1}^{Z_1^i(r)} \left|\frac{Z_{l,1,k}^r(j)}{M_{1,k}(r,j)} - 1\right|\right)^p \middle| Z_1^i(r)\right]\right) \\
\leqslant \;& B_p^p \mathbb{E}\left(\sum_{r=1}^{d} \frac{Z_1^i(r)}{M_0(i,r)^p} \mathbb{E}_\xi \left|\frac{Z_k^r(j)}{M_{0,k-1}(r,j)} - 1\right|^p\right) \\
= \;& B_p^p \sum_{r=1}^{d} \mathbb{E} M_0(i,r)^{1-p} \mathbb{E}\left|\frac{Z_k^r(j)}{M_{0,k-1}(r,j)} - 1\right|^p. \qquad (2.10.8)
\end{aligned}$$

Combining the relations (2.10.6)-(2.10.8), we obtain that for all $1 \leqslant i, j \leqslant d$ and $k \geqslant 1$,

$$\begin{aligned}
\mathbb{E}\left(\frac{Z_{k+1}^i(j)}{M_{0,k}(i,j)}\right)^p \leqslant\; & 2^{p-1} B_p^p \sum_{r=1}^{d} \mathbb{E} M_0(i,r)^{1-p} \mathbb{E}\left|\frac{Z_k^r(j)}{M_{0,k-1}(r,j)} - 1\right|^p \\
&+ 2^{p-1} \sum_{r=1}^{d} \mathbb{E}\left(\frac{Z_1^i(r)}{M_0(r,j)}\right)^p. \qquad (2.10.9)
\end{aligned}$$

By (2.7.1), for all $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-a.s., it holds

$$M_0(i,j) \geqslant \frac{1}{dD} \|M_0(\cdot, j)\| \geqslant \frac{1}{dD^2} \|M_0\|.$$

Combining this with (2.10.9), we get that for all $1 \leqslant j \leqslant d$ and $k \geqslant 0$,

$$\begin{aligned}
\mathbb{E}\left(\frac{Z_{k+1}^i(j)}{M_{0,k}(i,j)}\right)^p \leqslant\; & 2^{p-1} B_p^p (dD^2)^{p-1} \mathbb{E}\|M_0\|^{1-p} \sum_{r=1}^{d} \mathbb{E}\left|\frac{Z_k^r(j)}{M_{0,k-1}(r,j)} - 1\right|^p \\
&+ 2^{p-1} \sum_{r=1}^{d} \mathbb{E}\left(\frac{Z_1^i(r)}{M_0(r,j)}\right)^p.
\end{aligned}$$

Using the condition (2.2.24), by induction on k we conclude that for all $1 \leqslant i, j \leqslant d$ and $k \geqslant 1$,

$$\mathbb{E}\left(\frac{Z_k^i(j)}{M_{0,k-1}(i,j)}\right)^p < +\infty. \qquad (2.10.10)$$

Putting together (2.10.5) and (2.10.10), we obtain (2.10.4), which implies (2.10.3), and ends the proof of Theorem 2.2.13. □

2.11 Appendix

In this section we prove the several implications among the conditions **A3**-**A7**.

Lemma 2.11.1. *Assume conditions **A1** and **A2**. Then the following implications hold :*

$$\textbf{A7} \Rightarrow \textbf{A5} \Rightarrow \textbf{A3}, \quad \text{and} \quad \textbf{A7} \Rightarrow \textbf{A6} \Rightarrow \textbf{A3}.$$

*If additionally condition **A4** holds, then we have the equivalences*

$$\textbf{A5} \Leftrightarrow \textbf{A3} \quad \text{and} \quad \textbf{A7} \Leftrightarrow \textbf{A6}.$$

*Moreover, when the environment (ξ_n) is i.i.d. and **A4** holds, then*

$$\textbf{A3} \Leftrightarrow \textbf{A5} \Leftrightarrow \textbf{A6} \Leftrightarrow \textbf{A7}.$$

Proof of Lemma 2.11.1. We first prove that **A5** $\Rightarrow$ **A3**. For all $C > 1$, $K > 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s., we have

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\{\langle N_n^i, U_{n+1,\infty}\rangle \geqslant C^n\}} \right) \\
= \; & \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\{\langle N_n^i, U_{n+1,\infty}\rangle \geqslant C^n\}} \mathbb{1}_{\{\lambda_n U_{n,\infty}(i) < K^n\}} \right) \\
& + \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\{\langle N_n^i, U_{n+1,\infty}\rangle \geqslant C^n\}} \mathbb{1}_{\{\lambda_n U_{n,\infty}(i) \geqslant K^n\}} \right) \\
\leqslant \; & \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\left\{ \frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \geqslant (CK)^n \right\}} \right) \\
& + \sum_{n=0}^{+\infty} \mathbb{1}_{\{\lambda_n U_{n,\infty}(i) \geqslant K^n\}}.
\end{aligned} \tag{2.11.1}$$

First, by **A2**, for all $K>1$ and $1\leqslant i\leqslant d$, it holds that

$$\begin{aligned}\mathbb{E}\Big[\sum_{n=0}^{+\infty}\mathbb{1}_{\{\lambda_n U_{n,\infty}(i)\geqslant K^n\}}\Big] &= \mathbb{E}\Big[\sum_{n=0}^{+\infty}\mathbb{1}_{\{\log^+(\lambda_0 U_{0,\infty}(i))\geqslant n\log K\}}\Big]\\ &\leqslant \mathbb{E}\Big[\frac{\log^+(\lambda_0 U_{0,\infty}(i))}{\log(K)}+1\Big]\\ &\leqslant \frac{\mathbb{E}\log^+\|M_0\|}{\log(K)}+1<+\infty. \qquad (2.11.2)\end{aligned}$$

By (2.2.5), we have $\sum_{j=1}^{d}\frac{M_n(i,j)U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(i)}=1$, so that the summands are bounded by 1. Therefore, for all $C>1$ and $1\leqslant i\leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}&\sum_{n=0}^{+\infty}\mathbb{E}_\xi\left(\frac{\langle N_n^i,U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)}\mathbb{1}_{\left\{\frac{\langle N_n^i,U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)}\geqslant C^n\right\}}\right)\\ =\;&\sum_{n=0}^{+\infty}\mathbb{E}_\xi\Bigg(\sum_{j=1}^{d}\frac{M_n(i,j)U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(i)}\frac{N_n^i(j)}{M_n(i,j)}\times\\ &\mathbb{1}_{\left\{\sum_{r=1}^{d}\frac{M_n(i,r)U_{n+1,\infty}(r)}{\lambda_n U_{n,\infty}(r)}\frac{N_n^i(r)}{M_n(i,r)}\geqslant C^n\right\}}\Bigg)\\ \leqslant\;&\sum_{n=0}^{+\infty}\sum_{j=1}^{d}\mathbb{E}_\xi\Big(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\max_{1\leqslant r\leqslant d}\left(\frac{N_n^i(r)}{M_n(i,r)}\right)\geqslant C^n\right\}}\Big)\\ \leqslant\;&\sum_{n=0}^{+\infty}\sum_{j=1}^{d}\sum_{r=1}^{d}\mathbb{E}_\xi\Big(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\frac{N_n^i(r)}{M_n(i,r)}\geqslant C^n\right\}}\Big). \qquad (2.11.3)\end{aligned}$$

By a symmetry argument, for all $C>1$ and $1\leqslant j,r\leqslant d$, $\mathbb{P}$-a.s., we have

$$\begin{aligned}&\mathbb{E}_\xi\Big(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\frac{N_n^i(r)}{M_n(i,r)}\geqslant C^n\right\}}\Big)\\ =\;&\mathbb{E}_\xi\Big(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\frac{N_n^i(r)}{M_n(i,r)}\geqslant C^n\right\}}\mathbb{1}_{\left\{\frac{N_n^i(j)}{M_n(i,j)}\geqslant\frac{N_n^i(r)}{M_n(i,r)}\right\}}\Big)\\ &+\mathbb{E}_\xi\Big(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\frac{N_n^i(r)}{M_n(i,r)}\geqslant C^n\right\}}\mathbb{1}_{\left\{\frac{N_n^i(j)}{M_n(i,j)}<\frac{N_n^i(r)}{M_n(i,r)}\right\}}\Big)\\ \leqslant\;&\mathbb{E}_\xi\Big(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\frac{N_n^i(j)}{M_n(i,j)}\geqslant C^n\right\}}\Big)+\mathbb{E}_\xi\Big(\frac{N_n^i(r)}{M_n(i,r)}\mathbb{1}_{\left\{\frac{N_n^i(r)}{M_n(i,r)}\geqslant C^n\right\}}\Big). \qquad (2.11.4)\end{aligned}$$

Combining the inequalities (2.11.1)-(2.11.4), this shows that **A5** $\Rightarrow$ **A3**.

We next prove that **A7**$\Rightarrow$**A5** and **A6**$\Rightarrow$**A3**. Since the sequence of the environments

(ξ_n) is stationary, for all $C > 1$ and $1 \leqslant i, j \leqslant d$ we have

$$\begin{aligned}&\mathbb{E}\left(\sum_{n=0}^{+\infty}\mathbb{E}_\xi\left(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\frac{N_n^i(j)}{M_n(i,j)}\geqslant C^n\right\}}\right)\right)\\ &= \sum_{n=0}^{+\infty}\mathbb{E}\Big(\frac{Z_1^i(j)}{M_0(i,j)}\mathbb{1}_{\left\{\log^+\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)\geqslant n\log C\right\}}\Big).\end{aligned}$$

Therefore, we deduce that

$$\frac{B(i,j)}{\log C} \leqslant \mathbb{E}\left(\sum_{n=0}^{+\infty}\mathbb{E}_\xi\left(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\frac{N_n^i(j)}{M_n(i,j)}\geqslant C^n\right\}}\right)\right) \leqslant \frac{B(i,j)}{\log C}+1, \tag{2.11.5}$$

with

$$B(i,j) := \mathbb{E}\Big(\frac{Z_1^i(j)}{M_0(i,j)}\log^+\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)\Big).$$

The implication **A7**⇒**A5** follows. The implication **A6**⇒**A3** can be obtained by a similar argument.

We now prove that **A7**⇒**A6**. By the convexity of the function $x \mapsto x\log^+ x$ on $\mathbb{R}^+$ we obtain that for all $1 \leqslant i \leqslant d$,

$$\begin{aligned}&\mathbb{E}\left(\frac{\langle Z_1^i, U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(i)}\log^+\langle Z_1^i, U_{1,\infty}\rangle\right)\\ \leqslant\ &\mathbb{E}\left(\frac{\langle Z_1^i, U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(i)}\log^+\left(\frac{\langle Z_1^i, U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(i)}\right)\right)+\mathbb{E}\log^+(\lambda_0 U_{0,\infty}(i))\\ \leqslant\ &\mathbb{E}\left(\sum_{j=1}^d\frac{M_0(i,j)U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)}\frac{Z_1^i(j)}{M_0(i,j)}\log^+\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)\right)+\mathbb{E}\log^+\|M_0\|\\ \leqslant\ &\sum_{j=1}^d\mathbb{E}\Big(\frac{Z_1^i(j)}{M_0(i,j)}\log^+\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)\Big)+\mathbb{E}\log^+\|M_0\|.\end{aligned}$$

Using **A2**, this proves that **A7**⇒**A6**.

From now assume additionally the Furstenberg-Kesten condition **A4**. Then **A1** holds, so from the conclusions above we see that **A5**⇒**A3** and **A7**⇒**A6**. We will prove below the inverse implications **A3**⇒**A5** and **A7**⇒**A6**.

We first prove that **A3**⇒**A5**. For all $1 < K < C$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s., we have

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\left\{ \frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \geqslant C^n \right\}} \right) \\
= &\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\left\{ \frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \geqslant C^n \right\}} \mathbb{1}_{\{\langle N_n^i, U_{n+1,\infty}\rangle > K^n\}} \right) \\
&+ \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\left\{ \frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \geqslant C^n \right\}} \mathbb{1}_{\{\langle N_n^i, U_{n+1,\infty}\rangle \leqslant K^n\}} \right) \\
\leqslant &\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\{\langle N_n^i, U_{n+1,\infty}\rangle \geqslant (CK)^n\}} \right) \\
&+ \sum_{n=0}^{+\infty} \mathbb{1}_{\left\{ \lambda_n U_{n,\infty}(i) \leqslant (\frac{K}{C})^n \right\}}.
\end{aligned} \tag{2.11.6}$$

Notice that, by (2.7.3),

$$\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} = \sum_{j=1}^d \frac{M_n(i,j) U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(i)} \frac{N_n^i(j)}{M_n(i,j)} \geqslant \frac{1}{dD^2} \sum_{j=1}^d \frac{N_n^i(j)}{M_n(i,j)}. \tag{2.11.7}$$

Therefore we get that for all $C > 0$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \mathbb{1}_{\left\{ \frac{\langle N_n^i, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(i)} \geqslant C^n \right\}} \right) \\
&\qquad \geqslant \frac{1}{dD^2} \sum_{j=1}^d \sum_{n=0}^{+\infty} \mathbb{E}_\xi \left(\frac{N_n^i(j)}{M_n(i,j)} \mathbb{1}_{\left\{ \frac{N_n^i(j)}{M_n(i,j)} \geqslant dD^2 C^n \right\}} \right).
\end{aligned} \tag{2.11.8}$$

Moreover, for $1 < K < C$ and $1 \leqslant i \leqslant d$, it holds that

$$\begin{aligned}
\mathbb{E}\Big[\sum_{n=0}^{+\infty} \mathbb{1}_{\left\{ \lambda_n U_{n,\infty}(i) \leqslant (\frac{K}{C})^n \right\}} \Big] &= \mathbb{E}\Big[\sum_{n=0}^{+\infty} \mathbb{1}_{\left\{ \log(\lambda_0 U_{0,\infty}(i)) \leqslant n \log(\frac{K}{C}) \right\}} \Big] \\
&\leqslant \mathbb{E}\Big[\sum_{n=0}^{+\infty} \mathbb{1}_{\left\{ |\log(\lambda_0 U_{0,\infty}(i))| \geqslant n \log(\frac{C}{K}) \right\}} \Big] \\
&\leqslant \mathbb{E}\Big[\frac{|\log(\lambda_0 U_{0,\infty}(i))|}{\log(C/K)} + 1 \Big].
\end{aligned} \tag{2.11.9}$$

By (2.7.2) and Proposition (2.2.5), we get that

$$\begin{aligned}\mathbb{E}|\log(\lambda_0 U_{0,\infty}(i))| &\leqslant \mathbb{E}|\log\lambda_0| + \mathbb{E}|\log U_{0,\infty}(i)| \\ &\leqslant \mathbb{E}|\log\lambda_0| + \log(dD) < +\infty. \end{aligned} \tag{2.11.10}$$

From (2.11.9) and (2.11.10) we deduce that for $1 < K < C$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}\Big[\sum_{n=0}^{+\infty} \mathbb{1}_{\left\{\lambda_n U_{n,\infty}(i) \leqslant (\frac{K}{C})^n\right\}}\Big] \leqslant \frac{1}{\log(C/K)}\mathbb{E}|\log(\lambda_0 U_{0,\infty}(i))| + 1 < +\infty. \tag{2.11.11}$$

Combining the inequalities (2.11.6)-(2.11.11), we obtain the implication **A3**$\Rightarrow$**A5**.

It remains to prove that **A7**$\Rightarrow$**A6**. By (2.7.3), for all $1 \leqslant i, j \leqslant d$ we have

$$\begin{aligned}
&\mathbb{E}\Big(\frac{Z_1^i(j)}{M_0(i,j)}\log^+\Big(\frac{Z_1^i(j)}{M_0(i,j)}\Big)\Big) \\
\leqslant\ & \mathbb{E}\Big(dD^2\frac{U_{1,\infty}(j)Z_1^i(j)}{\lambda U_{0,\infty}(i)}\log^+\Big(dD^2\frac{U_{1,\infty}(j)Z_1^i(j)}{\lambda U_{0,\infty}(i)}\Big)\Big) \\
\leqslant\ & \mathbb{E}\left(dD^2\frac{\langle Z_1^i, U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(i)}\log^+\left(dD^2\frac{\langle Z_1^i, U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(i)}\right)\right) \\
\leqslant\ & dD^2\left(\mathbb{E}\left(\frac{\langle Z_1^i, U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(i)}\log^+\langle Z_1^i, U_{1,\infty}\rangle\right) + \mathbb{E}\log^+\left(\frac{dD^2}{\lambda_0 U_{0,\infty}(i)}\right)\right) \\
\leqslant\ & dD^2\left(\mathbb{E}\left(\frac{\langle Z_1^i, U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(i)}\log^+\langle Z_1^i, U_{1,\infty}\rangle\right) + \mathbb{E}\left|\log\left(\frac{\lambda_0 U_{0,\infty}(i)}{dD^2}\right)\right|\right).
\end{aligned}$$

This, together with (2.11.10), proves that **A7**$\Rightarrow$**A6**.

Finally, in addition to the condition **A4**, we suppose that the environment ξ is i.i.d. Using the implications proved above, to show that all the conditions **A3**-**A7** are equivalent, it suffices to prove that **A5**$\Leftrightarrow$**A7**. Let us prove this below. Since (ξ_n) is i.i.d., for all $C > 0$ and $1 \leqslant i, j \leqslant d$ the random variales

$$\mathbb{E}_\xi\left(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\frac{N_n^i(j)}{M_n(i,j)} \geqslant C^n\right\}}\right), \quad n \geqslant 0,$$

are independent, and bounded by 1. By the Kolmogorov's Three Series Theorem, we deduce that the condition **A5** holds if and only if for all $C > 0$ and $1 \leqslant i, j \leqslant d$,

$$\mathbb{E}\left(\sum_{n=0}^{+\infty}\mathbb{E}_\xi\left(\frac{N_n^i(j)}{M_n(i,j)}\mathbb{1}_{\left\{\frac{N_n^i(j)}{M_n(i,j)} \geqslant C^n\right\}}\right)\right) < +\infty.$$

Combining this with (2.11.5), it follows that **A5**⇔**A7**. This completes the proof of Lemma 2.11.1. □

Chapter 3

Convergence in L^p for a supercritical multi-type branching process in a random environment

Résumé. On considère un processus de branchement d-type surcritique $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d))$, $n \geqslant 0$, dans un environnement aléatoire indépendant et identiquement distribué $\xi = (\xi_0, \xi_1, \ldots)$, commençant avec une particule initiale de type i. Dans un précédent article on a établi un théorème de type Kesten-Stigum pour Z_n^i, qui implique que pour tout $1 \leqslant i, j \leqslant d$, $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j) \to W^i$ en probabilité quand $n \to +\infty$, où $\mathbb{E}_\xi Z_n^i(j)$ est l'espérance conditionnelle de $Z_n^i(j)$ sachant l'environnement ξ, et W^i est une variable alétoire postive et finie. Le but de cet article est d'obtenir une condition nécessaire et suffisante pour la convergence dans L^p de $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, et de montrer que la vitesse de convergence est exponentielle. Pour cela, on établit tout d'abord les résultats correspondant pour la martingale fondamentale (W_n^i) associée au processus de branchement (Z_n^i).

Abstract. Consider a d-type supercritical branching process $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d))$, $n \geqslant 0$, in an independent and identically distributed random environment $\xi = (\xi_0, \xi_1, \ldots)$, starting with one initial particle of type i. In a previous paper we have established a Kesten-Stigum type theorem for Z_n^i, which implies that for any $1 \leqslant i, j \leqslant d$, $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j) \to W^i$ in probability as $n \to +\infty$, where $\mathbb{E}_\xi Z_n^i(j)$ is the conditional expectation of $Z_n^i(j)$ given the environment ξ, and W^i is a non-negative and finite random variable. The goal of this paper is to obtain a necessary and sufficient condition for the convergence in L^p of $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, and to prove that the convergence rate is exponential. To this end, we first establish the corresponding results for the fundamental martingale (W_n^i) associated to the branching process (Z_n^i).

3.1 Introduction

A significant advancement in the theory of branching processes was made with the introduction of a random environment such that the offspring distribution of generation n depends on some random environment ξ_n at time n, in contrast to a constant distribution assumed in the Galton-Watson process. This allows a more adequate modeling, and turned out to be very fruitful in theoretical as well as in practical senses. For the first fundamental results on branching processes in random environments, see Athreya and Karlin [5, 6]. The importance of the study of branching processes in random environments is mainly due to its wide application background, both in theory and in practical problems. For example, Kesten, Kozlov and Spitzer [47] used such a process to study limit properties of for random walks in random environments; biologists are currently paying special attention to the problems of genetic transformation, and such problems can be studied via a multi-type branching process in a random environment; see Bansaye [8] for application in cell contamination. Due to huge applications and important technique challenge, in recent years, there is a great progress in the study of branching processes in random environments, see e.g. the recent papers [57, 70, 74], the recent book by Kersting and Vatutin [50] and many references therein. In an earlier work [32], for a supercritical multi-type branching process in an independent and identically distributed random environment, we have studied the convergence of the normalized population size and the non-degeneracy of its limit. In this paper, we will consider its convergence in L^p.

Let $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, be a d-type branching process in an independent and identically distributed (i.i.d.) random environment $\xi = (\xi_0, \xi_1, \cdots)$. For $n \geqslant 0$, denote by M_n the matrix of the conditioned means of the offspring distribution of n-th generation given the environment: the (i,j)-th entry of M_n is

$$M_n(i,j) = \mathbb{E}_\xi[Z_{n+1}(j) \mid Z_n = e_i],$$

where $\mathbb{E}_\xi$ denotes the conditional expectation given the environment ξ. Let $M_{0,n} = M_0 \cdots M_n$ be the product matrix. The process $(Z_n)_{n\geqslant 0}$ will be denoted $(Z_n^i)_{n\geqslant 0}$ when it starts with one initial particle of type i, that is when $Z_0 = e_i$, where e_i is the unit vector whose i-th component is 1 and all the others are 0. In [32] we obtained an extension of the famous Kesten-Stigum result on the Galton-Watson process to the multi-type branching process in random environment (MBPRE). Assume that the MBPRE $(Z_n^i)_{n\geqslant 0}$ is in the supercritical regime, in the sense that

$$\gamma := \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| > 0,$$

where $\|M_{0,n-1}\|$ is the L_1-norm of the matrix $M_{0,n-1}$. Under the Furstengerg-Kesten condition **M1** (see Section 3.2), we proved in [32, Theorem 2.11] that for all $1 \leqslant i, j \leqslant d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \to W^i \quad \text{in probability,} \tag{3.1.1}$$

where W^i is a non-negative random variable independent of j; moreover, W^i is non-degenerate for all i if and only if

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty \quad \forall i, j = 1, \cdots, d; \tag{3.1.2}$$

in addition $\mathbb{E}_\xi W^i = 1$ almost surely (a.s.) when (3.1.2) holds. By Sheffé's theorem, it follows that $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \to W^i$ in L^1 if and only if (3.1.2) holds.

The main objective in this paper is to find a necessary and sufficient condition under which the normalized population size $Z_n^i(j)/M_{0,n-1}(i,j)$ converges to W^i in L^p, $p > 1$, and to prove that the convergence rate is exponential, for all $1 \leqslant i, j \leqslant d$. In the single type case, it is known that such kind of results play an important role in the study of asymptotic properties of large deviations and Berry Esseen bounds in the central limit theorem on the process (Z_n^i), see [43, 31]. The situation is the same in the multi-type case, as can be seen in the preprints [34, 35].

For a single type branching process in a random environment $(Z_n)_{n\geqslant 0}$, Guivarc'h and Liu [38, Theorem 1.3] established the (annealed) L^p convergence criterion: they showed that when $d = 1$, for each given $p > 1$, $(Z_n/m_{0,n-1})_{n\geqslant 0}$ converges in L^p to a non negative random variable W if and only if

$$\mathbb{E}\left(\frac{Z_1}{m_0}\right)^p < +\infty \quad \text{and} \quad \mathbb{E} m_0^{1-p} < 1, \tag{3.1.3}$$

where $m_{0,n-1} = m_0 \cdots m_{n-1}$, and m_k denotes the conditioned mean of the offspring distribution at time k given the environment. Huang and Liu [44, Theorem 1.5] proved that the L^p convergence rate is exponential: if (3.1.3) holds, then

$$\lim_{n\to+\infty} \delta^{-n}\left(\mathbb{E}\left|\frac{Z_n}{m_{0,n-1}} - W\right|^p\right)^{1/p} = 0 \quad \forall \delta > \delta_c(p), \tag{3.1.4}$$

with

$$\delta_c(p) = \begin{cases} (\mathbb{E}m_0^{1-p})^{1/p} & \text{if } p \in (1,2), \\ \max\left\{(\mathbb{E}m_0^{1-p})^{1/p}, (\mathbb{E}m_0^{-p/2})^{1/p}\right\} & \text{if } p \geqslant 2. \end{cases} \tag{3.1.5}$$

Afanasyev [2] also gave sufficient conditions for the L^p-convergence of $(Z_n/m_{0,n-1})$, and estimated the rate of convergence.

For the MBPRE's case, the only result about the annealed L^p convergence is a claim by Cohn [17] which concerns the L^2 convergence. Assume that the supercritical condition $\gamma > 0$ holds, that each entry of M_0 is bounded a.s. from below and above by two positive constants, and that all the conditional second moments of the offspring distributions given the environment are bounded a.s. by a constant. Assume also the integrability condition $\mathbb{E}|\log\sum_{i=1}^d(1-\mathbb{P}(\|Z_1^i\|=0))| < \infty$. Under these conditions Cohn [17] claimed that for each $j = 1, \cdots, d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \to W^i \quad \text{ in } L^2 \tag{3.1.6}$$

under the annealed law $\mathbb{P}$, where W^i is a non degenerate random variable satisfying $\mathbb{E}W^i = 1$. However, the claim of Cohn [17] is false. To see this, it suffices to notice that when $d = 1$, (3.1.6) holds if and only if $\mathbb{E}\left(\frac{Z_1}{m_0}\right)^2 < +\infty$ and $\mathbb{E}m_0^{-1} < 1$ by the criterion (3.1.3) of Guivarc'h and Liu [38, Theorem 1.3]. A quantitative condition (which ensures $\mathbb{E}m_0^{-1} < 1$ for $d = 1$) is missing in the claim of Cohn [17]. This shows that the annealed L^p convergence is rather delicate even for $p = 2$. We mention that Jones [45], Biggins, Cohn and Nerman [12] have studied respectively the L^2 and L^p convergence of multi-type branching processes in varying environment. Their results give sufficient conditions for quenched L^p convergence for multi-type branching processes in random environments. In this paper, we deal with the annealed L^p convergence, which is in general more delicate because there is an additional integral operation. Since we will always dear with the annealed L^p convergence, for simplicity we will just say L^p convergence in the following.

More precisely, we will find a necessary and sufficient condition for the L^p convergence which extends the criterion (3.1.3) to the multi-type case, and establish the exponential convergence rate. Let $p > 1$ be such that $\mathbb{E}M_0(i,j)^{1-p} < +\infty$ for all $1 \leqslant i,j \leqslant d$, and define

$$\kappa(1-p) = \lim_{n\to+\infty} \Big(\mathbb{E}\|M_{0,n-1}\|^{1-p}\Big)^{1/n}.$$

It will be shown that the limit exists and is finite. Under the Furstenberg-Kesten condition **M1**, we will prove that $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \to W^i$ in L^p for any $1 \leqslant i,j \leqslant d$ if and only if

$$\max_{1\leqslant i,j\leqslant d} \mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)^p < +\infty \quad \text{and} \quad \kappa(1-p) < 1. \tag{3.1.7}$$

(cf. Theorem 3.2.1); moreover, if (3.1.7) holds, then there exists $\delta \in (0,1)$ such that

$$\lim_{n\to+\infty} \delta^{-n} \left(\mathbb{E}\left|\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} - W^i\right|^p\right)^{1/p} = 0 \tag{3.1.8}$$

(cf. Theorem 3.2.2). For a single type branching process in random environment, we have $\kappa(1-p) = \mathbb{E}m_0^{1-p}$, so (3.1.7) coincides with (3.1.3), and (3.1.8) corresponds to (3.1.4) but with less information on the exact exponential rate.

The proof of (3.1.7) and (3.1.8) is based on the corresponding results for the associated fundamental martingale (W_n^i) introduced in [32]. Let us recall briefly its construction. For any $n,k \geqslant 0$, let $\rho_{n,n+k}$ be the spectral radius of $M_{n,n+k}$. Applying the famous Perron-Frobenius theorem (see e.g. [7]), $\rho_{n,n+k}$ is a positive eigenvalue of $M_{n,n+k}$, for which there exist positive right and left eigenvectors $U_{n,n+k}$ and $V_{n,n+k}$ with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k}\rangle = 1$, where $\|x\|$ denotes the L^1 norm of the vector x, and $\langle x,y\rangle$ the scalar product of the vectors x,y. Then, under certain conditions, by the results of Hennion [40, Lemma 3.3 and Theorem 1] the limit

$$U_{n,\infty} := \lim_{k\to\infty} U_{n,n+k} \tag{3.1.9}$$

exists a.s., with $U_{n,\infty} > 0$ a.s. and $\|U_{n,\infty}\| = 1$; moreover, there exist random scalars $\lambda_n > 0$ a.s. called the pseudo-spectral radii of the random matrices (M_n), which satisfy a.s. the relation

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty}. \tag{3.1.10}$$

The relation (3.1.10) can be iterated to obtain

$$M_{n,n+k}U_{n+k+1,\infty} = \lambda_{n,n+k}U_{n,\infty}, \tag{3.1.11}$$

where $\lambda_{n,n+k} = \prod_{r=n}^{n+k} \lambda_r$ for $n, k \geqslant 0$. Then, the non-negative martingale (W_n^i) is defined as follows [32] :

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, U_{n,\infty}\rangle}{\lambda_{0,n-1}U_{0,\infty}(i)}, \quad n \geqslant 1. \tag{3.1.12}$$

Assume for simplicity that the Furstenberg-Kesten condition **M1** is satisfied. Assume also that $p > 1$ is such that $\mathbb{E}M_0(i,j)^{1-p} < +\infty$ for all $1 \leqslant i, j \leqslant d$. Then we show that W_n^i converges in L^p to the random variable W^i for any $1 \leqslant i \leqslant d$ if and only if (3.1.7) holds (cf. Theorem 3.2.3); moreover, if (3.1.7) is satisfied, then

$$\lim_{n\to+\infty} \delta^{-n}\Big(\mathbb{E}|W_n^i - W^i|^p\Big)^{1/p} = 0 \quad \forall \delta > \delta_c(p), \tag{3.1.13}$$

with

$$\delta_c(p) = \begin{cases} \kappa(1-p)^{1/p} & \text{if } p \in (1,2), \\ \max\Big\{\kappa(1-p)^{1/p}, \kappa(-p/2)^{1/p}\Big\} & \text{if } p \geqslant 2 \end{cases} \tag{3.1.14}$$

(cf. Theorem 3.2.4). In the case of the single type branching process, the martingale (W_n) coincides with the normalized population size $(Z_n/m_{0,n-1})$, so the relations (3.1.13) and (3.1.14) coincide exactly with (3.1.4) and (3.1.5). It is known that when $d = 1$, the critical value $\delta_c(p)$ is the best possible for (3.1.13) to hold (see Huang and Liu [44]).

For the proof, we develop the approach in [44] where the case $d = 1$ was considered. In addition to the complexity related to the products of random matrices, the main difficulty for the multi-dimensional case resides in the fact that W_n^i depends on the whole environment sequence $\xi = (\xi_0, \xi_1, \cdots)$, not just on the environment sequence until the present $(\xi_0, \cdots, \xi_{n-1})$, contrary to the one-dimensional case. Let us give a short description of the approach. For $p \in (1,2]$, we first control the quenched L^p norm of the martingale difference $W_{n+1}^i - W_n^i$, using the branching property and the Marcinkiewicz-Zygmund inequality on the L^p norm of sums of independent random variables. This permits us to obtain a bound of $\mathbb{E}_\xi|W_{n+1}^i - W_n^i|^p$ in terms of $(\lambda_{0,n-1}U_{0,\infty}(i))^{1-p}$. To overcome the difficulty related to the dependence on the whole environment sequence, we condition on the future $T^n\xi = (\xi_n, \xi_{n+1}, \cdots)$ to obtain $\mathbb{E}_{T^n\xi}(\lambda_{0,n-1}U_{0,\infty}(i))^{1-p} \leqslant C\kappa(1-p)^n$, which gives the correct convergence rate in L^p for the martingale (W_n^i). For $p > 2$, we use an argument by induction. To get the convergence rate of the normalized population size $Z_n^i(j)/M_{0,n-1}(i,j)$, we prove that the difference $Z_n^i(j)/M_{0,n-1}(i,j) - W_n^i$ goes to 0 in L^p exponentially fast, using the exponential convergence of the products of stochastic matrices due to Seneta [64]. For the necessity, we first establish some spectral properties of the important transfer operator P_s for $s \leqslant 0$ (see Section 3.3).

The main results will be presented in Section 3.2. In Section 3.3 we establish the spectral properties of the transfer operator P_s that we will need. In Section 3.4 we prove the criterion for the convergence in L^p of the martingales (W_n^i), as well as their exponential convergence rate. Similar results for the normalized population size $Z_n^i(j)/M_{0,n-1}(i,j)$ are proved in Section 3.5.

3.2 Notation and main results

Let $\mathbb{N} = \{0, 1, \ldots\}$ be the set of non-negative integers. The indicator of an event A is denoted by $\mathbb{1}_A$. The symbol $\xrightarrow{\mathbb{P}}$ denotes the convergence in probability with respect to the annealed law $\mathbb{P}$. For an integer $d \geqslant 1$, let $\mathbb{R}^d$ be the d-dimensional space of vectors with real coordinates, equipped with the scalar product and the L^1-norm respectively defined by

$$\langle x, y\rangle := \sum_{i=1}^{d} x(i)\, y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^{d} |x(i)|, \quad x, y \in \mathbb{R}^d.$$

Let e_i be the d-dimensional vector with 1 in the i-th place and 0 elsewhere. Define also $\mathcal{M}_d(\mathbb{R})$ the set of $d \times d$ matrices with entries in $\mathbb{R}$, and the operator norm on $\mathcal{M}_d(\mathbb{R})$:

$$\|M\| := \sup_{\|x\|=1} \|Mx\|, \quad M \in \mathcal{M}_d(\mathbb{R}).$$

For a matrix or a vector X, we write $X > 0$ to mean that each entry of X is strictly positive.

Let us define precisely the multi-type branching process in random environment (MBPRE). Let $\xi = (\xi_n)_{n\geqslant 0}$ be the random environment, which is an independent and identically distributed (i.i.d.) sequence with values in an abstract space $\mathbb{X}$. To each realization of ξ_n, we associate d probability generating functions : for $1 \leqslant r \leqslant d$,

$$f_n^r(s) = \sum_{k_1,\cdots,k_d=0}^{\infty} p^r_{k_1,\cdots,k_d}(\xi_n) s_1^{k_1} \cdots s_d^{k_d}, \quad s = (s_1, \cdots, s_d) \in [0,1]^d.$$

A MBPRE (Z_n) in the random environment ξ is a process with values in $\mathbb{N}^d$ such that for all $n \geqslant 0$,

$$Z_{n+1} = \sum_{r=1}^{d} \sum_{l=1}^{Z_n(r)} N^r_{l,n}, \tag{3.2.1}$$

where $Z_0 \in \mathbb{N}^d$ is fixed, $Z_n(j)$ represents the number of particles of type j of some population in generation n, and $N^r_{l,n}(j)$ is the offspring of type j at time $n+1$ of the l-th particle of type r in generation n. The random vectors $N^r_{l,n} = (N^r_{l,n}(1), \cdots, N^r_{l,n}(d))$, indexed by $l \geqslant 1$, $n \geqslant 0$, $r \in \{1, \cdots, d\}$, are conditionally independent and have the same probability generating function f_n^r, given the environment ξ. Set $f_n := (f_n^1, \cdots, f_n^d)$. When the process starts with one initial particle of type i, that is, when $Z_0 = e_i$, we write Z_n^i instead of Z_n.

Denote by $\mathbb{P}_\xi$ the underline probability when the environment ξ is given; it is called quenched law. Let τ be the law of the environment ξ. Then, the total probability $\mathbb{P}$, called annealed law, is defined by $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$. The expectation with respect to $\mathbb{P}_\xi$ and $\mathbb{P}$ are denoted respectively by $\mathbb{E}_\xi$ and $\mathbb{E}$. By our notation the quenched probability generating function of $N^r_{l,n}$ is

$$f_n^r(s) = \mathbb{E}_\xi\left(\prod_{j=1}^{d} s_j^{N^r_{l,n}(j)} \right), \quad s = (s_1, \ldots, s_d) \in [0,1]^d.$$

We introduce the random mean matrices $M_n \in \mathcal{M}_d(\mathbb{R})$ whose entries are defined by

$$M_n(i,j) := \frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi\Big[Z_{n+1}(j)\Big| Z_n = e_i\Big], \quad 1 \leqslant i,j \leqslant d, \ \ n \geqslant 0,$$

where $\frac{\partial f_n^i}{\partial s_j}(\mathbf{1})$ is the left derivative at $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^d$ of the d-dimensional probability generating function f_n^i with respect to s_j. For each $1 \leqslant i,j \leqslant d$, $M_n(i,j)$ represents the conditioned mean of the number of children of type j produced by a particle of type i at time n. The sequence of random matrices (M_n) is i.i.d. (because the sequence (ξ_n) is i.i.d.). We define the products of these matrices by

$$M_{k,n} := M_k \cdots M_n, \quad 0 \leqslant k \leqslant n.$$

Notice that we have

$$\mathbb{E}_\xi Z_{n+1}^i(j) = M_{0,n}(i,j), \quad n \geqslant 0, \;\; 1 \leqslant i,j \leqslant d. \tag{3.2.2}$$

For $n,k \geqslant 0$, denote by $\rho_{n,n+k}$ the spectral radius of $M_{n,n+k}$. By the Perron-Frobenius theorem (see e.g. [7]), $\rho_{n,n+k}$ is an eigenvalue of $M_{n,n+k}$. Let $U_{n,n+k}$ and $V_{n,n+k}$ be respectively the positive right and left eigenvectors associated to the eigenvalue $\rho_{n,n+k}$, with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k}\rangle = 1$.

Let $\mathcal{G}_+^0$ be the subset of the matrices of $\mathcal{M}_d(\mathbb{R})$ with strictly positive entries. According to the results of Hennion [40, Lemma 3.3 and Theorem 1], if M_0 is allowable in the sense that every row and column contains a strictly positive element, and if the positivity condition

$$\mathbb{P}\bigg(\bigcup_{n\geqslant 0} \Big\{ M_{0,n} \in \mathcal{G}_+^0 \Big\} \bigg) > 0 \tag{3.2.3}$$

holds, then the random vectors $U_{n,\infty}$ and the random scalars λ_n are well defined by (3.1.9) and (3.1.10), and satisfy (3.1.11). Note that the sequences $(U_{n,\infty})$ and (λ_n) are stationary ergodic. It is proved in [32, Theorem 1] that the sequence $(W_n^i)_{n\geqslant 0}$ defined by (3.1.12) is a non-negative martingale under $\mathbb{P}_\xi$ and $\mathbb{P}$, with respect to the filtration

$$\mathcal{F}_0 = \sigma(\xi), \;\; \mathcal{F}_n = \sigma\Big(\xi, N_{l,k}^r(j), 0 \leqslant k \leqslant n-1, 1 \leqslant r,j \leqslant d, l \geqslant 1\Big) \quad \text{for } n \geqslant 1.$$

Thus $\mathbb{P}$-a.s. for all $1 \leqslant i \leqslant d$, the limit

$$W^i := \lim_{n\to+\infty} W_n^i \tag{3.2.4}$$

exists and $\mathbb{E}_\xi W^i \leqslant 1$ by Fatou's lemma.

Now we introduce a classification of MBPRE's. Under the following moment condition

$$\mathbb{E} \log^+ \|M_0\| < +\infty, \tag{3.2.5}$$

by an argument of sub-additivity, the limite

$$\gamma := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\| = \inf_{n\geqslant 1}\frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\|,$$

exists; it is called Lyapunov exponent of the sequence $(M_n)_{n\geqslant 0}$. Moreover, Furstenberg and Kesten established in [26] a strong law of large numbers for $\log\|M_{0,n-1}\|$:

$$\lim_{n\to+\infty}\frac{1}{n}\log\|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.}$$

We say that a MBPRE is supercritical if $\gamma > 0$, critical if $\gamma = 0$, and subcritical if $\gamma < 0$. In this paper, the process (Z_n) will always be supercritical, i.e. $\gamma > 0$, under the conditions that we will assume.

The non-degeneracy of the limit variables W^i has been studied in [32]. In particular, when $\gamma > 0$, it has been proved in [32, Theorem 2.6] that the $X\log X$ condition

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\log^+\frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty \quad \forall 1\leqslant i,j\leqslant d \tag{3.2.6}$$

is sufficient for the non-degeneracy of each W^i in the sense that $\mathbb{P}(W^i > 0) > 0$, and that this condition is also necessary under the additional condition **M1** that we will introduce below. Moreover, when W^i are non-degenerate, then

$$\mathbb{P}_\xi(W^i > 0) > 0 \quad \text{and} \quad \mathbb{E}_\xi W^i = 1 \quad \text{a.s.,} \qquad \text{and} \quad W_n^i \to W^i \text{ in } L^1. \tag{3.2.7}$$

In this paper, for a given $p > 1$, we study the convergence in L^p of the fundamental martingale $(W_n^i)_{n\geqslant 0}$ and the normalized population size $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$, for all $i,j = 1,\cdots,d$.

We first consider the martingale $(W_n^i)_{n\geqslant 0}$, $1\leqslant i\leqslant d$. To formulate our results, we need to introduce some notation and condition. Set

$$I = \Big\{s\leqslant 0:\ \mathbb{E}M_0(i,j)^s < +\infty \quad \forall i,j = 1,\cdots,d\Big\}.$$

Obviously, by Hölder's inequality, I is an interval, and if there exists $s\in I$ with $s < 0$, then $M_0 > 0$ $\mathbb{P}$-a.s., so that condition (3.2.3) is satisfied. It will be seen in Proposition

3.3.1 that for $s \in I$ the limit

$$\kappa(s) := \lim_{n\to+\infty} \Big(\mathbb{E}\|M_{0,n-1}\|^s\Big)^{1/n} \tag{3.2.8}$$

exists, with $\kappa(s) < +\infty$. Notice that κ is a log-convex function on I. We will need the following condition of Furstenberg and Kesten [26]:

M1. There exists a constant $D > 1$ such that

$$1 \leqslant \frac{\max_{1\leqslant i,j\leqslant d} M_0(i,j)}{\min_{1\leqslant i,j\leqslant d} M_0(i,j)} \leqslant D.$$

Note that condition **M1** implies condition (3.2.3).

Our first theorem gives sufficient and necessary conditions for the L^p convergence of the martingales (W_n^i), $1 \leqslant i \leqslant d$.

Theorem 3.2.1. *Let $p > 1$ be such that $1-p \in I$. If*

$$\max_{1\leqslant i,j\leqslant d} \mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)^p < +\infty \quad \text{and} \quad \kappa(1-p) < 1, \tag{3.2.9}$$

*then $W_n^i \underset{n\to+\infty}{\longrightarrow} W^i$ in L^p for any $1 \leqslant i \leqslant d$. The converse is also valid when the Furstenberg-Kesten condition **M1** holds.*

It is clear that condition (3.2.9) implies (3.2.6). Moreover, (3.2.9) also implies the supercritical condition $\gamma > 0$ when condition (3.2.5) holds, since by Jensen's inequality we have $\log \kappa(1-p) \geqslant (1-p)\gamma$.

Our second theorem shows that the L^p convergence of W_n^i has an exponential rate.

Theorem 3.2.2. *Let $p > 1$ be such that $1-p \in I$. Assume (3.2.9).*

1. *If $1 < p \leqslant 2$, then denoting $\delta_c(p) = \kappa(1-p)^{1/p}$ we have*

$$\limsup_{n\to+\infty} \delta_c(p)^{-n}\Big(\mathbb{E}|W_n^i - W^i|^p\Big)^{1/p} < +\infty. \tag{3.2.10}$$

2. *If $p > 2$, then $\delta_c(p) := \max\Big\{\kappa(1-p)^{1/p}, \kappa(-p/2)^{1/p}\Big\} < 1$, and*

$$\lim_{n\to+\infty} \delta^{-n}\Big(\mathbb{E}|W_n^i - W^i|^p\Big)^{1/p} = 0 \quad \forall \delta > \delta_c(p). \tag{3.2.11}$$

In the proof we shall see that in Part 1 the moment condition $\mathbb{E}\Big(\frac{Z_1^i(j)}{M_0(i,j)}\Big)^p < +\infty$ for all $1 \leqslant i, j \leqslant d$ can be relaxed to $\mathbb{E}(W_1^i)^p < +\infty$ for all $1 \leqslant i \leqslant d$.

Note that for $p \geqslant 2$, by applying Hölder's inequality to $\mathbb{E}\|M_{0,n-1}\|^{-p/2}$ and then letting $n \to +\infty$, it is easy to see that $\kappa(-p/2)^{2/p} \leqslant \kappa(1-p)^{1/(p-1)}$. Thus $\kappa(1-p) < 1$ implies $\kappa(-p/2) < 1$, so that $\delta_c(p) < 1$.

Now we investigate the convergence in L^p of the normalized population size $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)}$. Recall that under condition (3.2.5), **M1** and the supercriticality condition $\gamma > 0$, by the Kesten-Stigum type theorem for a supercritical MBPRE [32, Theorem 2.11], for all $1 \leqslant i, j \leqslant d$,

$$\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i, \tag{3.2.12}$$

and the convergence holds a.s. if additionally $\mathbb{E}(Z_1^i(j)/M_0(i,j))^p < +\infty$ and $\mathbb{E}\|M_0\|^{1-p} < +\infty$ for some $p > 1$ and all $1 \leqslant i, j \leqslant d$ (see [32, Theorem 2.13]). By [32, Theorem 2.11] and Sheffé's theorem, under the supercritical condition $\gamma > 0$ and the Furstengerg-Kesten condition **M1**, $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \to W^i$ in L^1 if and only if (3.2.6) holds. From Theorem 3.2.1 and under condition **M1**, we obtain a criterion for the convergence in L^p of $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)}$.

Theorem 3.2.3. *Assume condition* ***M1****. Let $p > 1$ be such that $1 - p \in I$. Then $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow[n\to+\infty]{} W^i$ in L^p for any $1 \leqslant i, j \leqslant d$ if and only if* (3.2.9) *holds.*

Finally, from Theorem 3.2.2, we deduce an exponential rate of the convergence in L^p of $\frac{Z_n^i(j)}{M_{0,n-1}(i,j)}$.

Theorem 3.2.4. *Assume condition* ***M1****. Let $p > 1$ be such that $1-p \in I$ and that* (3.2.9) *holds. Then there exists $\delta \in (0,1)$ such that*

$$\lim_{n\to+\infty} \delta^{-n} \left(\mathbb{E}\left|\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} - W^i\right|^p\right)^{1/p} = 0. \tag{3.2.13}$$

3.3 Spectral properties of the transfer operator P_s

We start this section by giving some notation. Denote by $\mathcal{S} = \{x \in \mathbb{R}^d : x \geqslant 0, \|x\| = 1\}$. For $x \in \mathcal{S}$ and $M \in \mathcal{G}_+^0$ (the set of matrices with strictly positive entries), define the projective action of M on $\mathcal{S}$ by $M \cdot x := \frac{Mx}{\|Mx\|}$. Let $\mathcal{C}(\mathcal{S})$ be the space of continuous functions on $\mathcal{S}$ with real values. For any $\varphi \in \mathcal{C}(\mathcal{S})$, set

$$\|\varphi\|_\infty = \sup_{x\in\mathcal{S}} \|\varphi x\|.$$

For $s \in I$, define the transfer operator P_s as follows : for all $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s\varphi(x) := \mathbb{E}\Big[\|M_0 x\|^s \varphi(M_0 \cdot x)\Big], \quad x \in \mathcal{S}. \tag{3.3.1}$$

Define also the conjugate operator P_s^*, such that for $s \in I$ and $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s^*\varphi(x) := \mathbb{E}\Big[\|M_0^T x\|^s \varphi(M_0^T \cdot x)\Big], \quad x \in \mathcal{S}. \tag{3.3.2}$$

In this section, we investigate the spectral properties of the transfer operator P_s and its conjugate P_s^* for $s \leqslant 0$. These results extend some properties known in the case $s \geqslant 0$ (see [56, 13]) to the case $s < 0$. We also give some properties of $\kappa(s)$. The main result is given by the following proposition. We use the notation $\mu(\psi) = \int \psi d\mu$ to denote the integral of ψ with respect the measure μ.

Proposition 3.3.1. *Assume that $s \in I$. Then $\kappa(s) < +\infty$, the spectral radius of P_s is equal to $\kappa(s)$, and there exists a probability measure ν_s on $\mathcal{S}$ and a strictly positive function $r_s \in \mathcal{C}(\mathcal{S})$ such that*

$$\nu_s P_s = \kappa(s)\nu_s \quad \textit{and} \quad P_s r_s = \kappa(s) r_s,$$

where $\nu_s P_s$ denotes the measure on $\mathcal{S}$ such that $(\nu_s P_s)(\psi) = \nu_s(P_s\psi)$ for all $\psi \in \mathcal{C}(\mathcal{S})$. Moreover, $\kappa(s)$ is also the spectral radius of P_s^*, and there exists a probability measure ν_s^* on $\mathcal{S}$ and a strictly positive function $r_s^* \in \mathcal{C}(\mathcal{S})$ such that

$$\nu_s^* P_s^* = \kappa(s)\nu_s^* \quad \textit{and} \quad P_s^* r_s^* = \kappa(s) r_s^*.$$

To prove the above proposition, we will use the following Lemma about the properties of $\kappa(s)$, $s \leqslant 0$.

Lemma 3.3.2. *Assume that $s \in I$. Then*

$$\kappa(s) = \lim_{n\to+\infty} \Big(\mathbb{E}\|M_{0,n-1}\|^s\Big)^{1/n} = \sup_{n\geqslant 1} \Big(\mathbb{E}\|M_{0,n-1}\|^s\Big)^{1/n} < +\infty,$$

and there exists $C_s > 0$ such that for all $x, y \in \mathcal{S}$ and $n \geqslant 1$,

$$\mathbb{E}\|M_{0,n-1}\|^s \leqslant \mathbb{E}\|M_{0,n-1}x\|^s \leqslant \mathbb{E}\langle M_{0,n-1}x, y\rangle^s \leqslant C_s \mathbb{E}\|M_{0,n-1}\|^s.$$

Proof. Notice that the sequence $\left(\mathbb{E}\|M_{0,n-1}\|^s\right)_{n\geqslant 1}$ is super-multiplicative for $s \in I$, so the limit $\kappa(s) = \lim_{n\to\infty} \left(\mathbb{E}\|M_{0,n-1}\|^s\right)^{1/n}$ exists, and

$$\kappa(s) = \sup_{n\geqslant 1} \left(\mathbb{E}\|M_{0,n-1}\|^s)\right)^{1/n} \in \mathbb{R}_+ \cup \{+\infty\}.$$

Clearly, for all $x, y \in \mathcal{S}$ and $n \geqslant 1$, we have the inequalities

$$\mathbb{E}\|M_{0,n-1}\|^s \leqslant \mathbb{E}\|M_{0,n-1}x\|^s \leqslant \mathbb{E}\langle M_{0,n-1}x, y\rangle^s \leqslant \mathbb{E}\Big(\max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^s\Big). \tag{3.3.3}$$

Moreover, since the sequence of matrices (M_n) is i.i.d, for all $n, k \geqslant 1$ we have

$$\begin{aligned} & \mathbb{E} \max_{1\leqslant i,j\leqslant d} M_{0,n+k-1}(i,j)^s \\ \leqslant\ & \mathbb{E} \max_{1\leqslant i,j,l\leqslant d} \Big(M_{0,n-1}(i,l)^s M_{n,n+k-1}(l,j)^s\Big) \\ \leqslant\ & \mathbb{E} \max_{1\leqslant i,l\leqslant d} M_{0,n-1}(i,l)^s \ \mathbb{E} \max_{1\leqslant l,j\leqslant d} M_{0,k-1}(l,j)^s. \end{aligned}$$

Hence $\Big(\mathbb{E} \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^s\Big)_{n\geqslant 1}$ is sub-multiplicative, so that

$$\lim_{n\to+\infty} \Big(\mathbb{E} \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^s\Big)^{1/n} = \inf_{n\geqslant 1} \Big(\mathbb{E} \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^s\Big)^{1/n}.$$

Combining this with (3.3.3), and letting $n \to +\infty$, we obtain

$$\kappa(s) \leqslant \lim_{n\to+\infty} \Big(\mathbb{E} \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^s\Big)^{1/n} \leqslant \mathbb{E} \max_{1\leqslant i,j\leqslant d} M_0(i,j)^s < +\infty.$$

Furthermore by simple calculations, for all $n \geqslant 3$ it holds that

$$\begin{aligned}
&\mathbb{E} \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^s \\
= \ & \mathbb{E} \max_{1\leqslant i,j\leqslant d}\left(\sum_{1\leqslant l_1,l_2\leqslant d} M_0(i,l_1)M_{1,n-2}(l_1,l_2)M_{n-1}(l_2,j)\right)^s \\
\leqslant \ & \mathbb{E}\left(\min_{1\leqslant i,l_1\leqslant d} M_0(i,l_1) \min_{1\leqslant l_2,j\leqslant d} M_{n-1}(l_2,j) \sum_{1\leqslant l_1,l_2\leqslant d} M_{1,n-2}(l_1,l_2)\right)^s \\
\leqslant \ & \left(\mathbb{E} \max_{1\leqslant i,j\leqslant d} M_0(i,j)^s\right)^2 \mathbb{E}\|M_{0,n-3}\|^s.
\end{aligned}$$

It follows that for all $n \geqslant 3$,

$$\begin{aligned}
&\mathbb{E} \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^s \\
\leqslant \ & \left(\mathbb{E} \max_{1\leqslant i,j\leqslant d} M_0(i,j)^s\right)^2 \frac{\mathbb{E}(\|M_{0,n-3}\|\|M_{n-2}\|\|M_{n-1}\|)^s}{(\mathbb{E}\|M_0\|^s)^2} \\
\leqslant \ & \left(\frac{\mathbb{E}\max_{1\leqslant i,j\leqslant d} M_0(i,j)^s}{\mathbb{E}\|M_0\|^s}\right)^2 \mathbb{E}\|M_{0,n-1}\|^s.
\end{aligned}$$

This, together with (3.3.3), proves the inequalities of Lemma 3.3.2 for $n \geqslant 3$ with

$$C_s = \left(\frac{\mathbb{E}\max_{1\leqslant i,j\leqslant d} M_0(i,j)^s}{\mathbb{E}\|M_0\|^s}\right)^2,$$

which is finite since $s \in I$. It is clear that the inequalities remain valid for $1 \leqslant n \leqslant 2$ by modifying slightly the value of C_s (choosing it large enough). This concludes the proof of Lemma 3.3.2. □

Proof of Proposition 3.3.1. We shall use an argument similar to that in the proof of [13, Proposition 4.4] where the case $s \geqslant 0$ is considered. Let $M^1(\mathcal{S})$ be the set of all probability measures on $\mathcal{S}$, and $M_b^1(\mathbb{R})$ the set of all finite signed measures on $\mathbb{R}$ equipped with the total variation norm. Since $M^1(\mathcal{S})$ is a compact convex subset of the Banach space $M_b^1(\mathbb{R})$, by the Schauder-Tychonoff theorem applied to the continuous map $\nu \mapsto \nu P_s/\nu P_s\big(\mathcal{S}\big)$, there exists an invariant probability measure $\nu_s \in M^1(\mathcal{S})$ of this map. Consequently, ν_s is an eigenmeasure of P_s:

$$\nu_s P_s = [\nu_s P_s\big(\mathcal{S}\big)]\, \nu_s. \tag{3.3.4}$$

In the same way there exists an probability eigenmeasure ν_s^* of the operator P_s^*, associated to the eigenvalue $k(s) = \nu_s^* P_s^*\big(\mathcal{S}\big)$:

$$\nu_s^* P_s^* = k(s)\, \nu_s^*. \tag{3.3.5}$$

Set

$$r_s(x) := \frac{1}{k(s)} \int_{\mathcal{S}} \mathbb{E}\langle M_0 x, y\rangle^s \nu_s^*(dy), \quad x \in \mathcal{S}.$$

Since $s \in I$, it is clear that for all $x \in \mathcal{S}$,

$$0 < \frac{1}{k(s)} \mathbb{E} \min_{1 \leqslant i,j \leqslant d} M_0(i,j)^s \leqslant r_s(x) \leqslant \frac{1}{k(s)} \mathbb{E} \max_{1 \leqslant i,j \leqslant d} M_0(i,j)^s < +\infty,$$

and that $r_s \in \mathcal{C}(\mathcal{S})$. Moreover, for all $x \in \mathcal{S}$ we have

$$\begin{aligned} r_s(x) &= \frac{1}{k(s)} \int_{\mathcal{S}} \mathbb{E}\langle x, M_0^T y\rangle^s \nu_s^*(dy) \\ &= \frac{1}{k(s)} \int_{\mathcal{S}} \mathbb{E}[\|M_0^T y\|^s \langle x, M_0^T \cdot y\rangle^s] \nu_s^*(dy) \\ &= \frac{1}{k(s)} \int_{\mathcal{S}} \langle x, y\rangle^s (\nu_s^* P_s^*)(dy) \\ &= \int_{\mathcal{S}} \langle x, y\rangle^s \nu_s^*(dy). \end{aligned} \tag{3.3.6}$$

Using Fubini's theorem, it follows from the definition of r_s and (3.3.6) that for all $x \in \mathcal{S}$,

$$\begin{aligned} r_s(x) &= \frac{1}{k(s)} \mathbb{E}\Big[\|M_0 x\|^s \int_{\mathcal{S}} \langle M_0 \cdot x, y\rangle^s \nu_s^*(dy)\Big] \\ &= \frac{1}{k(s)} \mathbb{E}\Big[\|M_0 x\|^s r_s(M_0 \cdot x)\Big] \\ &= \frac{1}{k(s)} P_s r_s(x). \end{aligned} \tag{3.3.7}$$

So we have proved that $P_s r_s = k(s) r_s$. Now we show that

$$k(s) = \rho(P_s) = \kappa(s),$$

where $\rho(P_s)$ is the spectral radius of P_s.

First we have $k(s) \leqslant \rho(P_s)$, since $k(s)$ is a positive eigenvalue of P_s.

Next, we prove that $\rho(P_s) \leqslant \kappa(s)$. By iteration of the operator P_s (using the fact that

$M_{0,n-1}$ has the same law as $M_{n-1}\cdots M_0$) and Lemma 3.3.2, it holds that for all $\varphi \in \mathcal{C}(\mathcal{S})$ and $x \in \mathcal{S}$,

$$P_s^n\varphi(x) = \mathbb{E}\Big[\|M_{0,n-1}x\|^s\varphi(M_{0,n-1}\cdot x)\Big] \leqslant C_s\|\varphi\|_\infty \mathbb{E}\|M_{0,n-1}\|^s.$$

This implies that

$$\rho(P_s) = \lim_{n\to\infty}\sup\{\|P_s^n\varphi\|_\infty^{1/n} : \|\varphi\|_\infty = 1\} \leqslant \lim_{n\to+\infty}\Big(\mathbb{E}\|M_{0,n-1}\|^s\Big)^{1/n} = \kappa(s).$$

We then prove that $\kappa(s) \leqslant k(s)$. Iterating the relation $\nu_s^* P_s^* = k(s)\nu_s^*$, we obtain $\nu_s^* P_s^{*n} = k(s)^n\nu_s^*$, so that

$$k(s)^n = \nu_s^* P_s^{*n}(\mathcal{S}) = \int_{\mathcal{S}} \mathbb{E}\|M_{0,n-1}^T y\|^s \nu_s^*(dy) \geqslant \mathbb{E}\|M_{0,n-1}^T\|^s.$$

This implies $\kappa(s) \leqslant k(s)$, since

$$\kappa(s) = \lim_{n\to+\infty}\Big(\mathbb{E}\|M_{0,n-1}\|^s\Big)^{1/n} = \lim_{n\to+\infty}\Big(\mathbb{E}\|M_{0,n-1}^T\|^s\Big)^{1/n}.$$

So we have proved the equalities $k(s) = \rho(P_s) = \kappa(s)$. This together with (3.3.5) and (3.3.7) implies

$$\nu_s^* P_s^* = \kappa(s)\nu_s^* \quad \text{and} \quad P_s r_s = \kappa(s) r_s.$$

Changing the roles of P_s and P_s^*, by the same arguments we can prove that

$$\nu_s P_s = \kappa(s)\nu_s \quad \text{and} \quad P_s^* r_s^* = \kappa(s) r_s^*$$

for some strictly positive function $r_s^* \in \mathcal{C}(\mathcal{S})$, and that $\kappa(s)$ is also the spectral radius of P_s^*. This concludes the proof of Proposition 3.3.1. □

3.4 Convergence in L^p of the martingale W_n^i

In this section, we prove Theorems 3.2.1 and 3.2.2 giving sufficient and necessary conditions for the L^p convergence of W_n^i, $1 \leqslant i \leqslant d$, with an exponential speed. First we formulate the following result.

Theorem 3.4.1. *Let $p > 1$ be such that $1-p \in I$. Consider the assertions:*

$$W_n^i \underset{n\to+\infty}{\longrightarrow} W^i \quad \text{in } L^p \quad \forall i = 1, \cdots, d \tag{3.4.1}$$

$$\mathbb{E}(W_1^i)^p < +\infty \quad \forall i = 1, \cdots, d \quad \text{and} \quad \kappa(1-p) < 1. \tag{3.4.2}$$

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)^p < +\infty \quad \forall i, j = 1, \cdots, d \quad \text{and} \quad \kappa(1-p) < 1. \tag{3.4.3}$$

*If $1 < p \leqslant 2$, then we have the implications: (3.4.3) $\Rightarrow$ (3.4.1) $\leftrightarrow$ (3.4.2). If $p > 2$, then we have: (3.4.3) $\Rightarrow$ (3.4.1) $\Rightarrow$ (3.4.2). When the Furstenberg-Kesten condition **M1** holds, then for each $p > 1$, (3.4.3) $\leftrightarrow$ (3.4.1) $\leftrightarrow$ (3.4.2).*

It is clear that the assertions of Theorem 3.2.1 follow from Theorem 3.4.1. Theorem 3.4.1 is slightly stronger in the sense that for $1 < p \leqslant 2$, it gives a sufficient and necessary condition without assuming the Furstenberg-Kesten condition **M1**.

The following Lemma will be useful to investigate the convergence in L^p, and it is a direct consequence of the Marcinkiewicz-Zygmund inequality in [16, Theorem 1.5], as stated in [60, Lemma 1.4].

Lemma 3.4.2. *Let $(X_k)_{k\geqslant 1}$ be a sequence of i.i.d. random centered variables. Then for all $n \geqslant 1$ and $p > 1$:*

$$\mathbb{E}\left|\sum_{k=1}^n X_k\right|^p \leqslant \begin{cases} (B_p)^p \mathbb{E}|X_k|^p n, & \text{if } 1 < p \leqslant 2, \\ (B_p)^p \mathbb{E}|X_k|^p n^{\frac{p}{2}}, & \text{if } p > 2, \end{cases}$$

where $B_p = 2\min\left\{k^{1/2} : k \in \mathbb{N}, k \geqslant \frac{p}{2}\right\}$.

In a last article, we proved the following result [32, Lemma 7.1]. It gives some properties on the products of random matrices $M_{n,n+k}$ under the Furstenberg-Kesten condition **M1**.

Lemma 3.4.3. *Assume condition **M1**. Then:*

1. *for all $n, k \geqslant 0$ and $1 \leqslant i, j, r \leqslant d$, $\mathbb{P}$-a.s.,*

$$\frac{1}{D} \leqslant \frac{M_{n,n+k}(i,j)}{M_{n,n+k}(i,r)} \leqslant D \quad \text{and} \quad \frac{1}{D} \leqslant \frac{M_{n,n+k}(i,j)}{M_{n,n+k}(r,j)} \leqslant D; \tag{3.4.4}$$

2. *for all $n, k \geqslant 0$ and $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-a.s.,*

$$\frac{1}{dD^2} \leqslant \frac{M_{n,n+k}(i,j)U_{n+k+1,\infty}(j)}{\lambda_{n,n+k}U_{n,\infty}(i)} \leqslant 1. \tag{3.4.5}$$

Proof of Theorems 3.4.1 and 3.2.2. By iterating (3.2.1), it is easy to see that the process $(Z_n)_{n\geqslant 0}$ satisfies the relation

$$Z_{n+k} = \sum_{j=1}^{d} \sum_{l=1}^{Z_n(j)} Z^j_{l,n,k}, \quad n \geqslant 0, k \geqslant 1, \tag{3.4.6}$$

where $Z^j_{l,n,k}(r)$ denotes the number of the offspring of type r at time $n+k$ of the l-th particle of type j in the generation n; conditional on the environment ξ, the random vectors $Z^j_{l,n,k} = (Z^j_{l,n,k}(1), \cdots, Z^j_{l,n,k}(d))$ indexed by $l \in \mathbb{N}^*$ and $j \in \{1, \cdots, d\}$ (for fixed n and k) are independent, each has the probability generating function $f^j_n \circ f_{n+1} \circ \cdots \circ f_{n+k-1}$. Combining (3.4.6), (3.1.12) and (3.1.11), we have, for all $n, k \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} W^i_{n+k} - W^i_n &= \sum_{r=1}^{d} \frac{U_{n+k,\infty}(r)}{\lambda_{0,n+k-1} U_{0,\infty}(i)} \sum_{j=1}^{d} \sum_{l=1}^{Z^i_n(j)} Z^j_{l,n,k}(r) - W^i_n \\ &= \sum_{j=1}^{d} \frac{U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \sum_{l=1}^{Z^i_n(j)} \sum_{r=1}^{d} \frac{U_{n+k,\infty}(r) Z^j_{l,n,k}(r)}{\lambda_{n,n+k-1} U_{n,\infty}(j)} - W^i_n \\ &= \sum_{j=1}^{d} \frac{U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \sum_{l=1}^{Z^i_n(j)} (W^j_{l,n,k} - 1), \end{aligned} \tag{3.4.7}$$

where

$$W^j_{l,n,k} := \frac{\langle Z^j_{l,n,k}, U_{n+k,\infty} \rangle}{\lambda_{n,n+k-1} U_{n,\infty}(j)}.$$

Let T be the shift operator of the environment sequence:

$$T\xi = (\xi_1, \xi_2, \cdots) \quad \text{if} \quad \xi = (\xi_0, \xi_1, \cdots),$$

and let T^n be its n-fold iteration. It is clear that, given the environment ξ, the random variables $W^j_{l,n,k}$, $l \geqslant 1$, are i.i.d.; they are independent of $\xi_0, \ldots, \xi_{n-1}$ and Z^i_n, and have the same distribution as $W^j_{n,k}$, where $\left(W^j_{n,k}\right)_{k\geqslant 0}$ is the martingale associated to a MBPRE starting with one individual of type j, in the shift random environment $T^n\xi$.

We divide the proof into 5 steps.

Step 1. We first prove that for $1 < p \leqslant 2$, we have the implications (3.4.2) $\Rightarrow$ (3.4.1) of Theorem 3.4.1, and (3.4.2) $\Rightarrow$ (3.2.10) of Theorem 3.2.2. We assume that $1 < p \leqslant 2$ and (3.4.2). Applying (3.4.7), the convexity of the function $x \mapsto x^p$ (together with the fact that $\sum_{j=1}^d U_{n,\infty}(j) = 1$), Lemma 3.4.2 and (3.1.11), for all $n \geqslant 0$, $k \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s., we have (for $1 < p \leqslant 2$),

$$\begin{aligned}
\mathbb{E}_\xi |W_{n+k}^i - W_n^i|^p &\leqslant \mathbb{E}_\xi \left(\sum_{j=1}^d \frac{U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \left| \sum_{l=1}^{Z_n^i(j)} \left(W_{l,n,k}^j - 1 \right) \right| \right)^p \\
&\leqslant \sum_{j=1}^d \frac{U_{n,\infty}(j)}{(\lambda_{0,n-1} U_{0,\infty}(i))^p} \mathbb{E}_\xi \left(\left| \sum_{l=1}^{Z_n^i(j)} \left(W_{l,n,k}^j - 1 \right) \right| \right)^p \\
&\leqslant B_p^p \sum_{j=1}^d \frac{U_{n,\infty}(j)}{(\lambda_{0,n-1} U_{0,\infty}(i))^p} \mathbb{E}_\xi Z_n^i(j) \mathbb{E}_\xi \left| W_{n,k}^j - 1 \right|^p \\
&\leqslant B_p^p \sigma_{n,k}(p) \sum_{j=1}^d \frac{M_{0,n-1}(i,j) U_{n,\infty}(j)}{(\lambda_{0,n-1} U_{0,\infty}(i))^p} \\
&= B_p^p \sigma_{n,k}(p) (\lambda_{0,n-1} U_{0,\infty}(i))^{1-p}, \qquad (3.4.8)
\end{aligned}$$

with

$$\sigma_{n,k}(p) = \max_{1 \leqslant j \leqslant d} \mathbb{E}_\xi \left| W_{n,k}^j - 1 \right|^p. \qquad (3.4.9)$$

Using again (3.1.11) and Lemma 3.3.2 together with the fact that $M_{0,n-1}$ is independent of $T^n\xi$, we get that for all $s \in I$, $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.

$$\begin{cases} \mathbb{E}_{T^n\xi} \lambda_{0,n-1}^s = \mathbb{E}_{T^n\xi} \| M_{0,n-1} U_{n,\infty} \|^s \leqslant C_s \kappa(s)^n; \\ \mathbb{E}_{T^n\xi} (\lambda_{0,n-1} U_{0,\infty}(i))^s = \mathbb{E}_{T^n\xi} \langle M_{0,n-1} U_{n,\infty}, e_i \rangle^s \leqslant C_s \kappa(s)^n. \end{cases} \qquad (3.4.10)$$

Taking expectation in (3.4.8), by (3.4.10) we get that for all $n \geqslant 0$, $k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
\mathbb{E}|W_{n+k}^i - W_n^i|^p &\leqslant B_p^p \mathbb{E}\Big[\sigma_{n,k}(p) \mathbb{E}_{T^n\xi} (\lambda_{0,n-1} U_{0,\infty}(i))^{1-p} \Big] \\
&\leqslant B_p^p C_{1-p} \mathbb{E}\sigma_{0,k}(p) \kappa(1-p)^n. \qquad (3.4.11)
\end{aligned}$$

From condition (3.4.2) we have $\mathbb{E}\sigma_{0,1}(p) < +\infty$ and $\kappa(1-p) < 1$. So by the triangular inequality of L^p, it follows from (3.4.11) that for all $1 \leqslant i \leqslant d$, with $C = B_p [C_{1-p} \mathbb{E}\sigma_{0,1}(p)]^{1/p}$,

$$\sup_{n \geqslant 0} \left(\mathbb{E}(W_n^i)^p \right)^{1/p} \leqslant 1 + C \sum_{n=0}^{+\infty} \kappa(1-p)^{n/p} < +\infty. \qquad (3.4.12)$$

Therefore for all $1 \leqslant i \leqslant d$, (W_n^i) is a martingale bounded in L^p, so that it converges in

L^p. This proves the implication (3.4.2) $\Rightarrow$ (3.4.1) of Theorem 3.4.1. Furthermore (3.4.12) implies that $\sup_{k\geqslant 0}\mathbb{E}\sigma_{0,k}(p)<+\infty$. So, by letting $k\to+\infty$ in (3.4.11) we get (3.2.10) of Theorem 3.2.2.

Step 2. We next prove the implication (3.4.3) $\Rightarrow$ (3.4.2) of Theorem 3.4.1 for any $p>1$, which, in particular, will conclude the proof of Theorem 3.4.1 for $1<p\leqslant 2$. By (3.1.10) we have $0\leqslant \frac{M_0(i,j),U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)}\leqslant 1$ a.s. for all $1\leqslant i,j\leqslant d$. So by the triangular inequality of L^p, it follows that for $p>1$ and $1\leqslant i\leqslant d$,

$$\Big(\mathbb{E}(W_1^i)^p\Big)^{1/p} = \left(\mathbb{E}\left(\frac{\langle Z_1^i,U_{1,\infty}\rangle}{\lambda_0 U_{0,\infty}(i)}\right)^p\right)^{1/p} \tag{3.4.13}$$

$$= \left(\mathbb{E}\left(\sum_{j=1}^d \frac{M_0(i,j),U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)}\frac{Z_1^i(j)}{M_0(i,j)}\right)^p\right)^{1/p} \tag{3.4.14}$$

$$\leqslant \sum_{j=1}^d\left(\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\right)^p\right)^{1/p}. \tag{3.4.15}$$

Therefore the implication (3.4.3) $\Rightarrow$ (3.4.2) of Theorem 3.4.1 holds.

Step 3. We now prove that for $p>2$, we have the implications (3.4.3) $\Rightarrow$ (3.4.1) of Theorem 3.4.1, and (3.4.3) $\Rightarrow$ (3.2.11) of Theorem 3.2.2. Assume $p>2$ and (3.4.3). In the following $C>0$ will be a constant which may depend on p and which may differ from line to line. Applying (3.4.7), the inequality $(\sum_{j=1}^d x_j)^p\leqslant d^{p-1}\sum_{j=1}^d x_j^p$, $x_j\geqslant 0$ for any $1\leqslant j\leqslant d$, and Lemma 3.4.2, for all $n\geqslant 0$, $k\geqslant 1$ and $1\leqslant i\leqslant d$, $\mathbb{P}$-a.s. we have

$$\begin{aligned}
&\mathbb{E}_\xi|W_{n+k}^i-W_n^i|^p\\
\leqslant\ & d^{p-1}B_p^p\sum_{j=1}^d\left(\frac{U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)}\right)^p\mathbb{E}_\xi(Z_n^i(j))^{p/2}\mathbb{E}_\xi\Big|W_{n,k}^j-1\Big|^p\\
\leqslant\ & C\sigma_{n,k}(p)\sum_{j=1}^d(U_{n,\infty}(j))^{p/2}\mathbb{E}_\xi\left(\frac{U_{n,\infty}(j)Z_n^i(j)}{\lambda_{0,n-1}U_{0,\infty}(i)}\right)^{p/2}(\lambda_{0,n-1}U_{0,\infty}(i))^{-p/2}\\
\leqslant\ & C\sigma_{n,k}(p)\Big(\sum_{j=1}^d U_{n,\infty}(j)\Big)\mathbb{E}_\xi(W_n^i)^{p/2}(\lambda_{0,n-1}U_{0,\infty}(i))^{-p/2}\\
\leqslant\ & C\sigma_{n,k}(p)\mathbb{E}_\xi(W_n^i)^{p/2}(\lambda_{0,n-1}U_{0,\infty}(i))^{-p/2},
\end{aligned}\tag{3.4.16}$$

with $\sigma_{n,k}(p)$ defined as in (3.4.9) (for $p>2$). Set $j_p\in\mathbb{N}$ the unique integer such that

$1 < \frac{p}{2^{j_p}} \leqslant 2$. For all $n \geqslant 0$, $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant j_p$, define

$$a^i_{n,j}(p) := (\lambda_{0,n-1} U_{0,\infty}(i))^{p/2^j - p}\, \mathbb{E}_\xi (W_n^i)^{p/2^j}. \tag{3.4.17}$$

Taking expectation in (3.4.16), we obtain that for all $n \geqslant 0$, $k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}|W^i_{n+k} - W^i_n|^p \leqslant C\mathbb{E}\Big[\sigma_{n,k}(p)\mathbb{E}_{T^n\xi} a^i_{n,1}(p)\Big]. \tag{3.4.18}$$

To prove (3.4.1) of Theorem 3.4.1, and (3.2.11) of Theorem 3.2.2, it is enough to show that there exists a constant $C_1 > 0$ (which may depend on p) such that for all $n \geqslant 0$, $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant j_p$ and $\delta > \delta_c(p)$,

$$\delta^{-n}\Big(\mathbb{E}_{T^n\xi} a^i_{n,j}(p)\Big)^{1/p} \leqslant C_1 \quad \mathbb{P}\text{-a.s.} \tag{3.4.19}$$

In fact, combining (3.4.18) and (3.4.19) for $j = 1$, it follows that for all $1 \leqslant i \leqslant d$ and $\delta > \delta_c(p)$,

$$\sup_{n \geqslant 0} \Big(\mathbb{E}(W^i_n)^p\Big)^{1/p} \leqslant 1 + C\Big(\mathbb{E}\sigma_{0,1}(p)\Big)^{1/p} \sum_{n=0}^{+\infty} \delta^n. \tag{3.4.20}$$

Condition (3.4.3) implies that $\mathbb{E}\sigma_{0,1}(p) < +\infty$ and $\delta_c(p) < 1$. Therefore, applying (3.4.20) with $\delta \in (\delta_c(p), 1)$, we deduce that (W^i_n) is a martingale bounded in L^p, for all $1 \leqslant i \leqslant d$. Hence, (W^i_n), $1 \leqslant i \leqslant d$, converge in L^p, and we have $\sup_{k \geqslant 0} \mathbb{E}\sigma_{0,k}(p) < +\infty$. This proves the implication (3.4.3) $\Rightarrow$ (3.4.1). Moreover, combining again (3.4.18) and (3.4.19), and letting $k \to +\infty$, we obtain that for all $n \geqslant 0$, $1 \leqslant i \leqslant d$ and $\delta \in (\delta_c(p), 1)$,

$$\Big(\mathbb{E}|W^i - W^i_n|^p\Big)^{1/p} \leqslant C\delta^n,$$

which implies (3.2.11) and ends the proof of Theorem 3.2.2 in the case $p > 2$.

It remains to prove (3.4.19). We will prove it by iteration on j. First consider the case $j = j_p$. By definition of j_p we have $1 < p/2^{j_p} \leqslant 2$. So, by the triangular inequality in

$L^{p/2^{j_p}}$ under $\mathbb{P}_\xi$ and (3.4.8), it follows that for all $1 \leqslant i \leqslant d$ and $n \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned} a^i_{n,j_p}(p)^{2^{j_p}/p} &\leqslant (\lambda_{0,n-1}U_{0,\infty}(i))^{1-2^{j_p}} \\ &\quad + (\lambda_{0,n-1}U_{0,\infty}(i))^{1-2^{j_p}} \sum_{l=0}^{n-1} \Big(\mathbb{E}_\xi|W^i_{l+1} - W^i_l|^{p/2^{j_p}}\Big)^{2^{j_p}/p} \\ &\leqslant (\lambda_{0,n-1}U_{0,\infty}(i))^{1-2^{j_p}} \\ &\quad + C\sum_{l=0}^{n-1}\Big[\sigma_{l,1}\Big(\frac{p}{2^{j_p}}\Big)\Big]^{2^{j_p}/p} (\lambda_{0,l-1}U_{0,\infty}(i))^{2^{j_p}/p-2^{j_p}} \lambda^{1-2^{j_p}}_{l,n-1}. \end{aligned} \tag{3.4.21}$$

Taking the $L^{p/2^{j_p}}$-norm under $\mathbb{P}_{T^n\xi}$ on both sides, and using the triangular inequality in $L^{p/2^{j_p}}$ and inequalities (3.4.10), we obtain that for all $1 \leqslant i \leqslant d$ and $n \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned} &\Big(\mathbb{E}_{T^n\xi} a^i_{n,j_p}(p)\Big)^{2^{j_p}/p} \\ &\leqslant \Big(\mathbb{E}_{T^n\xi}(\lambda_{0,n-1}U_{0,\infty}(i))^{p/2^{j_p}-p}\Big)^{2^{j_p}/p} \\ &+ C\sum_{l=0}^{n-1}\Big\{\mathbb{E}_{T^n\xi}\Big[\mathbb{E}_{T^l\xi}\Big[(\lambda_{0,l-1}U_{0,\infty}(i))^{1-p}\Big]\sigma_{l,1}\Big(\frac{p}{2^{j_p}}\Big)\lambda^{p/2^{j_p}-p}_{l,n-1}\Big]\Big\}^{2^{j_p}/p} \\ &\leqslant C\,\kappa\Big(\frac{p}{2^{j_p}} - p\Big)^{\frac{n2^{j_p}}{p}} + C\sum_{l=0}^{n-1}\Big\{\kappa(1-p)^l \mathbb{E}_{T^n\xi}\Big[\sigma_{l,1}\Big(\frac{p}{2^{j_p}}\Big)\lambda^{p/2^{j_p}-p}_{l,n-1}\Big]\Big\}^{2^{j_p}/p}. \end{aligned} \tag{3.4.22}$$

Notice that if $1 \leqslant j \leqslant j_p$, then we have $1 - p < \frac{p}{2^j} - p < -\frac{p}{2}$. Since κ is log-convexe on I, we obtain that

$$\max_{1\leqslant j\leqslant j_p}\Big\{\kappa\Big(\frac{p}{2^j} - p\Big)\Big\} \leqslant \max\Big\{\kappa(1-p), \kappa\Big(-\frac{p}{2}\Big)\Big\} = \delta_c(p)^p. \tag{3.4.23}$$

We now deal with the second term in (3.4.22), by calculating first the conditional expectation $\mathbb{E}_{T^{l+1}\xi}$. By the triangular inequalities of $L^{p/2^j}$ under $\mathbb{P}_\xi$ and $\mathbb{P}_{T^{l+1}\xi}$, and inequalities (3.4.10), it holds that for all $l \geqslant 0$ and $1 \leqslant j \leqslant j_p$, $\mathbb{P}$-a.s.,

$$\begin{aligned} &\Big\{\mathbb{E}_{T^{l+1}\xi}\Big(\sigma_{l,1}\Big(\frac{p}{2^j}\Big)\lambda_l^{p/2^j-p}\Big)\Big\}^{2^j/p} \\ =\;& \Big\{\mathbb{E}_{T^{l+1}\xi}\Big(\max_{1\leqslant r\leqslant d}\mathbb{E}_\xi\Big|W^r_{l,k}-1\Big|^{p/2^j}\lambda_l^{p/2^j-p}\Big)\Big\}^{2^j/p} \\ \leqslant\;& \Big\{\mathbb{E}_{T^{l+1}\xi}\Big(\max_{1\leqslant r\leqslant d}\mathbb{E}_\xi\Big(W^r_{l,1}\Big)^{p/2^j}\lambda_l^{p/2^j-p}\Big)\Big\}^{2^j/p} + \Big(\mathbb{E}_{T^{l+1}\xi}\lambda_l^{p/2^j-p}\Big)^{2^j/p} \\ \leqslant\;& \Big\{\sum_{r=1}^d \mathbb{E}_{T^{l+1}\xi}\Big[\mathbb{E}_\xi\Big(W^r_{l,1}\Big)^{p/2^j}\lambda_l^{p/2^j-p}\Big]\Big\}^{2^j/p} + C\,\kappa\Big(\frac{p}{2^j}-p\Big)^{2^j/p}. \end{aligned}$$

Therefore, using inequality (3.4.23), condition (3.4.3) and again the triangular inequalities of $L^{p/2^j}$ under $\mathbb{P}_\xi$ and $\mathbb{P}_{T^{l+1}\xi}$, we see that for all $l \geqslant 0$ and $1 \leqslant j \leqslant j_p$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\left(\mathbb{E}_{T^{l+1}\xi}\left(\sigma_{l,1}\left(\frac{p}{2^j}\right)\lambda_l^{p/2^j-p}\right)\right)^{2^j/p} \\
\leqslant\ & d^{2^j/p}\max_{1\leqslant r\leqslant d}\left(\mathbb{E}_{T^{l+1}\xi}\left[\lambda_l^{-p}\mathbb{E}_\xi\left(\lambda_l W_{l,1}^r\right)^{p/2^j}\right]\right)^{2^j/p} + C \\
\leqslant\ & C\max_{1\leqslant r\leqslant d}\left(\mathbb{E}_{T^{l+1}\xi}\left[\lambda_l^{-p}\mathbb{E}_\xi\left(\lambda_l W_{l,1}^r \mathbb{1}_{\{\lambda_l W_{l,1}^r\leqslant 1\}}\right)^{p/2^j}\right]\right)^{2^j/p} \\
&+C\max_{1\leqslant r\leqslant d}\left(\mathbb{E}_{T^{l+1}\xi}\left[\lambda_l^{-p}\mathbb{E}_\xi\left(\lambda_l W_{l,1}^r \mathbb{1}_{\{\lambda_l W_{l,1}^r> 1\}}\right)^{p/2^j}\right]\right)^{2^j/p} + C \\
\leqslant\ & C\max_{1\leqslant r\leqslant d}\left(\mathbb{E}_{T^{l+1}\xi}\left[\lambda_l^{1-p}\mathbb{E}_\xi W_{l,1}^r\right]\right)^{2^j/p} + C\max_{1\leqslant r\leqslant d}\left(\mathbb{E}_{T^{l+1}\xi}\left(W_{l,1}^r\right)^p\right)^{2^j/p} + C.
\end{aligned}$$

We know that $\left(W_{l,k}^r\right)_{k\geqslant 0}$ is the martingale associated to a MBPRE starting with one individual of type r, in the shift random environment $T^l\xi$. In particular we have $\mathbb{E}_\xi W_{l,1}^r = 1$ a.s. Therefore, applying again (3.4.10), (3.4.23) and condition (3.4.3), it follows that for all $l \geqslant 0$ and $1 \leqslant j \leqslant j_p$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\left(\mathbb{E}_{T^{l+1}\xi}\left(\sigma_{l,1}\left(\frac{p}{2^j}\right)\lambda_l^{p/2^j-p}\right)\right)^{2^j/p} \\
\leqslant\ & C\max_{1\leqslant r\leqslant d}\left(\mathbb{E}_{T^{l+1}\xi}\lambda_l^{1-p}\right)^{2^j/p} + C\max_{1\leqslant r\leqslant d}\left(\mathbb{E}_{T^{l+1}\xi}\left(W_{l,1}^r\right)^p\right)^{2^j/p} + C \\
\leqslant\ & C\,\kappa(1-p)^{2^j/p} + C\max_{1\leqslant r\leqslant d}\left(\mathbb{E}_{T^{l+1}\xi}\left(W_{l,1}^r\right)^p\right)^{2^j/p} + C \\
\leqslant\ & C\left(1+\max_{1\leqslant r\leqslant d}\left(\mathbb{E}_{T^{l+1}\xi}\left(W_{l,1}^r\right)^p\right)^{2^j/p}\right). \qquad (3.4.24)
\end{aligned}$$

Then, by a similar calculation as in (3.4.13), for all $1 \leqslant r \leqslant d$ and $l \geqslant 0$, $\mathbb{P}$-a.s., we have

$$\begin{aligned}\Big(\mathbb{E}_{T^{l+1}\xi}(W^r_{l,1})^p\Big)^{1/p} &= \left(\mathbb{E}_{T^{l+1}\xi}\left(\frac{\langle N^r_l, U_{l+1,\infty}\rangle}{\lambda_l U_{l,\infty}(r)}\right)^p\right)^{1/p}\\ &= \left(\mathbb{E}_{T^{l+1}\xi}\left(\sum_{j=1}^d \frac{M_l(r,j)U_{l+1,\infty}(j)}{\lambda_l U_{l,\infty}(r)}\frac{N^r_l}{M_l(r,j)}\right)^p\right)^{1/p}\\ &\leqslant \sum_{j=1}^d \left(\mathbb{E}_{T^{l+1}\xi}\left(\frac{N^r_l(j)}{M_l(r,j)}\right)^p\right)^{1/p}\\ &= \sum_{j=1}^d \left(\mathbb{E}\left(\frac{Z^r_1(j)}{M_0(r,j)}\right)^p\right)^{1/p} < +\infty. \end{aligned} \tag{3.4.25}$$

Putting together (3.4.24) and (3.4.25), we get that for all $l \geqslant 0$ and $1 \leqslant j \leqslant j_p$, $\mathbb{P}$-a.s.,

$$\mathbb{E}_{T^{l+1}\xi}\left(\sigma_{l,1}\left(\frac{p}{2^j}\right)\lambda_l^{p/2^j - p}\right) \leqslant C. \tag{3.4.26}$$

Therefore, for all $n \geqslant 0$ and $0 \leqslant l \leqslant n-1$, $\mathbb{P}$-a.s., (the value of the constant C may change from line to line),

$$\begin{aligned}\mathbb{E}_{T^n\xi}\left[\sigma_{l,1}\left(\frac{p}{2^{j_p}}\right)\lambda_{l,n-1}^{p/2^{j_p}-p}\right] &= \mathbb{E}_{T^n\xi}\left[\mathbb{E}_{T^{l+1}\xi}\left(\sigma_{l,1}\left(\frac{p}{2^{j_p}}\right)\lambda_l^{p/2^{j_p}-p}\right)\lambda_{l+1,n-1}^{p/2^{j_p}-p}\right]\\ &\leqslant C\mathbb{E}_{T^n\xi}\lambda_{l+1,n-1}^{p/2^{j_p}-p}\\ &\leqslant C[\kappa(\frac{p}{2^{j_p}} - p)]^{n-1-l}\\ &\leqslant C\delta_c(p)^{(n-1-l)p},\end{aligned}$$

where the last two inequalities hold by (3.4.10) and (3.4.23). Combining this with (3.4.22) and (3.4.23), we obtain that for all $1 \leqslant i \leqslant d$ and $n \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\Big(\mathbb{E}_{T^n\xi}a^i_{n,j_p}(p)\Big)^{2^{j_p}/p} &\leqslant C\delta_c(p)^{n2^{j_p}}\\ &\leqslant C\delta_c(p)^{n2^{j_p}} + C\sum_{l=0}^{n-1}\Big(\delta_c(p)^{lp}\delta_c(p)^{(n-1-l)p}\Big)^{2^{j_p}/p}\\ &\leqslant C(1+\delta_c(p)^{-2^{j_p}}n)\delta_c(p)^{n2^{j_p}}.\end{aligned}$$

So (3.4.19) holds for $j = j_p$.

Now suppose that (3.4.19) holds for $j+1 \leqslant j_p$ with $j \geqslant 1$. We will prove that it still holds for j. By recurrence this will prove that (3.4.19) holds for all $j = 1, \cdots, j_p$. Since

$j+1$ satisfies (3.4.19), for all $n \geqslant 0$, $1 \leqslant i \leqslant d$, and $\delta > \delta_c(p)$,

$$\delta^{-n}\Big(\mathbb{E}_{T^n\xi} a^i_{n,j+1}(p)\Big)^{1/p} \leqslant C \quad \mathbb{P}\text{-a.s.} \tag{3.4.27}$$

By definition of j_p we have $p/2^j > 2$. Corresponding to (3.4.22), with the same argument as in its proof but applying (3.4.16) instead of (3.4.8), we obtain that for all $1 \leqslant i \leqslant d$ and $n \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned} &\Big(\mathbb{E}_{T^n\xi} a^i_{n,j}(p)\Big)^{2^j/p} \\ \leqslant\ & C\,\kappa\Big(\frac{p}{2^j}-p\Big)^{n2^j/p} + C\times \\ & \sum_{l=0}^{n-1}\Big\{\mathbb{E}_{T^n\xi}\Big[\sigma_{l,1}\Big(\frac{p}{2^j}\Big)\mathbb{E}_\xi(W_l^i)^{\frac{p}{2^{j+1}}}(\lambda_{0,l-1}U_{0,\infty}(i))^{\frac{p}{2^{j+1}}-p}\lambda_{l,n-1}^{p/2^j-p}\Big]\Big\}^{2^j/p} \\ =\ & C\,\kappa\Big(\frac{p}{2^j}-p\Big)^{n2^j/p} + C\sum_{l=0}^{n-1}\Big(\mathbb{E}_{T^n\xi}\Big[\sigma_{l,1}\Big(\frac{p}{2^j}\Big)\lambda_{l,n-1}^{p/2^j-p}\mathbb{E}_{T^l\xi}a^i_{l,j+1}(p)\Big]\Big)^{2^j/p}. \end{aligned} \tag{3.4.28}$$

This enables us to obtain the desired bound of $\mathbb{E}_{T^n\xi}a^i_{n,j}(p)$ from that of $\mathbb{E}_{T^n\xi}a^i_{n,j+1}(p)$. In fact, combining this with the recurrence hypothesis (3.4.27), together with (3.4.23), (3.4.26) and (3.4.10), we obtain that for all $n \geqslant 0$, $1 \leqslant i \leqslant d$, and $\delta > \delta_c(p)$,

$$\begin{aligned} \Big(\mathbb{E}_{T^n\xi}a^i_{n,j}(p)\Big)^{2^j/p} &\leqslant C\,\delta_c(p)^{n2^j} + C\times \\ &\quad \sum_{l=0}^{n-1}\delta^{l2^j}\Big\{\mathbb{E}_{T^n\xi}\Big[\mathbb{E}_{T^{l+1}\xi}\Big[\sigma_{l,1}\Big(\frac{p}{2^j}\Big)\lambda_l^{p/2^j-p}\Big]\lambda_{l+1,n-1}^{p/2^j-p}\Big]\Big\}^{2^j/p} \\ &\leqslant C\,\delta_c(p)^{n2^j} + C\sum_{l=0}^{n-1}\delta^{l2^j}\kappa\Big(\frac{p}{2^j}-p\Big)^{(n-1-l)2^j/p} \\ &\leqslant C\Big(\delta_c(p)^{n2^j} + n\delta^{(n-1)2^j}\Big) \\ &\leqslant C\Big(1+\delta^{-2^j}n\Big)\delta^{n2^j}. \end{aligned} \tag{3.4.29}$$

So (3.4.19) also holds for j. Therefore, by recurrence, we have proved that (3.4.19) holds for all $j = 1, \cdots, j_p$.

Step 4. For any $p > 1$, we prove the implication (3.4.1) $\Rightarrow$ (3.4.2) of Theorem 3.4.1. Assume $p > 1$ and condition (3.4.1), that is, the martingale $(W_n^i)_{n\geqslant 0}$ converges in L^p, for all $1 \leqslant i \leqslant d$. In particular we have $\mathbb{E}(W_1^i)^p < +\infty$, $\mathbb{E}(W^i)^p < +\infty$, and $\mathbb{E}(W^i) = 1$ for all $1 \leqslant i \leqslant d$. It was observed in [32, Theorem 2.6] that $\mathbb{E}_\xi(W^i) = 1$ a.s. when W^i are non-degenerate. In fact $\mathbb{E}_\xi(W^i) = 1$ a.s. whenever $\mathbb{E}(W^i) = 1$, because $\mathbb{E}_\xi(W^i) \leqslant 1$ by Fatou's lemma.

Notice that for all $1 \leqslant i \leqslant d$, the limit variables W^i satisfy

$$W^i = \sum_{j=1}^{d} \frac{U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)} \sum_{l=1}^{Z_1^j} W^j(l, T\xi) \quad \mathbb{P}\text{-a.s.},$$

where for all $1 \leqslant j \leqslant d$, under the conditional law $\mathbb{P}_\xi$, $\Big(W^j(l, T\xi)\Big)_{l \geqslant 1}$ is a sequence of i.i.d. random variables, also independent of Z_1^j, with common distribution $\mathbb{P}_\xi(W^j(l, T\xi) \in \cdot) = \mathbb{P}_{T\xi}(W^j \in \cdot)$. So, by the strict sub-additivity of the function $x \mapsto x^p$ on $\mathbb{R}_+^*$, we get that for all $1 \leqslant i \leqslant d$,

$$\mathbb{E}_\xi(W^i)^p > \sum_{j=1}^{d} \frac{M_0(i,j) U_{1,\infty}(j)^p}{\lambda_0^p U_{0,\infty}(i)^p} \mathbb{E}_{T\xi}(W^j)^p \quad \mathbb{P}\text{-a.s.},$$

using the fact that $\mathbb{E}_\xi(W^i) = 1$ a.s. This, together with (3.1.10), implies that for all $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\mathbb{E}_\xi(W^i)^p U_{0,\infty}(i)^{p-1} &> \lambda_0^{1-p} \sum_{j=1}^{d} \frac{M_0(i,j) U_{1,\infty}(j)}{\lambda_0 U_{0,\infty}(i)} \mathbb{E}_{T\xi}(W^j)^p U_{1,\infty}(j)^{p-1} \\
&> \lambda_0^{1-p} \min_{1 \leqslant j \leqslant d} \Big(\mathbb{E}_{T\xi}(W^j)^p U_{1,\infty}(j)^{p-1} \Big).
\end{aligned}$$

Therefore we obtain

$$\min_{1 \leqslant i \leqslant d} \Big(\mathbb{E}_\xi(W^i)^p U_{0,\infty}(i)^{p-1} \Big) > \lambda_0^{1-p} \min_{1 \leqslant i \leqslant d} \Big(\mathbb{E}_{T\xi}(W^i)^p U_{1,\infty}(i)^{p-1} \Big) \quad \mathbb{P}\text{-a.s.} \tag{3.4.30}$$

On the other hand, by (3.3.1) and (3.1.10), the transfer operator P_{1-p} satisfies the following property: for all $\varphi \in \mathcal{C}(\mathcal{S})$, $\mathbb{P}$-a.s.,

$$P_{1-p}\varphi(U_{1,\infty}) = \mathbb{E}_{T\xi}\Big[\|M_0 U_{1,\infty}\|^{1-p} \varphi(M_0 \cdot U_{1,\infty}) \Big] = \mathbb{E}_{T\xi}\Big[\lambda_0^{1-p} \varphi(U_{0,\infty}) \Big]. \tag{3.4.31}$$

Using (3.4.31) with $\varphi = r_{1-p}$, and combining this with Proposition 3.3.1, we get

$$\mathbb{E}_{T\xi}\Big[\lambda_0^{1-p} r_{1-p}(U_{0,\infty}) \Big] = \kappa(1-p) r_{1-p}(U_{1,\infty}) \quad \mathbb{P}\text{-a.s.} \tag{3.4.32}$$

Moreover, by Proposition 3.3.1 we know that r_{1-p} is a strictly positive continuous function on $\mathcal{S}$. This implies that

$$0 < \mathbb{E}\Big[\min_{1\leqslant i\leqslant d}\Big(\mathbb{E}_\xi (W^i)^p U_{0,\infty}(i)^{p-1}\Big) r_{1-p}(U_{0,\infty})\Big] \leqslant \|r_{1-p}\|_\infty \min_{1\leqslant i\leqslant d} \mathbb{E}(W^i)^p < +\infty. \tag{3.4.33}$$

Therefore, putting together (3.4.30), (3.4.32) and (3.4.33), we obtain

$$\begin{aligned}
& \mathbb{E}\Big[\min_{1\leqslant i\leqslant d}\Big(\mathbb{E}_\xi (W^i)^p U_{0,\infty}(i)^{p-1}\Big) r_{1-p}(U_{0,\infty})\Big] \\
> \ & \mathbb{E}\Big[\mathbb{E}_{T\xi}\Big[\lambda_0^{1-p} r_{1-p}(U_{0,\infty})\Big] \min_{1\leqslant i\leqslant d}\Big(\mathbb{E}_{T\xi}(W^i)^p U_{1,\infty}(i)^{p-1}\Big)\Big] \\
= \ & \kappa(1-p)\mathbb{E}\Big[\min_{1\leqslant i\leqslant d}\Big(\mathbb{E}_{T\xi}(W^i)^p U_{1,\infty}(i)^{p-1}\Big) r_{1-p}(U_{1,\infty})\Big] \\
= \ & \kappa(1-p)\mathbb{E}\Big[\min_{1\leqslant i\leqslant d}\Big(\mathbb{E}_{\xi}(W^i)^p U_{0,\infty}(i)^{p-1}\Big) r_{1-p}(U_{0,\infty})\Big],
\end{aligned}$$

so $\kappa(1-p) < 1$. This ends the proof of (3.4.1) $\Rightarrow$ (3.4.2).

Step 5. To conclude the proof, it remains to show that under the Furstenberg-Kesten condition **M1** we have (3.4.2) $\Rightarrow$ (3.4.3) for all $p > 1$. By (3.4.5) in Lemma 3.4.3, we know that, under **M1**, we have for all $n \geqslant 0$ and $1 \leqslant i, j \leqslant d$

$$\frac{1}{dD^2} \leqslant \frac{M_{0,n-1}(i,j) U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \leqslant 1 \quad \mathbb{P}\text{-a.s.}$$

Therefore we obtain that for all $n \geqslant 0$ and $1 \leqslant i, j \leqslant d$,

$$\begin{aligned}
\frac{Z_n^i(j)}{M_{0,n-1}(i,j)} &\leqslant dD^2 \frac{M_{0,n-1}(i,j) U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \\
&= dD^2 \frac{Z_n^i(j) U_{n,\infty}(j)}{\lambda_{0,n-1} U_{0,\infty}(i)} \leqslant dD^2 W_n^i.
\end{aligned} \tag{3.4.34}$$

The implication (3.4.2) $\Rightarrow$ (3.4.3) follows from (3.4.34) with $n = 1$.

This ends the proof of Theorems 3.4.1 and 3.2.2. □

3.5 Convergence in L^p of the normalized population size $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$

In this section we give proof of Theorems 3.2.3 and 3.2.4 about the convergence in L^p of $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$ and its exponential convergence rate, under the Furstenberg-Kesten condition

M1.

3.5.1 Auxiliary results

We need some preliminary results concerning the products of random matrices $M_{n,n+k}$. The following proposition was established by Hennion in [40, Theorem 1], which provides an analog of the Perron-Frobenius theorem for products of random matrices.

Proposition 3.5.1. *Assume that $M_0 > 0$ $\mathbb{P}$-a.s. Then for all $n \geqslant 0$ and $1 \leqslant i, j \leqslant d$, as $k \to +\infty$, $\mathbb{P}$-a.s.,*

$$M_{n,n+k}(i,j) \underset{k\to+\infty}{\sim} \rho_{n,n+k} U_{n,n+k}(i) V_{n,n+k}(j).$$

For $1 \leqslant i \leqslant d$, let $(\Pi_n^i)_{n\geqslant 0}$ be the sequence of random matrices in $\mathcal{M}_d(\mathbb{R})$ such that for all $1 \leqslant j, r \leqslant d$,

$$\Pi_0^i(j,r) := \delta_{i,r}, \quad \Pi_n^i(j,r) := \frac{M_{0,n-1}(i,r)M_n(r,j)}{M_{0,n}(i,j)}, \quad n \geqslant 1.$$

By definition all the entries of the i-th column of Π_0^i are equal to 1, the others are 0; each Π_n^i is a stochastic matrix. For $n, k \geqslant 0$ let

$$\Pi_{n+k,n}^i := \Pi_{n+k}^i \cdots \Pi_n^i$$

be the products of the matrices Π_n^i. Clearly each $\Pi_{n+k,n}^i$ is a non-negative stochastic random matrix. The following lemma concerns the convergence of the products $\Pi_{n+k,n}^i$ of random matrices and its exponential rate as $k \to +\infty$, which will be very useful for the study of the L^p convergence of the normalized population size $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)}$.

Lemma 3.5.2. *Assume the Furstenberg-Kesten condition **M1**. Then for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, as $k \to +\infty$, the sequence $(\Pi_{n+k,n}^i)_{k\geqslant 0}$ converges $\mathbb{P}$-a.s. to the random matrix $\Pi_{\infty,n}^i$ such that:*

(1) For all $1 \leqslant j, r \leqslant d$,

$$\Pi_{\infty,0}^i(j,r) := \Pi_0^i(j,r), \quad \Pi_{\infty,n}^i(j,r) := \frac{M_{0,n-1}(i,r)U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)}, \ n \geqslant 1.$$

(2) There exist $C > 0$ and $\delta \in (0,1)$ such that for all $k \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{n\geqslant 0} \|\Pi^i_{n+k,n} - \Pi^i_{\infty,n}\| \leqslant C\delta^k \quad \mathbb{P}\text{-}a.s. \tag{3.5.1}$$

For the proof of Lemma 3.5.2, we will use the following result established by Seneta [64, Theorem 4.19], which gives the convergence of products of stochastic matrices with an exponential rate.

Lemma 3.5.3 ([64])**.** *Assume that $(P_n)_{n\geqslant 0}$ is a sequence of stochastic matrices such that for some $\varepsilon \in (0,1)$, all $n \geqslant 0$ and all $1 \leqslant i,j \leqslant d$,*

$$P_n(i,j) \geqslant \varepsilon.$$

Then, for all $n \geqslant 0$ the product $P_{n+k,n} := P_{n+k}\cdots P_n$ converges as $k \to +\infty$ to some matrix $P_{\infty,n}$, and there exist two constants $C > 0$ and $\delta \in (0,1)$ depending only on ε such that

$$\sup_{n\geqslant 0} \left\| P_{n+k,n} - P_{\infty,n} \right\| \leqslant C\delta^k.$$

Proof of Lemma 3.5.2. (1) First we show by induction on k that for all $n \geqslant 1$, $k \geqslant 0$ and $1 \leqslant i,j,r \leqslant d$,

$$\Pi^i_{n+k,n}(j,r) = \frac{M_{0,n-1}(i,r)M_{n,n+k}(r,j)}{M_{0,n+k}(i,j)}. \tag{3.5.2}$$

Obviously, by definition of Π^i_n, (3.5.2) holds for $k = 0$. Assume that (3.5.2) holds for some $k \geqslant 0$. Then, for all $n \geqslant 1$ and $1 \leqslant i,j,r \leqslant d$ we have

$$\begin{aligned}
\Pi^i_{n+k+1,n}(j,r) &= \sum_{l=1}^d \Pi^i_{n+k+1}(j,l)\Pi^i_{n+k,n}(l,r) \\
&= \sum_{l=1}^d \frac{M_{0,n+k}(i,l)M_{n+k+1}(l,j)}{M_{0,n+k+1}(i,j)} \frac{M_{0,n-1}(i,r)M_{n,n+k}(r,l)}{M_{0,n+k}(i,l)} \\
&= \frac{M_{0,n-1}(i,r)}{M_{0,n+k+1}(i,j)} \sum_{l=1}^d M_{n,n+k}(r,l)M_{n+k+1}(l,j) \\
&= \frac{M_{0,n-1}(i,r)M_{n,n+k+1}(r,j)}{M_{0,n+k+1}(i,j)}.
\end{aligned}$$

So (3.5.2) holds for $k+1$. Therefore by reduction (3.5.2) holds for all $k \geqslant 0$.

Combining (3.5.2) with Proposition 3.5.1, we deduce that for all $n \geqslant 1$ and $1 \leqslant i, j, r \leqslant d$, $\mathbb{P}$-a.s. as $k \to +\infty$,

$$\begin{aligned}
\Pi^i_{n+k,n}(j,r) &\sim \frac{M_{0,n-1}(i,r)\rho_{n,n+k}U_{n,n+k}(r)V_{n,n+k}(j)}{\rho_{0,n+k}U_{0,n+k}(i)V_{0,n+k}(j)} \\
&\sim \frac{\rho_{n,n+k}V_{n,n+k}(j)}{\rho_{0,n+k}V_{0,n+k}(j)} \frac{M_{0,n-1}(i,r)U_{n,\infty}(r)}{U_{0,\infty}(i)} \\
&= \Big(\prod_{l=0}^{n-1} \frac{\rho_{l+1,n+k}V_{l+1,n+k}(j)}{\rho_{l,n+k}V_{l,n+k}(j)} \Big) \frac{M_{0,n-1}(i,r)U_{n,\infty}(r)}{U_{0,\infty}(i)}. \qquad (3.5.3)
\end{aligned}$$

By [32, Proposition 2.2] we know that for all $l, n \geqslant 0$ and $1 \leqslant j \leqslant d$,

$$\lambda_l = \lim_{k\to+\infty} \frac{\rho_{l,n+k}V_{l,n+k}(j)}{\rho_{l+1,n+k}V_{l+1,n+k}(j)} \quad \mathbb{P}\text{-a.s.}$$

This, together with (3.5.3), implies that for $n \geqslant 1$ and $1 \leqslant j, r \leqslant d$, as $k \to +\infty$,

$$\Pi^i_{n+k,n}(j,r) \to \Pi^i_{\infty,n}(j,r) = \frac{M_{0,n-1}(i,r)U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)} \quad \mathbb{P}\text{-a.s.}$$

Hence, as $k \to \infty$, $\mathbb{P}$-a.s., $\Pi^i_{k,0} = \Pi^i_{k,1}\Pi^i_0 \to \Pi^i_{\infty,1}\Pi^i_0 = \Pi^i_{\infty,0}$, where

$$\Pi^i_{\infty,0}(j,r) = \sum_{l=1}^d \Pi^i_{\infty,1}(j,l)\Pi^i_0(l,r) = \sum_{l=1}^d \Pi^i_{\infty,1}(j,l)\delta_{i,r} = \delta_{i,r}, \ 1 \leqslant j, r \leqslant d.$$

(2) By (3.4.4) in Lemma 3.4.3, we have, for all $n \geqslant 1$ and $1 \leqslant i, j, r \leqslant d$,

$$\frac{1}{\Pi^i_n(j,r)} = \sum_{l=1}^d \frac{M_{0,n-1}(i,l)M_n(l,j)}{M_{0,n-1}(i,r)M_n(r,j)} \leqslant dD^2 \qquad \mathbb{P}\text{-a.s.},$$

or equivalently

$$\Pi^i_n(j,r) \geqslant \frac{1}{dD^2} \quad \mathbb{P}\text{-a.s.} \qquad (3.5.4)$$

Since $(\Pi^i_n)_{n\geqslant 0}$ is a sequence of positive stochastic matrices satisfying (3.5.4), by Lemma 3.5.3, there exist two constants $C > 0$ and $\delta \in (0,1)$ such that for all $k \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{n\geqslant 0} \|\Pi^i_{n+k,n} - \Pi^i_{\infty,n}\| \leqslant C\delta^k, \quad \mathbb{P}\text{-a.s.}$$

This concludes the proof of Lemma 3.5.2. □

3.5.2 Proof of Theorems 3.2.3 and 3.2.4

For all $n \geqslant 0$ and $1 \leqslant i, j \leqslant d$, set

$$\overline{Z}_n^i(j) := \frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)}.$$

First we show that (3.2.9) is a necessary condition for the convergence in L^p of the normalized population size $\overline{Z}_n^i(j)$, $1 \leqslant i, j \leqslant d$. Assume that $(\overline{Z}_n^i(j))_{n\geqslant 0}$ converges in L^p for all $1 \leqslant i, j \leqslant d$. For $n \geqslant 0$ and $1 \leqslant i \leqslant d$ we have, by the definition of W_n^i and (3.1.11),

$$W_n^i = \sum_{j=1}^d \frac{M_{0,n-1}(i,j)U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \overline{Z}_n^i(j) \leqslant \max_{1\leqslant j\leqslant d} \overline{Z}_n^i(j).$$

This implies that the martingale $(W_n^i)_{n\geqslant 0}$, $1 \leqslant i \leqslant d$, is bounded in L^p, hence convergences in L^p. So by Theorem 3.2.1, condition (3.2.9) holds.

Now we prove that (3.2.9) is sufficient for the convergence in L^p of $\overline{Z}_n^i(j)$, $1 \leqslant i, j \leqslant d$, and establish meanwhile Theorem 3.2.4 about its convergence rate. Assume (3.2.9). By the definition of the branching process (Z_n^i) (cf. (3.4.6)), we have the following decomposition: for all $1 \leqslant i, j \leqslant d$ and $n, k \geqslant 1$,

$$\begin{aligned}\overline{Z}_{n+k}^i(j) &= \sum_{r=1}^d \frac{M_{n,n+k-1}(r,j)}{M_{0,n+k-1}(i,j)} \sum_{l=1}^{Z_n^i(r)} \frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} \\ &= \sum_{r=1}^d \frac{M_{n,n+k-1}(r,j)}{M_{0,n+k-1}(i,j)} Z_n^i(r) \\ &\quad + \sum_{r=1}^d \frac{M_{n,n+k-1}(r,j)}{M_{0,n+k-1}(i,j)} \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right). \end{aligned} \tag{3.5.5}$$

Combining (3.5.5) and (3.5.2), we get that for all $1 \leqslant i, j \leqslant d$ and $n, k \geqslant 1$,

$$\begin{aligned}\overline{Z}^i_{n+k}(j) &= \sum_{r=1}^{d} \Pi^i_{n+k-1,n}(j,r)\overline{Z}^i_n(r) \\ &\quad + \sum_{r=1}^{d} \frac{\Pi^i_{n+k-1,n}(j,r)}{M_{0,n-1}(i,r)} \sum_{l=1}^{Z^i_n(r)} \left(\frac{Z^r_{l,n,k}(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \\ &= \left\langle \Pi^i_{n+k-1,n}\overline{Z}^i_n, e_j \right\rangle + R^i_{n,k}(j), \end{aligned} \tag{3.5.6}$$

with

$$R^i_{n,k}(j) := \sum_{r=1}^{d} \frac{\Pi^i_{n+k-1,n}(j,r)}{M_{0,n-1}(i,r)} \sum_{l=1}^{Z^i_n(r)} \left(\frac{Z^r_{l,n,k}(j)}{M_{n,n+k-1}(r,j)} - 1 \right).$$

Notice that by the definition of W^i_n and that of $\Pi^i_{\infty,n}$ (cf. Lemma 3.5.2 (1)),

$$W^i_n = \sum_{r=1}^{d} \Pi^i_{\infty,n}(j,r)\overline{Z}^i_n(r) = \left\langle \Pi^i_{\infty,n}\overline{Z}^i_n, e_j \right\rangle \tag{3.5.7}$$

for any $1 \leqslant i, j \leqslant d$. Using (3.5.6) and (3.5.7), together with the triangular inequality in L^p under $\mathbb{P}$, we obtain that for all $1 \leqslant i, j \leqslant d$ and $n, k \geqslant 1$,

$$\begin{aligned} &\left(\mathbb{E}\left|\overline{Z}^i_{n+k}(j) - W^i\right|^p\right)^{1/p} \\ = \; &\left(\mathbb{E}\left|\left\langle \Pi^i_{n+k-1,n}\overline{Z}^i_n, e_j \right\rangle - W^i + R^i_{n,k}(j)\right|^p\right)^{1/p} \\ = \; &\left(\mathbb{E}\left|W^i_n - W^i + \left\langle (\Pi^i_{n+k-1,n} - \Pi^i_{\infty,n})\overline{Z}^i_n, e_j \right\rangle + R^i_{n,k}(j)\right|^p\right)^{1/p} \\ \leqslant \; &\left(\mathbb{E}|W^i_n - W^i|^p\right)^{1/p} + \left(\mathbb{E}\|(\Pi^i_{n+k-1,n} - \Pi^i_{\infty,n})\overline{Z}^i_n\|^p\right)^{1/p} \\ &+ \max_{1\leqslant j\leqslant d}\left(\mathbb{E}|R^i_{n,k}(j)|^p\right)^{1/p} \\ = \; &J^i_1(n) + J^i_2(n,k) + J^i_3(n,k). \end{aligned} \tag{3.5.8}$$

In the following $C > 0$ will be a constant which may depend on p and which may differ from line to line.

Control of $J^i_1(n)$. By condition (3.2.9) and Theorem 3.2.2 we get that there exists

$\delta_1 \in (\delta_c(p), 1)$ such that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$J_1^i(n) = \Big(\mathbb{E}|W_n^i - W^i|^p\Big)^{1/p} \leqslant C\delta_1^n. \tag{3.5.9}$$

Control of $J_2^i(n,k)$. Applying the relation (3.5.1) of Lemma 3.5.2, we get that there exists $\delta_2 \in (0,1)$ such that for all $n, k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$J_2^i(n,k) = \Big(\mathbb{E}\|(\Pi^i_{n+k-1,n} - \Pi^i_{\infty,n})\overline{Z}_n^i\|^p\Big)^{1/p} \leqslant C\Big(\mathbb{E}\|\overline{Z}_n^i\|^p\Big)^{1/p}\delta_2^k. \tag{3.5.10}$$

Combining (3.4.34) and Theorem 3.2.1, and using condition (3.2.9), we obtain that, for all $1 \leqslant i \leqslant d$,

$$\sup_{n\geqslant 0}\Big(\mathbb{E}\|\overline{Z}_n^i\|^p\Big)^{1/p} \leqslant d^2D^2 \sup_{n\geqslant 0}\Big(\mathbb{E}(W_n^i)^p\Big)^{1/p} < +\infty. \tag{3.5.11}$$

This, together with (3.5.10), implies that for all $n, k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$J_2^i(n,k) \leqslant C\delta_2^k. \tag{3.5.12}$$

Control of $J_3^i(n,k)$ for $1 < p \leqslant 2$. Assume that $1 < p \leqslant 2$. Using the convexity of the function $x \mapsto x^p$ (together with $\sum_{r=1}^d \Pi^i_{n+k-1,n}(j,r) = 1$) and Lemma 3.4.2, for all $n, k \geqslant 1$ and $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-a.s., we have

$$\begin{aligned}
\mathbb{E}_\xi|R^i_{n,k}(j)|^p &\leqslant \mathbb{E}_\xi\left(\sum_{r=1}^d \frac{\Pi^i_{n+k-1,n}(j,r)}{M_{0,n-1}(i,r)}\left|\sum_{l=1}^{Z_n^i(r)}\left(\frac{Z^r_{l,n,k}(j)}{M_{n,n+k-1}(r,j)} - 1\right)\right|\right)^p \\
&\leqslant \mathbb{E}_\xi\left(\sum_{r=1}^d \frac{\Pi^i_{n+k-1,n}(j,r)}{M_{0,n-1}(i,r)^p}\left|\sum_{l=1}^{Z_n^i(r)}\left(\frac{Z^r_{l,n,k}(j)}{M_{n,n+k-1}(r,j)} - 1\right)\right|^p\right) \\
&\leqslant B_p^p \sum_{r=1}^d \frac{\Pi^i_{n+k-1,n}(j,r)}{M_{0,n-1}(i,r)^p}\mathbb{E}_\xi Z_n^i(r)\mathbb{E}_\xi\left|\frac{Z^r_{1,n,k}(j)}{M_{n,n+k-1}(r,j)} - 1\right|^p \\
&= B_p^p \sum_{r=1}^d \Pi^i_{n+k-1,n}(j,r)M_{0,n-1}(i,r)^{1-p}\mathbb{E}_\xi\left|\frac{Z^r_{1,n,k}(j)}{M_{n,n+k-1}(r,j)} - 1\right|^p \\
&\leqslant B_p^p\overline{\sigma}_{n,k}(p)\max_{1\leqslant r\leqslant d} M_{0,n-1}(i,r)^{1-p},
\end{aligned}$$

where

$$\overline{\sigma}_{n,k}(p) = \max_{1\leqslant r,j\leqslant d}\mathbb{E}_\xi\left|\frac{Z^r_{1,n,k}(j)}{M_{n,n+k-1}(r,j)} - 1\right|^p.$$

So, by taking expectation and using the independence between $\overline{\sigma}_{n,k}(p)$ and $M_{0,n-1}$, we get that for all $n, k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$J_3^i(n,k)^p \leqslant B_p^p \mathbb{E}\overline{\sigma}_{0,k}(p) \sum_{r=1}^{d} \mathbb{E}\Big[M_{0,n-1}(i,r)^{1-p}\Big]. \tag{3.5.13}$$

By (3.5.11) we have

$$\sup_{k\geqslant 0} \mathbb{E}\overline{\sigma}_{0,k}(p) \leqslant d^2 \max_{1\leqslant r,j\leqslant d} \sup_{k\geqslant 0} \mathbb{E}\Big|\overline{Z}_k^r(j) - 1\Big|^p < +\infty. \tag{3.5.14}$$

Therefore, putting together the relations (3.5.13) and (3.5.14) with Lemma 3.3.2, we get that for $n, k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} J_3^i(n,k)^p &\leqslant C \sup_{k\geqslant 0} \mathbb{E}\overline{\sigma}_{0,k}(p) \sum_{r=1}^{d} \mathbb{E}\Big[\langle M_{0,n-1}e_r, e_i\rangle^{1-p}\Big] \\ &\leqslant C\kappa(1-p)^n \\ &\leqslant C\delta_c(p)^{np} \end{aligned} \tag{3.5.15}$$

(recall that the value of C may change from line to line by our convention).

Control of $J_3^i(n,k)$ for $p \geqslant 2$. Assume that $p \geqslant 2$. Similar to the preceding case, by the convexity of $x \mapsto x^p$ (together with $\sum_{r=1}^d \Pi_{n+k-1,n}^i(j,r) = 1$) and Lemma 3.4.2, for all $n, k \geqslant 1$ and $1 \leqslant i, j \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \mathbb{E}_\xi |R_{n,k}^i(j)|^p &\leqslant \mathbb{E}_\xi \left(\sum_{r=1}^{d} \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)} \left| \sum_{l=1}^{Z_n^i(r)} \left(\frac{Z_{l,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right) \right| \right)^p \\ &\leqslant B_p^p \sum_{r=1}^{d} \frac{\Pi_{n+k-1,n}^i(j,r)}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi (Z_n^i(r))^{p/2} \mathbb{E}_\xi \left| \frac{Z_{1,n,k}^r(j)}{M_{n,n+k-1}(r,j)} - 1 \right|^p \\ &\leqslant B_p^p \overline{\sigma}_{n,k}(p) \sum_{r=1}^{d} \Pi_{n+k-1,n}^i(j,r) \mathbb{E}_\xi \Big(\overline{Z}_n^i(r)\Big)^{p/2} M_{0,n-1}(i,r)^{-p/2} \\ &\leqslant B_p^p \overline{\sigma}_{n,k}(p) \max_{1\leqslant r\leqslant d} \Big\{ \mathbb{E}_\xi \Big(\overline{Z}_n^i(r)\Big)^{p/2} M_{0,n-1}(i,r)^{-p/2} \Big\}. \end{aligned} \tag{3.5.16}$$

Notice that (3.5.14) still holds when $p \geqslant 2$. Therefore, taking expectation in (3.5.16) and using (3.5.14), we obtain that for all $n, k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$J_3^i(n,k)^p \leqslant B_p^p \mathbb{E}\overline{\sigma}_{0,k}(p) \sum_{1\leqslant r\leqslant d} \mathbb{E}\Big[\mathbb{E}_\xi\Big(\overline{Z}_n^i(r)\Big)^{p/2} M_{0,n-1}(i,r)^{-p/2}\Big]$$
$$\leqslant C\sum_{r=1}^{d} \mathbb{E}\Big[\mathbb{E}_\xi\Big(\overline{Z}_n^i(r)\Big)^{p/2} M_{0,n-1}(i,r)^{-p/2}\Big]. \tag{3.5.17}$$

Using (3.4.4) in Lemma 3.4.3 and (3.1.11), for all $n \geqslant 1$ and $1 \leqslant i, r \leqslant d$, $\mathbb{P}$-a.s., we have

$$\begin{aligned} M_{0,n-1}(i,r) &\geqslant \frac{1}{dD}\|M_{0,n-1}(i,\cdot)\| \\ &\geqslant \frac{1}{dD}\langle M_{0,n-1}(i,\cdot), U_{n,\infty}\rangle \\ &= \frac{1}{dD}\lambda_{0,n-1}U_{0,\infty}(i). \end{aligned}$$

Combining this with (3.5.17) and (3.4.34), we get that for $p > 2$, $n, k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$J_3^i(n,k)^p \leqslant C\mathbb{E}\Big[\mathbb{E}_\xi(W_n^i)^{p/2}(\lambda_{0,n-1}U_{0,\infty}(i))^{-p/2}\Big] = C\mathbb{E}a_{n,1}^i(p),$$

where $a_{n,1}^i(p)$ is defined in (3.4.17) with $j = 1$. This, together with (3.4.19) (which holds under condition (3.2.9)), implies that there exists $\delta_3 \in (\delta_c(p), 1)$ such that for all $n, k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$J_3^i(n,k)^p \leqslant C\delta_3^{np}. \tag{3.5.18}$$

Combining (3.5.8), (3.5.9), (3.5.12), (3.5.15) and (3.5.18), we obtain that for all $n, k \geqslant 1$ and $1 \leqslant i, j \leqslant d$,

$$\Big(\mathbb{E}\Big|\overline{Z}_{n+k}^i(j) - W^i\Big|^p\Big)^{1/p} \leqslant C(\delta_1^n + \delta_2^k + \delta_3^n).$$

Applying this inequality with n replaced by $\lfloor n/2 \rfloor$ (the integral part of $n/2$) and taking $k = n - \lfloor n/2 \rfloor$, we see that for all $n \geqslant 1$ and $1 \leqslant i, j \leqslant d$,

$$\Big(\mathbb{E}\Big|\overline{Z}_n^i(j) - W^i\Big|^p\Big)^{1/p} \leqslant C(\delta_1^{n/2} + \delta_2^{n/2} + \delta_3^{n/2}) \leqslant C\delta^n,$$

with $\delta = \max\{\delta_1^{1/2}, \delta_2^{1/2}, \delta_3^{1/2}\} < 1$. Therefore, for any $1 \leqslant i, j \leqslant d$ the normalized population size $\overline{Z}_n^i(j)$ convergences in L^p to W^i with an exponential speed, which gives

(3.2.11). This concludes the proof of Theorems 3.2.3 and 3.2.4.

Chapter 4

Berry-Esseen's bound and harmonic moments for supercritical multi-type branching processes in random environments

Résumé. On considère un processus de branchement multi-type surcritique dans un environnement aléatoire indépendant et identiquement distribué. On établit une borne de type Berry-Esseen pour la vitesse de convergence dans le théorème central limite pour la taille de la population au temps n quand n tend vers l'infini. Pour cela, on commence par trouver des conditions simples pour l'existence des moments harmoniques de la variable aléatoire limite de la martingale fondamentale.

Abstract. Consider a supercritical multi-type branching process in an independent and identically distributed random environment. We establish a Berry-Esseen type bound for the rate of convergence in the central limit theorem on the population size at time n as n goes to infinity. To this end we first find simple conditions for the existence of harmonic moments of the limit variable of the fundamental martingale.

4.1 Introduction

A branching process in a random environment is a natural and important extension of the Galton-Watson process. In such a process, the offspring distributions of particles in n-th generation depend on an environment ξ_n at time n. This process was first introduced by Smith and Wilkinson [65] when the environment sequence (ξ_n) is independent and identically distributed, and by Athreya and Karlin [5, 6] when the environment sequence is stationary and ergodic, where basic results have been established. This process has attracted the attention of many authors in the last two decades, see for example the recent book by Kersting and Vatutin [50] and many references therein. The interest of study of such processes is growing in recent years, thanks to a large number of applications and interactions to other scientific fields. See for example [3, 31, 29, 10, 74, 20, 25] for the single-type case, and [70, 57, 73, 72, 42, 71] for the multi-type case. The current interest of researchers mainly focuses on the multi-type case, as in this case many important problems are open and challenging. All the papers cited above on multi-type branching processes in random environments (MBPRE) concern the critical or sub-critical cases (mainly on the survival probability), except the paper [42] where asymptotic properties of $\mathbb{P}(Z_n = z)$ are studied for a super-critical MBPRE (Z_n). Very recently, in [32, 33] we obtained a theorem of Kesten-Stigum type and a criterion of L^p- convergence $(p > 1)$ for a suitable norming of the population size, for a super-critical MBPRE (Z_n). In this paper, also for a supercritical MBPRE (Z_n), we will establish a Berry-Esseen type bound for the rate of convergence in a central limit theorem on (Z_n), and prove the existence of harmonic moments of the limit of the normalized population size. These results will play very important role in the study of moderate and large deviations of (Z_n), as we will see in [35], to obtain results similar to those in [9, 43, 31] where the single type case was considered.

Let us give a quick presentation of the model with some preliminary results, and some explanations of the main results with key ideas in the proof. For an integer $d \geqslant 1$, consider a d-type branching process $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, in an independent and identically distributed (i.i.d.) random environment $\xi = (\xi_0, \xi_1, \cdots)$. In the sequel, we will denote by (Z_n^i) the process (Z_n) starting with one initial particle of type i, which means that $Z_0 = e_i$, where e_i is the vector with 1 in the i-th place and 0 elsewhere. Denote by $\mathcal{M}_d(\mathbb{R})$ the set of $d \times d$ matrices. We equip the space $\mathbb{R}^d$ with the L^1-norm $\| \cdot \|$. Let $M_n \in \mathcal{M}_d(\mathbb{R})$ be the random matrix of the conditioned means of the offspring distribution of n-th generation given the environment, that is

$$M_n(i,j) = \mathbb{E}_\xi[Z_{n+1}(j) \mid Z_n = e_i], \quad 1 \leqslant i, j \leqslant d,$$

where $\mathbb{E}_\xi$ denotes the conditional expectation given the environment ξ. Define the product matrix $M_{0,n} = M_0 \cdots M_n$, and the associated Lyapunov exponent

$$\gamma := \lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|,$$

where $\|M_{0,n-1}\|$ is the L_1-norm of the matrix $M_{0,n-1}$.

Recently, the asymptotic behaviour of the MBPRE (Z_n^i) under the supercritical condition $\gamma > 0$ has been studied in [32]. In particular, a strong law of large number for $\log \|Z_n^i\|$ is proved: under appropriate conditions, it holds that on the explosion event

$\{\|Z_n^i\| \to +\infty\}$,

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Z_n^i\| = \gamma \quad \text{a.s.} \tag{4.1.1}$$

The main objective in this paper is to establish a Berry-Esseen type theorem on the rate of convergence in the central limit theorem for $\log \|Z_n^i\|$: we will show (cf. Theorem 4.2.5) that

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{C}{\sqrt{n}}, \tag{4.1.2}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\, dt$ is the standard normal distribution function, $\sigma^2 = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[(\log \|M_{0,n-1}^T x\| - n\gamma)^2]$ is the asymptotic variance which is independent of x and $C > 0$ is a constant. This result is new for $d \geqslant 2$; for a single type branching process ($d = 1$), Grama, Liu and Miqueu proved (4.1.2) in [31, Theorem 1.1].

Let us briefly explain our approach for proving (4.1.2). It is heavily based on the fundamental martingale (W_n^i) associated to the process (Z_n^i) defined in [32]. For each $n, k \geqslant 0$, denote by $\rho_{n,n+k}$ the spectral radius of the matrix $M_{n,n+k} = M_n \cdots M_{n+k}$. By the Perron-Frobenius theorem, $\rho_{n,n+k}$ is an eigenvalue of $M_{n,n+k}$, and there exists a non negative eigenvector $U_{n,n+k}$ associated to $\rho_{n,n+k}$ with $\|U_{n,n+k}\| = 1$. According to Hennion [40, Lemma 3.3 and Theorem 1], under conditions, the limit

$$U_{n,\infty} := \lim_{k\to\infty} U_{n,n+k} \tag{4.1.3}$$

exists a.s., with $U_{n,\infty} > 0$ and $\|U_{n,\infty}\| = 1$; moreover the sequence $(U_{n,\infty})$ satisfies the relation

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty}, \tag{4.1.4}$$

where $\lambda_n, n \geqslant 0$ are positive random scalars called the pseudo-spectral radii of the random matrices (M_n). Set $\lambda_{0,n} = \lambda_0 \cdots \lambda_n$. By iteration of (4.1.4), we obtain

$$M_{n,n+k} U_{n+k+1,\infty} = \lambda_{n,n+k} U_{n,\infty}, \quad n, k \geqslant 0. \tag{4.1.5}$$

Then, we define the martingale (W_n^i) as follows (see [32]):

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geqslant 1. \tag{4.1.6}$$

From (4.1.6) the following relations between $\log \|Z_n^i\|$ and $\log \|M_{0,n-1}(i,\cdot)\|$ hold:

$$\log \|Z_n^i\| \leqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i - \min_{1 \leqslant j \leqslant d} \log U_{n,\infty}(j), \tag{4.1.7}$$

$$\log \|Z_n^i\| \geqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i + \min_{1 \leqslant j \leqslant d} \log U_{n,\infty}(j). \tag{4.1.8}$$

From these relations, since the limit $W^i = \lim_{n \to +\infty} W_n^i$ exists a.s. (as (W_n^i) is a non-negative martingale), and $(U_{n,\infty})$ is a stationary sequence of random variables, these two terms will be negligible in the limit properties that we consider. Actually, using the relations (4.1.7) and (4.1.8) we can infer the limiting behaviour of $\log \|Z_n^i\|$ from that of $\log \|M_{0,n-1}(i,\cdot)\|$ by giving a tight control of the quantities $\log W_n^i$ and $\log U_{n,\infty}(j)$. For $\log \|M_{0,n-1}(i,\cdot)\|$ we use the Berry-Esseen bound proved in Xiao, Grama and Liu [75].

An important step in our approach is to establish sufficient conditions for the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ of the limit $W^i = \lim_{n \to +\infty} W_n^i$. This is the second objective of the paper. Actually the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ will give us a suitable control of the sequence $(\log W_n^i)$, which will be one of the key arguments to prove (4.1.2).

Our study of the harmonic moments $\mathbb{E}(W^i)^{-a}$ is composed of two steps.

In the first step, under a strong assumption on the offspring distributions given the environment ξ (see **P2**), we establish a necessary condition and a sufficient condition for the existence of $\mathbb{E}(W^i)^{-a}$ for all $1 \leqslant i \leqslant d$. Set

$$\kappa_1(a) := \lim_{n \to +\infty} \left(\mathbb{E} \|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k) \Big\|_\infty \right)^{1/n}, \quad a > 0, \tag{4.1.9}$$

where $\| \cdot \|_\infty$ denotes the L^∞-norm on $\mathcal{M}_d(\mathbb{R})$, and $P_1(\xi_k)$ is the random matrix whose (i,j)-th component is the probability to produce 1 particle which is of type j by a particle of type i in generation k, given the environment ξ. Then we will prove in Theorem 4.2.1 the following implications:

$$\kappa_1(a) < 1 \Rightarrow \max_{1 \leqslant i \leqslant d} \mathbb{E}(W^i)^{-a} < +\infty \Rightarrow \kappa_1(a) \leqslant 1. \tag{4.1.10}$$

In particular, the solution $a_0 > 0$ of the equation $\kappa_1(a_0) = 1$ is the critical value for the existence of the harmonic moments, in the sense that $\max_{1\leqslant i\leqslant d} \mathbb{E}(W^i)^{-a} < +\infty$ for $a < a_0$, and $\max_{1\leqslant i\leqslant d} \mathbb{E}(W^i)^{-a} = +\infty$ for $a \geqslant a_0$. For the single type case ($d = 1$), Huang and Liu proved in [43] that $\kappa_1(a) < 1$ is a necessary and sufficient for the existence of the harmonic moment $\mathbb{E}(W^1)^{-a}$. Therefore, our result (4.1.10) generalizes that of the single type case, except that we don't know if the harmonic moments exists for the critical exponent a_0.

In the second step, we assume weaker conditions than in the first step (see **P3**), and we prove the existence of a small exponent $a > 0$ such that for all $1 \leqslant i \leqslant d$,

$$\mathbb{E}(W^i)^{-a} < +\infty, \tag{4.1.11}$$

(cf. Theorem 4.2.3). Unfortunately, in this case we have no information on the maximal value of the exponent $a > 0$ for which (4.1.11) holds: we cannot identify the critical exponent.

The outline of the paper is as follows. We introduce some necessary notation and present the main results in Section 4.2. Section 4.3 is devoted to the study of the harmonic moments of W^i for any $1 \leqslant i \leqslant d$. We prove in Section 4.5 the Berry-Esseen type theorem for $\log \|Z_n^i\|$.

4.2 Notation and main results

For $d \geqslant 1$, let $\mathbb{R}^d$ be the d-dimensional space of vectors. We equip $\mathbb{R}^d$ with the scalar product and the L^1-norm respectively defined by

$$\langle x, y\rangle := \sum_{i=1}^{d} x(i)\, y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^{d} |x(i)|, \quad x, y \in \mathbb{R}^d.$$

Set $\mathcal{S} = \{x \in \mathbb{R}^d : x \geqslant 0, \|x\| = 1\}$ for the intersection of the unit sphere with the positive quadrant. For each $1 \leqslant i \leqslant d$, e_i will be the d-dimensional vector with 1 in the i-th place and 0 elsewhere. Let $\mathbf{0} = (0, \cdots, 0) \in \mathbb{R}^d$ be the vector with all coordinates equal to 0. Denote by $\mathbb{N} = \{0, 1, \ldots\}$ the set of non-negative integers. Set $\mathbb{1}_A$ for the indicator of an event A.

Denote by $\mathcal{M}_d(\mathbb{R})$ the set of $d \times d$ matrices with entries in $\mathbb{R}$, and define the operator norm with respect to L^1 vectorial norm by

$$\|M\| := \sup_{x \in \mathcal{S}} \|Mx\|, \quad M \in \mathcal{M}_d(\mathbb{R}).$$

In addition we equip $\mathbb{R}^d$ and $\mathcal{M}_d(\mathbb{R})$ with the L^∞-norms:

$$\|x\|_\infty := \max_{1 \leqslant i \leqslant d} |x(i)|, \quad x \in \mathbb{R}^d;$$
$$\|M\|_\infty := \sup_{\|x\|_\infty = 1} \|Mx\|_\infty, \quad M \in \mathcal{M}_d(\mathbb{R}).$$

For a matrix or a vector X, we write $X > 0$ when all the entries of X are strictly positive.

Now we give a precise definition of the multi-type branching process in random environment (MBPRE). The environment $\xi = (\xi_n)_{n \geqslant 0}$ is a sequence of independent and identically distributed (i.i.d.) random variables taking values in an abstract space $\mathbb{X}$. To each realization of ξ_n correspond d probability distributions on $\mathbb{N}^d$ identified by the probability generating functions

$$f_n^r(s) = \sum_{k_1, \cdots, k_d = 0}^{\infty} p^r_{k_1, \cdots, k_d}(\xi_n) s_1^{k_1} \cdots s_d^{k_d}, \quad s = (s_1, \ldots, s_d) \in [0,1]^d,$$

where $1 \leqslant r \leqslant d$. The d-type branching process $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, in the random environment ξ is a process with values in $\mathbb{N}^d$ such that $Z_0 \in \mathbb{N}^d$ is a fixed vector, and for all $n \geqslant 0$,

$$Z_{n+1} = \sum_{r=1}^{d} \sum_{l=1}^{Z_n(r)} N^r_{l,n}. \tag{4.2.1}$$

where $N^r_{l,n} = (N^r_{l,n}(1), \cdots, N^r_{l,n}(d))$ is a random vector whose j-th component $N^r_{l,n}(j)$ represents the offspring of type j at time $n+1$ of the l-th particle of type r in generation n, and $Z_n(j)$ is the total number of particles of type j in generation n. Conditioned on the environment ξ, the random vectors $N^r_{l,n}$ indexed by $l \geqslant 1$, $n \geqslant 0$ and $1 \leqslant r \leqslant d$ are independent, each $N^r_{l,n}$ has the same probability generating function f^r_n. In the sequel, the process Z_n will be noted Z^i_n when $Z_0 = e_i$, which corresponds to the MBPRE starting with one initial particle of type i.

Denote by $\mathbb{P}_\xi$ the quenched law, i.e. the probability under which the process is defined when the environment ξ is given. Let τ be the law of ξ. The total probability $\mathbb{P}$ of (Z_n), usually called annealed law, is defined by $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$. Denote by $\mathbb{E}_\xi$ and $\mathbb{E}$ the corresponding expectation with respect to $\mathbb{P}_\xi$ and $\mathbb{P}$. With our notation,

$$f_n^r(s) = \mathbb{E}_\xi\left(\prod_{j=1}^d s_j^{N_{l,n}^r(j)}\right), \quad s = (s_1, \dots, s_d) \in [0,1]^d$$

is the quenched probability generating function of $N_{l,n}^r$. For $n \geqslant 0$, let M_n be the $d \times d$ random matrix whose (i,j)-th entry $M_n(i,j)$ is the conditioned mean of the number of children of type j produced by a particle of type i at time n:

$$M_n(i,j) := \frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi\Big[Z_{n+1}(j)\Big|Z_n = e_i\Big],$$

where $\frac{\partial f}{\partial s_j}(\mathbf{1})$ denotes the left derivative at $\mathbf{1}$ of a d-dimensional probability generating function f with respect to s_j. Since the sequence of the environments (ξ_n) is i.i.d., the sequence of the mean matrices (M_n) is also i.i.d.. For $0 \leqslant k \leqslant n$, denote by

$$M_{k,n} := M_k \cdots M_n,$$

the product of the mean matrices $M_k, \dots, M_n$. It follows that, for $n \geqslant 0$ and $1 \leqslant i, j \leqslant d$,

$$\mathbb{E}_\xi Z_{n+1}^i(j) = M_{0,n}(i,j). \tag{4.2.2}$$

The main objective of this paper is to establish a Berry-Esseen bound for $\log \|Z_n^i\|$. To this end, the key tool will be the fundamental martingale we mentioned in the introduction. Let $\rho_{n,n+k}$ be the spectral radius of $M_{n,n+k}$. We know by the Perron-Frobenius theorem (see e.g. [7]) that $\rho_{n,n+k}$ is a positive eigenvalue of $M_{n,n+k}$, and there exist positive right and left eigenvectors $U_{n,n+k}$ and $V_{n,n+k}$ associated to $\rho_{n,n+k}$ with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k}\rangle = 1$. Denote by $\mathcal{G}_+^0$ the subset of the matrices of $\mathcal{M}_d(\mathbb{R})$ with strictly positive entries. Throughout the paper, we assume that M_0 is allowable (every row and column contains a strictly positive element), and that the following positivity property holds:

$$\mathbb{P}\left(\bigcup_{n \geqslant 0}\left\{M_{0,n} \in \mathcal{G}_+^0\right\}\right) > 0. \tag{4.2.3}$$

By the results of Hennion [40, Lemma 3.3 and Theorem 1], under condition (4.2.3) there exists the random vectors $U_{n,\infty}$ and the random scalars λ_n defined by (4.1.3) and (4.1.4), which satisfy the relation (4.1.5) and that $(U_{n,\infty})$ and (λ_n) are stationary ergodic; moreover, we proved in [32, Theorem 1] that the sequence (W_n^i) defined in (4.1.6) is a non-negative martingale under the measure $\mathbb{P}_\xi$ and $\mathbb{P}$, w.r.t. the filtration

$$\mathcal{F}_0 = \sigma(\xi), \quad \mathcal{F}_n = \sigma\Big(\xi, N_{l,k}^r(j), 0 \leqslant k \leqslant n-1, 1 \leqslant r, j \leqslant d, l \geqslant 1\Big) \text{ for } n \geqslant 1.$$

Let $W^i := \lim_{n\to+\infty} W_n^i$ be the a.s. limit of the martingale (W_n^i).

We will use the classification of MBPRE's defined in [32]. It is well known that, under the following moment condition

$$\mathbb{E} \log^+ \|M_0\| < +\infty, \tag{4.2.4}$$

the Lyapunov exponent γ of the sequence of matrices $(M_n)_{n\geqslant 0}$ exists, with

$$\gamma := \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| = \inf_{n\geqslant 1} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|.$$

Moreover, Furstenberg and Kesten established in [26] a strong law of large numbers for $\log \|M_{0,n-1}\|$:

$$\lim_{n\to+\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.}$$

According to the values of the Lyapunov exponent γ, we have the following classification of MBPRE's: a MBPRE is supercritical if $\gamma > 0$, critical if $\gamma = 0$, and subcritical if $\gamma < 0$. In this article, the process (Z_n) will always be supercritical, i.e. $\gamma > 0$.

Under the supercritical condition $\gamma > 0$, we established in [32, Theorem 2.6 and Corollary 2.8] that the condition

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty \quad \forall 1 \leqslant i, j \leqslant d \tag{4.2.5}$$

is sufficient for the non-degeneracy of each W^i (in the sense that $\mathbb{P}(W^i > 0) > 0$), which is equivalent to the L^1-convergence of W_n^i to W^i by Sheffé's theorem; moreover, when each W^i is non-degenerate, we have a.s.,

$$\mathbb{E}_\xi W^i = 1 \quad \text{and} \quad \mathbb{P}_\xi(W^i > 0) = \mathbb{P}_\xi\Big(\|Z_n^i\| \underset{n\to\infty}{\to} \infty\Big) = 1 - q^i(\xi) > 0, \tag{4.2.6}$$

where $q^i(\xi)$ is the quenched probability of extinction of the process (Z_n^i).

First we establish the existence of the harmonic moments of the limits W^i, $1 \leqslant i \leqslant d$. For $n \geqslant 0$, define the vector $p_0(\xi_n)$ and the matrix $P_1(\xi_n)$, whose components are

$$p_0(\xi_n)(i) = f_n^i(\mathbf{0}) \quad \text{and} \quad P_1(\xi_n)(i,j) = \frac{\partial f_n^i}{\partial s_j}(\mathbf{0}), \quad 1 \leqslant i,j \leqslant d.$$

Then, for $1 \leqslant i,j \leqslant d$,

$$p_0(\xi_n)(i) = \mathbb{P}_{T^n\xi}(\|Z_1^i\| = 0) \quad \text{and} \quad P_1(\xi_n)(i,j) = \mathbb{P}_{T^n\xi}(Z_1^i = e_j).$$

Throughout the paper, we will assume the following condition:

P1. The vector $p_0(\xi_0) = (f_0^1(\mathbf{0}), \dots, f_0^d(\mathbf{0}))$ satisfies

$$p_0(\xi_0) = \mathbf{0} \quad \mathbb{P}\text{-a.s.} \tag{4.2.7}$$

The condition **P1** means that each individual of the population gives birth to at least one child, so $q^i(\xi) = 0$ a.s. When (4.2.6) holds, this implies that $\|Z_n^i\| \to +\infty$ a.s. as $n \to +\infty$.

We introduce the following assumption :

P2. There exist constants $p \in (1,2]$, $A > A_1 > 1$ and $A_2 > 0$ such that for any $1 \leqslant i,j \leqslant d$, $\mathbb{P}$-a.s.

$$A_2 \leqslant M_0(i,j), \quad A_1 \leqslant \|M_0(i,\cdot)\| \quad \text{and} \quad \mathbb{E}_\xi(Z_1^i(j)^p) \leqslant A^p.$$

It is clear that **P2** implies the conditions (4.2.3), (4.2.4), (4.2.5) and $\gamma > 0$. From **P2** we have also that for all $1 \leqslant i,j \leqslant d$,

$$M_0(i,j) \leqslant A \quad \mathbb{P}\text{-a.s.}$$

Under condition **P2**, by the sub-multiplicative property of the norms $\|.\|$ and $\|.\|_\infty$ on $\mathcal{M}_d(\mathbb{R})$ and the subadditive ergodic theorem, it follows that, for all $a \geqslant 0$ the limit

$$\kappa_1(a) := \lim_{n\to+\infty} \Big(\mathbb{E}\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k) \Big\|_\infty \Big)^{1/n} \tag{4.2.8}$$

exists and is finite, with

$$\kappa_1(a) = \inf_{n\geqslant 1} \left(\mathbb{E}\|M_{0,n-1}\|^a \Big\| \prod_{k=0}^{n-1} P_1(\xi_k) \Big\|_\infty \right)^{1/n}.$$

We will establish (see Lemma 4.3.9) that, under **P1** and **P2**, the function κ_1 is continuous and increasing on $\mathbb{R}_+$ and $\kappa_1(0) = \rho\Big(\mathbb{E}P_1(\xi_0)\Big)$, where $\rho(M)$ denotes the spectral radius of the matrix M. For a random variable X, set $\|X\|_{L^\infty} := \operatorname{ess\,sup}(\mathrm{X})$ the essential supremum of X. Our first result gives a sufficient and a necessary condition for the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$, $a > 0$.

Theorem 4.2.1. *Assume conditions **P1**, **P2** and $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$. For each fixed $a > 0$, the following implications hold :*

(1) if $\kappa_1(a) < 1$ then $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$;

(2) if $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$, then $\kappa_1(a) \leqslant 1$.

Notice that in Part (2), we can prove more: we will see in the proof that the sequence $\mathbb{E}\|M_{0,n-1}\|^a \, \| \prod_{k=0}^{n-1} P_1(\xi_k)\|_\infty, n \geqslant 1$, is bounded.

From Theorem 4.2.1 we get the following corollary.

Corollary 4.2.2. *Under the conditions of Theorem 4.2.1, it holds:*

(1) $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$ and $a > 0$ if and only if $\mathbb{E}P_1(\xi_0)$ is nilpotent;

(2) if $\mathbb{E}P_1(\xi_0)$ is not nilpotent, then there exists a unique constant $a_0 > 0$ satisfying

$$\kappa_1(a_0) = 1, \tag{4.2.9}$$

and

$$\max_{1\leqslant i\leqslant d} \mathbb{E}(W^i)^{-a} \begin{cases} < +\infty & \text{if } a \in [0, a_0), \\ = +\infty & \text{if } a \in (a_0, +\infty). \end{cases}$$

Part (1) gives a necessary and sufficient condition to have the existence of all orders of the harmonic moments of each W^i, $1 \leqslant i \leqslant d$. Part (2) reveals that the quantity a_0 is the critical value for the existence of the harmonic moments of all the W^i, $1 \leqslant i \leqslant d$. We believe that at the critical value a_0 the harmonic moments do not exist, i.e., $\mathbb{E}(W^i)^{-a_0} = +\infty$. This is the case when $d = 1$, as shown by Chunmao and Liu [43].

Now we investigate the existence of harmonic moments for W^i, when the boundedness condition **P2** is relaxed to a moment condition. For all $n \geqslant 0$ and $p > 1$ denote by

$$\theta_n(p) := \max_{1\leqslant i,j\leqslant d} \mathbb{E}_\xi \left| \frac{N^i_{1,n}(j)}{M_n(i,j)} - 1 \right|^p.$$

The next result gives a sufficient condition for the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ of a small order $a > 0$. The single -type case has been considered in [30]. The multi-type case considered here is much more complicated.

To formulate it, we need the following moment condition :

P3. There exist two constants $p \in (1, 2]$ and $\eta \in (0, 1)$ such that

$$\mathbb{E}\|M_0\|^\eta < +\infty, \quad \max_{1\leqslant i,j\leqslant d} \mathbb{E}M_0(i,j)^{-\eta} < +\infty \quad \text{and} \quad \mathbb{E}\theta_0(p)^\eta < +\infty.$$

Like **P2**, condition **P3** also implies (4.2.3) and (4.2.4) and (4.2.5). The first two implications are evident; the third will be proved in Section 4.3.

Theorem 4.2.3. *Assume conditions **P1**, **P3** and $\gamma > 0$. Then there exists $a > 0$ such that $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$.*

Now we formulate a central limit theorem and a Berry-Esseen type theorem for $\log \|Z^i_n\|$, for all $1 \leqslant i \leqslant d$. We introduce the following assumption:

P4. The random matrix M_0 satisfies

$$\mathbb{E}(\log \|M_0\|)^2 < +\infty.$$

Obviously, condition **P4** implies (4.2.4). Using the central limit theorem due to Hennion [40, Theorem 3] for the norm cocycle $\log \|M^T_{0,n-1}x\|$, where $x \in \mathcal{S}$, we establish the following central limit theorem for $\log \|Z^i_n\|$. Notice that $\|Z^i_n\| = Z^i_n(1) + \cdots + Z^i_n(d)$ represents the population size of generation n.

Theorem 4.2.4. *Assume conditions (4.2.3) and **P4**. Assume also **P1**, (4.2.5) and $\gamma > 0$. Then there exists $\sigma \geqslant 0$ such that for all $1 \leqslant i \leqslant d$, as $n \to \infty$,*

$$\frac{\log \|Z^i_n\| - n\gamma}{\sqrt{n}} \to \mathcal{N}(0, \sigma^2) \quad \textit{in law,}$$

where $\mathcal{N}(0,\sigma^2)$ denotes the normal law with mean 0 and variance σ^2.

Notice that for the single type case $d=1$, this theorem was established in [43]. By [75, Proposition 3.14], under the condition **P3**, the asymptotic variance σ^2 defined in Theorem 4.2.4 satisfies

$$\sigma^2 = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[(\log \|M_{0,n-1}x\| - n\gamma)^2],$$

uniformly in $x \in \mathcal{S}$. Note that in Theorem 4.2.4 the limit variance σ^2 can be degenerated: $\sigma^2 = 0$. For the rate of convergence we need the following assumption :

P5. The asymptotic variance σ^2 satisfies

$$\sigma^2 > 0.$$

According to [14, Lemma 7.2], a sufficient condition under which **P5** holds is that μ is a non-arithmetic probability measure; the definition of arithmeticity is introduced below.

For $x \in \mathcal{S}$ and $M \in \mathcal{G}_+^0$, define the projective action of M on $\mathcal{S}$ by $M \cdot x := \frac{Mx}{\|Mx\|}$. Let μ be the law of M_0 and $\Gamma_\mu = [\operatorname{supp}\mu]$, the semi-group generated by the support of μ. By the Perron-Frobenius theorem, since any $M \in \Gamma_\mu$ is strictly positive under **P3**, the spectral radius ρ_M of M is the unique eigenvalue with the largest modulus, which is simple. Let $u_M \in \mathcal{S}$ be the associated unique right eigenvector with unit norm. Set $V(\Gamma_\mu) = \overline{\{\pm u_M : M \in \Gamma_\mu\}}$, where $\overline{A}$ denotes the closure of the set A. The measure μ is called arithmetic if there exist $t>0$, $\theta \in [0,2\pi)$ and a function $h : \mathcal{S} \to \mathbb{R}$ such that for all $M \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$, we have

$$\exp\{it\log\|Mx\| - i\theta + ih(M\cdot x) - ih(x)\} = 1.$$

In the following, denote by $\Phi : x \mapsto \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\,dt$ the standard normal distribution function on $\mathbb{R}$.

Theorem 4.2.5. *Assume conditions **P1**, **P3**, **P5** and $\gamma > 0$. Then there exists a constant $C>0$ such that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,*

$$\left|\mathbb{P}\left(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{C}{\sqrt{n}}.$$

For the single type case $d=1$, a version of this result exists under different conditions in

[31, Theorem 1.1]. Notice that, in this case, we have $\gamma = \mathbb{E}\log m_0$ and $\sigma^2 = \mathbb{E}(\log m_0 - \gamma)^2$ with $m_0 = \mathbb{E}_\xi Z_1$, and the condition **P3** can be simplified to the following: there exist two constants $p \in (1,2]$ and $\eta \in (0,1)$ such that

$$\mathbb{E}m_0^\eta < +\infty \quad \text{and} \quad \mathbb{E}\theta_0(p)^\eta < +\infty, \text{ where } \theta_0(p) = \mathbb{E}_\xi \left| \frac{Z_1}{m_0} - 1 \right|^p.$$

4.3 Harmonic moments of W^i

In this section we study the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ of all the random variables W^i, where $1 \leqslant i \leqslant d$, $a > 0$.

4.3.1 Auxiliary results

We start with four lemmas that we will need. The first lemma permits to compare the moment of $\phi(W^i)$ with the corresponding one of $\phi(W_n^i)$, with ϕ a positive convex function on $\mathbb{R}_+$.

Lemma 4.3.1. *Assume condition (4.2.5) and $\gamma > 0$. Then for all $1 \leqslant i \leqslant d$ and any convex function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$,*

$$\lim_{n\to+\infty} \mathbb{E}_\xi \phi(W_n^i) = \sup_{n\geqslant 0} \mathbb{E}_\xi \phi(W_n^i) = \mathbb{E}_\xi \phi(W^i), \tag{4.3.1}$$

and

$$\lim_{n\to+\infty} \mathbb{E}\phi(W_n^i) = \sup_{n\geqslant 0} \mathbb{E}\phi(W_n^i) = \mathbb{E}\phi(W^i). \tag{4.3.2}$$

Proof. The result is a direct consequence of the fact that, by [32, Theorem 2.6], under the conditions (4.2.5) and $\gamma > 0$, $(W_n^i, \mathcal{F}_n)$ is a martingale under $\mathbb{P}_\xi$ and $\mathbb{P}$, which converges in L^1. In fact, by Fatou's lemma, $\mathbb{E}\phi(W^i) \leqslant \sup_{n\geqslant 0} \mathbb{E}\phi(W_n^i)$; by the L^1- convergence of (W_n^i) and Jensen's inequality,

$$\mathbb{E}\phi(W^i) = \mathbb{E}[\mathbb{E}[\phi(W^i)|\mathcal{F}_n]] \geqslant \mathbb{E}[\phi(\mathbb{E}[W^i|\mathcal{F}_n])] = \mathbb{E}\phi(W_n^i).$$

Using the fact that $(\phi(W_n^i))$ is a sub-martingale, this concludes the proof of (4.3.2). The same argument with $\mathbb{E}$ replaced by $\mathbb{E}_\xi$ gives (4.3.1). □

The second lemma reveals the link between the harmonic moments and the Laplace transform of a positive random variable.

Lemma 4.3.2. *([58, Lemma 4.4]) Let X be a positive random variable, and $a > 0$. We have the following assertions:*

(1) if $\mathbb{E}X^{-a} < +\infty$ then $\mathbb{E}\,\mathrm{e}^{-tX} = O_{t\to+\infty}(t^{-a})$;

(2) if $\mathbb{E}\,\mathrm{e}^{-tX} = O_{t\to+\infty}(t^{-a})$ then $\mathbb{E}X^{-b} < +\infty$ for all $b \in (0, a)$;

(3) $\mathbb{E}\,\mathrm{e}^{-tX} = O_{t\to+\infty}(t^{-a})$ if and only if $\mathbb{P}(X \leqslant x) = O_{x\to 0}(x^a)$.

For all $1 \leqslant i \leqslant d$, let

$$\phi_\xi^i(t) = \mathbb{E}_\xi\,\mathrm{e}^{-tW^i} \quad \text{and} \quad \phi^i(t) = \mathbb{E}\phi_\xi^i(t) = \mathbb{E}\,\mathrm{e}^{-tW^i}, \qquad t \geqslant 0,$$

be the quenched and annealed Laplace transform of W^i. Denote by

$$\phi_\xi(t) = (\phi_\xi^1(t), \cdots, \phi_\xi^d(t)) \quad \text{and} \quad \phi(t) = (\phi^1(t), \cdots, \phi^d(t)), \quad t \geqslant 0.$$

We will study the decay rate of the Laplace transforms $\phi_\xi^i(t)$ and $\phi^i(t)$ when $t \to +\infty$, and then use Lemma 4.3.2 to estimate the corresponding harmonic moments. Let T be the shift operator of the environment sequence:

$$T\xi = (\xi_1, \xi_2, \cdots) \quad \text{if} \quad \xi = (\xi_0, \xi_1, \cdots),$$

and let T^n be its n-fold iteration. The third lemma, proved in [32, Theorem 2.4], gives the functional equations that the quenched Laplace transforms ϕ_ξ^i satisfy.

Lemma 4.3.3. *Assume condition (4.2.3). Then for all $i = 1, \cdots, d$, the quenched Laplace transform ϕ_ξ^i of W^i satisfies*

$$\phi_\xi^i(t) = f_0^i\left(\phi_{T\xi}^1\left(t\frac{U_{1,\infty}(1)}{\lambda_0 U_{0,\infty}(i)}\right), \cdots, \phi_{T\xi}^d\left(t\frac{U_{1,\infty}(d)}{\lambda_0 U_{0,\infty}(i)}\right)\right), \quad t \geqslant 0.$$

The fourth Lemma will be used to control the L^p-moments of the martingale (W_n^i). It is a direct consequence of the Marcinkiewicz-Zygmund inequality in [16, Theorem 1.5], as stated in [60, Lemma 1.4].

Lemma 4.3.4. *Let $(X_k)_{k\in\mathbb{N}^*}$ be a sequence of i.i.d. random centered variables. Then for all $n \in \mathbb{N}^*$ and $p > 1$:*

$$\mathbb{E}\Big|\sum_{k=1}^{n} X_k\Big|^p \leqslant \begin{cases} (B_p)^p\mathbb{E}|X_k|^p n, & \text{if } 1 < p \leqslant 2, \\ (B_p)^p\mathbb{E}|X_k|^p n^{\frac{p}{2}}, & \text{if } p > 2, \end{cases}$$

where $B_p = 2\min\{k^{1/2} : k \in \mathbb{N}, k \geqslant \frac{p}{2}\}$.

4.3.2 Existence of the quenched harmonic moments $\mathbb{E}_\xi(W^i)^{-a}$

In this section, under condition **P2**, we give an estimation of the decay rate of the quenched Laplace transforms $\phi_\xi^i(t)$ as $t \to +\infty$, which implies the uniform boundedness of the quenched harmonic moments $\mathbb{E}_\xi(W^i)^{-a}$, as indicated in the following theorem.

Theorem 4.3.5. *Assume conditions **P1**, **P2** and $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$. Then there exist two constants $C > 0$ and $a > 0$ such that for all $1 \leqslant i \leqslant d$, all $t > 0$ and all $x > 0$,*

$$\phi_\xi^i(t) \leqslant \frac{C}{t^a} \quad \mathbb{P}\text{-a.s.}, \tag{4.3.3}$$

$$\mathbb{P}_\xi(W^i \leqslant x) \leqslant Cx^a \quad \text{and} \quad \mathbb{E}_\xi(W^i)^{-a} \leqslant C. \tag{4.3.4}$$

For the proof of Theorem 4.3.5, we will need the following preliminary result about a control of $\phi_\xi^i(t)$, $1 \leqslant i \leqslant d$. For the case $d = 1$, this result was established in [43].

Lemma 4.3.6. *Assume condition **P2**. Then there exist two constants $\beta \in (0, 1)$ and $t_0 > 0$ such that for all $1 \leqslant i \leqslant d$ and $t \geqslant t_0$,*

$$\phi_\xi^i(t) \leqslant \beta \quad \mathbb{P}\text{-a.s.}$$

Proof. We will adapte the approach in [43] where the case $d = 1$ was considered. By

(4.2.1) and (4.1.6), we have that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} W_{n+1}^i - W_n^i &= \sum_{j=1}^d \frac{U_{n+1,\infty}(j)}{\lambda_{0,n} U_{0,\infty}(i)} \sum_{r=1}^d \sum_{l=1}^{Z_n^i(r)} N_{l,n}^r(j) - W_n^i \\ &= \sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(r)} \sum_{j=1}^d \frac{U_{n+1,\infty}(j) N_{l,n}^r(j)}{\lambda_n U_{n,\infty}(r)} - W_n^i \\ &= \sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1), \end{aligned} \tag{4.3.5}$$

where

$$W_{l,n}^r := \frac{\langle N_{l,n}^r, U_{n+1,\infty} \rangle}{\lambda_n U_{n,\infty}(r)}.$$

It is clear that, given the environment ξ, the random variables $W_{l,n}^r$, $l \geqslant 1$, are i.i.d., and independent of $\xi_0, \dots, \xi_{n-1}$ and Z_n^i. Let $p \in (1, 2]$ be such that condition **P2** holds. Notice that, by (4.1.5), we have $\sum_{r=1}^d \frac{M_{0,n-1}(i,r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} = 1$ a.s. for any $n \geqslant 1$ and $1 \leqslant i \leqslant d$. Therefore, applying (4.3.5), the convexity of the function $x \mapsto x^p$ on $\mathbb{R}_+$ and Lemma 4.3.4, for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s., we get

$$\begin{aligned} \mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p &\leqslant \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \left| \sum_{l=1}^{Z_n^i(r)} \left(W_{l,n}^r - 1 \right) \right| \right)^p \\ &\leqslant \sum_{r=1}^d \frac{M_{0,n-1}(i,r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \frac{1}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi \left| \sum_{l=1}^{Z_n^i(r)} \left(W_{l,n}^r - 1 \right) \right|^p \\ &\leqslant B_p^p \sum_{r=1}^d \frac{M_{0,n-1}(i,r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \cdot \frac{\mathbb{E}_\xi Z_n^i(r)}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi |W_{1,n}^r - 1|^p \\ &\leqslant B_p^p \max_{1 \leqslant r \leqslant d} \left\{ \frac{\mathbb{E}_\xi Z_n^i(r)}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi |W_{1,n}^r - 1|^p \right\} \\ &= B_p^p \max_{1 \leqslant r \leqslant d} \mathbb{E}_\xi |W_{1,n}^r - 1|^p \max_{1 \leqslant j \leqslant d} (M_{0,n-1}(i,j))^{1-p}. \end{aligned} \tag{4.3.6}$$

Again by the convexity of $x \mapsto x^p$, for all $1 \leqslant r \leqslant d$ and $n \geqslant 0$, $\mathbb{P}$-a.s., we have

$$\begin{aligned}\mathbb{E}_\xi|W_{1,n}^r - 1|^p &= \mathbb{E}_\xi\left|\frac{\langle N_{1,n}^r, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(r)} - 1\right|^p \\ &= \mathbb{E}_\xi\left|\sum_{j=1}^d \frac{M_n(r,j)U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(r)}\left(\frac{N_{1,n}^r(j)}{M_n(r,j)} - 1\right)\right|^p \\ &\leqslant \max_{1\leqslant i,j\leqslant d} \mathbb{E}_\xi\left|\frac{N_{1,n}^i(j)}{M_n(i,j)} - 1\right|^p = \theta_n(p). \end{aligned} \tag{4.3.7}$$

Combining this with (4.3.6), we obtain that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\mathbb{E}_\xi|W_{n+1}^i - W_n^i|^p \leqslant B_p^p \theta_n(p) \max_{1\leqslant j\leqslant d}(M_{0,n-1}(i,j))^{1-p}. \tag{4.3.8}$$

Using the triangular inequality in L^p under $\mathbb{P}_\xi$ and condition **P2**, for $n \geqslant 0$ we have

$$\theta_n(p) \leqslant \max_{1\leqslant i,j\leqslant d} \left(\frac{\left[\mathbb{E}_\xi\left(N_{1,n}^i(j)\right)^p\right]^{1/p}}{M_n(i,j)} + 1\right)^p \leqslant \left(\frac{A}{A_2} + 1\right)^p \quad \mathbb{P}\text{-a.s.} \tag{4.3.9}$$

Now we deal with the last factor in (4.3.8). For $M \in \mathcal{M}_d(\mathbb{R})$, set

$$\iota(M) := \inf_{\|x\|=1} \|Mx\| = \min_{1\leqslant j\leqslant d}\sum_{i=1}^d M(i,j). \tag{4.3.10}$$

Using **P2** we have $\iota(M_0^T) \geqslant A_1$ a.s. It can be easily seen that the application ι satisfies the inequality $\iota(AB) \geqslant \iota(A)\iota(B)$, for $A, B \in \mathcal{G}_+^0$. Therefore, we deduce that for all $n \geqslant 2$,

$$\begin{aligned} M_{0,n-1}(i,j) &= \sum_{1\leqslant r\leqslant d} M_{0,n-2}(i,r)M_{n-1}(r,j) \\ &\geqslant A_2\iota(M_{0,n-2}^T) \\ &\geqslant A_2\prod_{k=0}^{n-2}\iota(M_k^T) \\ &\geqslant A_2A_1^{n-1}. \end{aligned} \tag{4.3.11}$$

It is evident that the above inequality remains true for $n = 1$. Combining (4.3.8), (4.3.9) and (4.3.11), we obtain that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p \leqslant B_p^p \Big(\frac{A}{A_2} + 1 \Big)^p A_2^{1-p} (A_1^{1-p})^{n-1} \quad \mathbb{P}\text{-a.s.}, \tag{4.3.12}$$

with $A_1^{1-p} \in (0,1)$. By (4.3.7) and (4.3.9), it is clear that (4.3.12) holds for $n = 0$. Recall that condition **P2** implies (4.2.5) and $\gamma > 0$. Then, applying Lemma 4.3.1 with the convex function $x \mapsto x^p$ on $\mathbb{R}_+$, we have

$$\mathbb{E}_\xi (W^i)^p = \sup_{n \geqslant 0} \mathbb{E}_\xi (W_n^i)^p \quad \mathbb{P}\text{-a.s.}$$

This, together with (4.3.12) and the triangular inequality in L^p, implies that there exists a constant $C > 0$ such that

$$\mathbb{E}_\xi (W^i)^p = \sup_{n \geqslant 0} \mathbb{E}_\xi (W_n^i)^p \leqslant \Big(1 + \sum_{n=0}^{+\infty} \Big(\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p \Big)^{1/p} \Big)^p \leqslant C. \tag{4.3.13}$$

Since $x \to (\mathrm{e}^{-x} - 1 + x)/x^p$ is a positive bounded function on $\mathbb{R}_+^*$, it follows that there exists a constant $C_1 > 0$ such that for all $x > 0$,

$$\mathrm{e}^{-x} \leqslant 1 - x + C_1 x^p.$$

Combining this with (4.3.13), we see that for any $1 \leqslant i \leqslant d$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \phi_\xi^i(t) = \mathbb{E}_\xi \, \mathrm{e}^{-tW^i} &\leqslant \mathbb{E}_\xi \Big(1 - tW^i + C_1 t^p (W^i)^p \Big) \\ &= 1 - t + C_1 t^p \mathbb{E}_\xi (W^i)^p \\ &\leqslant 1 - t + CC_1 t^p. \end{aligned} \tag{4.3.14}$$

Let $h(t) = 1 - t + CC_1 t^p$, $t \geqslant 0$. We observe that the minimal value of $h(t)$ is $\beta = h(t_0)$, where $t_0 := (pCC_1)^{1/(1-p)}$, and we have

$$\beta = 1 - (pCC_1)^{1/(1-p)} + \frac{1}{p} (pCC_1)^{1/(1-p)} = 1 - \Big(1 - \frac{1}{p} \Big) t_0. \tag{4.3.15}$$

So $\beta \in (0,1)$. Since the quenched Laplace transform ϕ_ξ^i is decreasing on $\mathbb{R}_+$, we conclude from (4.3.14) that for all $t \geqslant t_0$,

$$\phi_\xi^i(t) \leqslant \phi_\xi^i(t_0) \leqslant h(t_0) = \beta \quad \mathbb{P}\text{-a.s.} \tag{4.3.16}$$

This concludes the proof of Lemma 4.3.6. □

Proof of Theorem 4.3.5. By Lemma 4.3.2, we have the implication (4.3.3) ⇒ (4.3.4) (but the values of a and C can be changed). Therefore, it remains to prove (4.3.3). Set

$$\psi_\xi^i(t) = \phi_\xi^i(tU_{0,\infty}(i)), \quad 1 \leqslant i \leqslant d, \quad t \geqslant 0,$$

and $\psi_\xi(t) = (\psi_\xi^1(t), \cdots, \psi_\xi^d(t))$. From Lemma 4.3.3 we obtain that ψ_ξ satisfies the following equation: for all $1 \leqslant i \leqslant d$ and $t \geqslant 0$,

$$\begin{aligned}\psi_\xi^i(t) &= f_0^i\left(\phi_{T\xi}^1\left(\frac{tU_{1,\infty}(1)}{\lambda_0}\right), \cdots, \phi_{T\xi}^d\left(\frac{tU_{1,\infty}(d)}{\lambda_0}\right)\right) \\ &= f_0^i\left(\psi_{T\xi}^1\left(\frac{t}{\lambda_0}\right), \cdots, \psi_{T\xi}^d\left(\frac{t}{\lambda_0}\right)\right) \\ &= f_0^i\left(\psi_{T\xi}\left(\frac{t}{\lambda_0}\right)\right).\end{aligned} \tag{4.3.17}$$

For $n \geqslant 0$, denote by $Q_1(\xi_n)$ the positive random matrix whose entries are, for $1 \leqslant i, j \leqslant d$,

$$Q_1(\xi_n)(i,j) = \mathbb{P}_\xi\Big(\|Z_{n+1}\| \geqslant 2, Z_{n+1}(j) \geqslant 1, Z_{n+1}(r) = 0, r < j \Big| Z_n = e_i\Big).$$

It is clear that for $n \geqslant 0$, $Q_1(\xi_n)$ depends only of ξ_n and that the events $\{Z_{n+1}(j) \geqslant 1, Z_{n+1}(r) = 0 \ \forall r < j\}$, $1 \leqslant j \leqslant d$, constitute a partition of $\{\|Z_{n+1}\| \geqslant 1\}$. Hence

$$\|Q_1(\xi_n)(i,\cdot)\| = \sum_{j=1}^d Q_1(\xi_n)(i,j) = \mathbb{P}_\xi\Big(\|Z_{n+1}\| \geqslant 2 \Big| Z_n = e_i\Big) \quad \forall 1 \leqslant i \leqslant d.$$

By **P1** and the fact that $\|P_1(\xi_n)(i,\cdot)\| = \mathbb{P}_\xi\Big(\|Z_{n+1}\| = 1 \Big| Z_n = e_i\Big)$, we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\|P_1(\xi_n)(i,\cdot) + Q_1(\xi_n)(i,\cdot)\| = 1, \tag{4.3.18}$$

which means that $P_1(\xi_n) + Q_1(\xi_n)$ is a stochastic matrix. Then, by definition of the matrix $Q_1(\xi_0)$ and using again **P1** and the partition $\{Z_{n+1}(j) \geqslant 1, Z_{n+1}(r) = 0 \ \forall r < j\}$

$(1 \leqslant j \leqslant d)$ of $\{\|Z_{n+1}\| \geqslant 1\}$, we see that for all $s = (s_1, \cdots, s_d) \in [0,1]^d$ and $1 \leqslant i \leqslant d$, we have

$$\begin{aligned} f_0^i(s) &= \sum_{j=1}^d \mathbb{P}_\xi(Z_1^i = e_j) s_j + \sum_{k \in \mathbb{N}^d, \|k\| \geqslant 2} \mathbb{P}_\xi(Z_1^i = k) s_1^{k(1)} \cdots s_d^{k(d)} \\ &\leqslant \sum_{j=1}^d P_1(\xi_0)(i,j) s_j + \|s\|_\infty \sum_{j=1}^d Q_1(\xi_0)(i,j) s_j \\ &= \Big\langle \Big[P_1(\xi_0) + \|s\|_\infty Q_1(\xi_0) \Big] s, e_i \Big\rangle. \end{aligned}$$

This, together with (4.3.17), implies that for all $t \geqslant 0$,

$$\psi_\xi(t) \leqslant \left(P_1(\xi_0) + \left\| \psi_{T\xi}\left(\frac{t}{\lambda_0}\right) \right\|_\infty Q_1(\xi_0) \right) \psi_{T\xi}\left(\frac{t}{\lambda_0}\right) \quad \mathbb{P}\text{-a.s.} \tag{4.3.19}$$

In particular, by (4.3.18) we get that for all $t \geqslant 0$,

$$\|\psi_\xi(t)\|_\infty \leqslant \left\| \psi_{T\xi}\left(\frac{t}{\lambda_0}\right) \right\|_\infty \quad \mathbb{P}\text{-a.s.}$$

By iteration, we see that for all $n \geqslant 1$ and $t \geqslant 0$,

$$\|\psi_\xi(t)\|_\infty \leqslant \left\| \psi_{T^n\xi}\left(\frac{t}{\lambda_{0,n-1}}\right) \right\|_\infty \quad \mathbb{P}\text{-a.s.} \tag{4.3.20}$$

By iteration of (4.3.19) and using inequality (4.3.20), we obtain that for all $n \geqslant 1$ and $t \geqslant 0$,

$$\psi_\xi(t) \leqslant \prod_{k=0}^{n-1} \left(P_1(\xi_k) + \left\| \psi_{T^n\xi}\left(\frac{t}{\lambda_{0,n-1}}\right) \right\|_\infty Q_1(\xi_k) \right) \psi_{T^n\xi}\left(\frac{t}{\lambda_{0,n-1}}\right) \quad \mathbb{P}\text{-a.s.} \tag{4.3.21}$$

Notice that by (4.1.5) and condition **P2**, for any $1 \leqslant i \leqslant d$ we have

$$1 \geqslant U_{0,\infty}(i) = \frac{\langle M_0(i,\cdot), U_{1,\infty} \rangle}{\|M_0 U_{1,\infty}\|} \geqslant \frac{\min_{1 \leqslant j \leqslant d} M_0(i,j)}{\|M_0\|} \geqslant \frac{A_2}{dA} \quad \mathbb{P}\text{-a.s.} \tag{4.3.22}$$

Therefore, since ϕ_ξ is a decreasing function on $\mathbb{R}_+$, we obtain that for all $1 \leqslant i \leqslant d$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\phi_\xi^i(t) \leqslant \psi_\xi^i(t) = \phi_\xi^i(t U_{0,\infty}(i)) \leqslant \phi_\xi^i\left(t \frac{A_2}{dA}\right).$$

Combining this with (4.3.21), it follows that for all $n \geqslant 1$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\phi_\xi(t) \leqslant \prod_{k=0}^{n-1} \left(P_1(\xi_k) + \left\| \phi_{T^n\xi}\Big(\frac{tA_2}{dA\lambda_{0,n-1}} \Big) \right\|_\infty Q_1(\xi_k) \right) \phi_{T^n\xi}\Big(\frac{tA_2}{dA\lambda_{0,n-1}} \Big). \tag{4.3.23}$$

Now, by Lemma 4.3.6, we know that there exist two constants $\beta \in (0,1)$ and $t_0 > 0$ such that $\|\phi_\xi(t)\|_\infty \leqslant \beta$ a.s. for all $t \geqslant t_0$. Using the inequality $\lambda_{0,n-1} \leqslant \|M_{0,n-1}\| \leqslant (dA)^n$, this implies that for $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$\left\| \phi_{T^n\xi}\Big(\frac{tA_2}{dA\lambda_{0,n-1}} \Big) \right\|_\infty \leqslant \beta, \quad t \geqslant t_1(dA)^n,$$

where $t_1 = \frac{dA}{A_2} t_0$. Combining this with (4.3.23) and (4.3.18), we get that for all $n \geqslant 1$ and $t \geqslant t_1(dA)^n$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\|\phi_\xi(t)\|_\infty &\leqslant \beta \prod_{k=0}^{n-1} \|P_1(\xi_k) + \beta Q_1(\xi_k)\|_\infty \\
&= \beta \prod_{k=0}^{n-1} \Big\| \beta[P_1(\xi_k) + Q_1(\xi_k)] + (1-\beta)P_1(\xi_k) \Big\|_\infty \\
&\leqslant \beta \prod_{k=0}^{n-1} \Big(\beta + (1-\beta)\|P_1(\xi_k)\|_\infty \Big) \\
&\leqslant \beta\alpha^n,
\end{aligned} \tag{4.3.24}$$

where $\alpha = \beta + (1-\beta)\| \|P_1(\xi_0)\|_\infty \|_{L^\infty}$. Clearly we have $\alpha \in (0,1)$, since $\beta \in (0,1)$ and $\| \|P_1(\xi_0)\|_\infty \|_{L^\infty} < 1$ by hypothesis. Set

$$N(t) := \left\lfloor \frac{\log t - \log t_1}{\log(dA)} \right\rfloor + 1, \quad t \geqslant t_1.$$

We observe that, if $t \geqslant t_1$, then $N(t) \geqslant 1$, $t \geqslant t_1(dA)^{N(t)}$ and $N(t) \geqslant (\log t - \log t_1)/\log(dA)$. Therefore, using the inequality (4.3.24) with $n = N(t)$, we deduce that for all $t \geqslant t_1$, $\mathbb{P}$-a.s.,

$$\|\phi_\xi(t)\|_\infty \leqslant \beta\alpha^{N(t)} \leqslant \beta\alpha^{\frac{\log t - \log t_1}{\log(dA)}} = \beta t_1^{-\frac{\log\alpha}{\log(dA)}} t^{\frac{\log\alpha}{\log(dA)}}.$$

Taking $a = -\frac{\log\alpha}{\log(dA)} > 0$ and $C = \beta t_1^a > 0$, we conclude that for all $t \geqslant t_1$,

$$\|\phi_\xi(t)\|_\infty \leqslant \frac{C}{t^a} \quad \mathbb{P}\text{-a.s.},$$

which implies (4.3.3). This ends the proof of Theorem 4.3.5. □

4.3.3 Existence of the annealed harmonic moments $\mathbb{E}(W^i)^{-a}$

The aim of this section is to prove the following theorem which gives the optimal value of a to have $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$, under condition **P2**. (But the proof of the optimality will be done in the next section.)

Theorem 4.3.7. *Assume conditions* ***P1***, ***P2*** *and* $\| \| P_1(\xi_0) \|_\infty \|_{L^\infty} < 1$. *Let* $a > 0$ *be such that* $\kappa_1(a) < 1$. *Then there exists a constant* $C > 0$ *such that for all* $1 \leqslant i \leqslant d$ *and* $t > 0$,

$$\phi^i(t) \leqslant \frac{C}{t^a}, \tag{4.3.25}$$

and that for all $1 \leqslant i \leqslant d$, $0 < b < a$ *and* $x > 0$,

$$\mathbb{P}(W^i \leqslant x) \leqslant C x^a \quad \text{and} \quad \mathbb{E}(W^i)^{-b} \leqslant C. \tag{4.3.26}$$

For the proof of Theorem 4.3.7, we shall need the following technical lemma about the decay late of a function which satisfies a functional inequality.

Lemma 4.3.8. *([58, Lemma 4.1]) Let* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a bounded function. Assume there exists a random variable* $A > 0$ *and constants* $\alpha \in (0,1)$, $C \geqslant 0$, $a > 0$ *and* $t_0 \geqslant 0$ *such that*

$$\phi(t) \leqslant \alpha \mathbb{E} \phi(At) + \frac{C}{t^a}, \quad t > t_0. \tag{4.3.27}$$

If $\alpha \mathbb{E} A^{-a} < 1$, *then* $\phi(t) = O_{t \to +\infty}(t^{-a})$.

Proof of Theorem 4.3.7. First, by Lemma 4.3.2, we have the implication (4.3.25) $\Rightarrow$ (4.3.26). So, it remains to prove (4.3.25).

Let $\varepsilon \in (0,1)$ and $a > 0$ be such that $\kappa_1(a) < 1$. By (4.3.3) in Theorem 4.3.5, we get that there exists a constant $t_\varepsilon > 0$ such that for all $t \geqslant t_\varepsilon$,

$$\| \phi_\xi(t) \|_\infty \leqslant \varepsilon \quad \mathbb{P}\text{-a.s.} \tag{4.3.28}$$

Recall that, by **P2**, we have $\lambda_{0,n-1} \leqslant \| M_{0,n-1} \| \leqslant (dA)^n$ a.s., $n \geqslant 1$. Then, combining

(4.3.28) and (4.3.23), we obtain that for all $n \geqslant 1$ and $t \geqslant \frac{(dA)^{n+1}}{A_2} t_\varepsilon$, $\mathbb{P}$-a.s.,

$$\phi_\xi(t) \leqslant \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \varepsilon Q_1(\xi_k) \Big) \phi_{T^n \xi} \Big(\frac{tA_2}{dA \|M_{0,n-1}\|} \Big). \tag{4.3.29}$$

Taking expectation in (4.3.29) and using the independence of the environments ξ_n, it follows that for all $n \geqslant 1$ and $t \geqslant \frac{(dA)^{n+1}}{A_2} t_\varepsilon$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \phi(t) &\leqslant \mathbb{E}\Big(\prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \varepsilon Q_1(\xi_k) \Big) \mathbb{E}\Big[\phi_{T^n\xi} \Big(\frac{tA_2}{dA\|M_{0,n-1}\|} \Big) \Big| \xi_k, 0 \leqslant k < n \Big] \Big) \\ &= \mathbb{E}\Big[\prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \varepsilon Q_1(\xi_k) \Big) \phi\Big(\frac{tA_2}{dA\|M_{0,n-1}\|} \Big) \Big]. \end{aligned}$$

This implies that for all $n \geqslant 1$ and $t \geqslant \frac{(dA)^{n+1}}{A_2} t_\varepsilon$,

$$\begin{aligned} \|\phi(t)\|_\infty &\leqslant \mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} (P_1(\xi_k) + \varepsilon Q_1(\xi_k)) \Big\|_\infty \Big\| \phi\Big(\frac{tA_2}{dA\|M_{0,n-1}\|} \Big) \Big\|_\infty \Big] \\ &= \alpha_{n,\varepsilon} \mathbb{E} \|\phi(\tilde{A}_{n,\varepsilon} t)\|_\infty, \end{aligned} \tag{4.3.30}$$

where $\alpha_{n,\varepsilon} = \mathbb{E}\Big\| \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \varepsilon Q_1(\xi_k) \Big) \Big\|_\infty > 0$ and $\tilde{A}_{n,\varepsilon}$ is a positive random variable whose distribution is determined by

$$\mathbb{E} h(\tilde{A}_{n,\varepsilon}) = \frac{1}{\alpha_{n,\varepsilon}} \mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \varepsilon Q_1(\xi_k) \Big) \Big\|_\infty h\Big(\frac{A_2}{dA\|M_{0,n-1}\|} \Big) \Big]$$

for all bounded function h on $\mathbb{R}_+$. Now we prove that there exist $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$ satisfying the two following conditions:

$$\alpha_{n,\varepsilon} < 1, \tag{4.3.31}$$

and

$$\alpha_{n,\varepsilon} \mathbb{E} \tilde{A}_{n,\varepsilon}^{-a} < 1. \tag{4.3.32}$$

If (4.3.31) and (4.3.32) hold, then by Lemma 4.3.8,

$$\|\phi(t)\|_\infty = O_{t \to +\infty}(t^{-a}), \tag{4.3.33}$$

which is equivalent to (4.3.25). So it remains to prove (4.3.31) and (4.3.32).

First we show (4.3.31). Notice that for any positive random matrix $M \in \mathcal{M}_d(\mathbb{R})$,

$$\|\mathbb{E}M\|_\infty = \max_{1\leqslant i\leqslant d}\mathbb{E}\Big(\sum_{j=1}^{d} M(i,j)\Big) \tag{4.3.34}$$

$$\leqslant \mathbb{E}\|M\|_\infty \leqslant \sum_{i,j=1}^{d} \mathbb{E}M(i,j) \leqslant d\|\mathbb{E}M\|_\infty. \tag{4.3.35}$$

Using this and the fact that $P_1(\xi_k)+Q_1(\xi_k)$ are stochastic matrices (so that $\| \prod_{k=0}^{n-1}(P_1(\xi_k)+Q_1(\xi_k))\|_\infty = 1$), we get for all $n \geqslant 1$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
\alpha_{n,\varepsilon} &\leqslant \mathbb{E}\Big\|\prod_{k=0}^{n-1} P_1(\xi_k) + \varepsilon\Big(\prod_{k=0}^{n-1}\Big(P_1(\xi_k)+Q_1(\xi_k)\Big) - \prod_{k=0}^{n-1} P_1(\xi_k)\Big)\Big\|_\infty \\
&\leqslant (1-\varepsilon)\mathbb{E}\Big\|\prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty + \varepsilon\mathbb{E}\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k)+Q_1(\xi_k)\Big)\Big\|_\infty \\
&\leqslant d(1-\varepsilon)\|(\mathbb{E}P_1(\xi_0))^n\|_\infty + \varepsilon.
\end{aligned}$$

The hypothesis $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$ implies that $\rho(\mathbb{E}P_1(\xi_0)) < 1$, so that $\|(\mathbb{E}P_1(\xi_0))^n\|_\infty \to_{n\to+\infty} 0$. Therefore, we obtain that for all $\varepsilon \in (0,1)$,

$$\limsup_{n\to+\infty} \alpha_{n,\varepsilon} \leqslant \varepsilon < 1. \tag{4.3.36}$$

Therefore (4.3.31) holds for all $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$ sufficiently large.

Now we prove (4.3.32). By definition of $\tilde{A}_{n,\varepsilon}$, for all $n \geqslant 1$ and $\varepsilon \in (0,1)$ we have

$$\alpha_{n,\varepsilon}\mathbb{E}\tilde{A}_{n,\varepsilon}^{-a} = \Big(\frac{dA}{A_2}\Big)^a \mathbb{E}\Big[\|M_{0,n-1}\|^a\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k)+\varepsilon Q_1(\xi_k)\Big)\Big\|_\infty\Big]. \tag{4.3.37}$$

Notice that $\|M_{0,n-1}\|^a\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k)+\varepsilon Q_1(\xi_k)\Big)\Big\|_\infty \leqslant \|M_{0,n-1}\|^a$ a.s., with $\mathbb{E}\|M_{0,n-1}\|^a < +\infty$. Therefore, by the Lebesgue dominated convergence theorem, letting $\varepsilon \to 0$ in (4.3.37), we get that for all $n \geqslant 1$,

$$\alpha_{n,\varepsilon}\mathbb{E}\tilde{A}_{n,\varepsilon}^{-a} \underset{\varepsilon\to 0}{\to} \Big(\frac{dA}{A_2}\Big)^a \mathbb{E}\Big[\|M_{0,n-1}\|^a\Big\|\prod_{k=0}^{n-1} P_1(\xi_k)\Big\|_\infty\Big].$$

This implies

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\left(\alpha_{n,\varepsilon}\mathbb{E}\tilde{A}_{n,\varepsilon}^{-a}\right)^{1/n}=\kappa_1(a)<1,$$

so (4.3.32) holds for $n\in\mathbb{N}$ sufficiently large and $\varepsilon>0$ small enough.

We have therefore proved (4.3.31) and (4.3.32), which implies (4.3.33). This concludes the proof of Theorem 4.3.7. □

4.3.4 Proofs of Theorem 4.2.1 and Corollary 4.2.2

In this section, using the sufficient condition for the existence of harmonic moments established in Theorem 4.3.7 of the preceding section, we prove Theorem 4.2.1 and Corollary 4.2.2 which give a necessary and a sufficient condition for the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$. To this end, we need the following lemma about the behaviour of the function κ_1 on $\mathbb{R}^+$. Recall that a matrix M is nilpotent if there is an integer $n\geqslant 1$ such that $M^n=0$.

Lemma 4.3.9. *Assume condition **P2**. Then the function κ_1 satisfies the following properties:*

(1) $\kappa_1(0)=\rho\Big(\mathbb{E}P_1(\xi_0)\Big)$*;*

(2) The following assertions are equivalent: (i) $\kappa_1(0)=0$*; (ii)* $\kappa_1(a)=0$ *for all* $a\geqslant 0$*; (iii)* $\mathbb{E}P_1(\xi_0)$ *is nilpotent;*

(3) if $\mathbb{E}P_1(\xi_0)$ *is not nilpotent, then* κ_1 *is a strictly increasing continuous function on* $\mathbb{R}_+$*, with* $\lim_{a\to+\infty}\kappa_1(a)=+\infty$*.*

Proof. (1) Part (1) follows because, using inequalities (4.3.34), we have

$$\begin{aligned}\kappa_1(0)&=\lim_{n\to+\infty}\left(\mathbb{E}\Big\|\prod_{k=0}^{n-1}P_1(\xi_k)\Big\|_\infty\right)^{1/n}\\&=\lim_{n\to+\infty}\left(\Big\|\mathbb{E}\Big[\prod_{k=0}^{n-1}P_1(\xi_k)\Big]\Big\|_\infty\right)^{1/n}\\&=\lim_{n\to+\infty}\Big(\|[\mathbb{E}P_1(\xi_0)]^n\|_\infty\Big)^{1/n}\\&=\rho\Big(\mathbb{E}P_1(\xi_0)\Big).\end{aligned}\tag{4.3.38}$$

(2) We next prove part (2). Notice that the matrix $\mathbb{E}P_1(\xi_0)$ is nilpotent if and only if $\rho\Big(\mathbb{E}P_1(\xi_0)\Big) = 0$. So by part (1), we only need to prove the equivalence between (i) and (ii). To this end we will prove that for all $a, b \geqslant 0$, there exist constants $c_1, c_2 > 0$ such that $\kappa_1(a+b) \leqslant c_1\kappa(a) \leqslant c_2\kappa_1(a+b)$.

By **P2** we have $\|M_{0,n-1}\| \leqslant (dA)^n$ a.s., so for all $a, b \geqslant 0$,

$$\begin{aligned}\kappa_1(a+b) &= \lim_{n\to+\infty}\Big(\mathbb{E}\|M_{0,n-1}\|^{a+b}\Big\|\prod_{k=0}^{n-1}P_1(\xi_k)\Big\|_\infty\Big)^{1/n} \\ &\leqslant \lim_{n\to+\infty}\Big((dA)^{nb}\mathbb{E}\|M_{0,n-1}\|^{a}\Big\|\prod_{k=0}^{n-1}P_1(\xi_k)\Big\|_\infty\Big)^{1/n} \\ &= (dA)^b\kappa_1(a). \end{aligned} \tag{4.3.39}$$

On the other hand, by (4.3.11) we get that for all $n \geqslant 0$,

$$\|M_{0,n-1}\| \geqslant \min_{1\leqslant i\leqslant d} M_{0,n-1}(i,j) \geqslant A_2A_1^{n-1} \quad \mathbb{P}\text{-a.s.}$$

Therefore, we obtain that for all $a, b \geqslant 0$, with $A_1 > 1$,

$$\begin{aligned}\kappa_1(a+b) &\geqslant \lim_{n\to+\infty}\Big((A_2A_1^{n-1})^b\mathbb{E}\|M_{0,n-1}\|^{a}\Big\|\prod_{k=0}^{n-1}P_1(\xi_k)\Big\|_\infty\Big)^{1/n} \\ &= A_1^b\kappa_1(a). \end{aligned} \tag{4.3.40}$$

From (4.3.39) and (4.3.40) we see that $\kappa_1(a) = 0$ for all $a \geqslant 0$ if and only if $\kappa_1(0) = 0$. This ends the proof of part (2).

(3) We finally prove part (3). Assume that $\mathbb{E}P_1(\xi_0)$ is not nilpotent. Then, by parts (1) and (2), we have $\kappa_1(0) = \rho\Big(\mathbb{E}P_1(\xi_0)\Big) > 0$. It follows from (4.3.39) and (4.3.40) that κ_1 is a strictly increasing continuous function on $\mathbb{R}_+$ with $\lim_{a\to+\infty}\kappa_1(a) = +\infty$. This concludes part (3).

The proof of Lemma 4.3.9 is complete. □

Proof of Theorem 4.2.1. First we assume that $\kappa_1(a) < 1$, where $a > 0$. By Lemma 4.3.9, κ_1 is a continuous function on $\mathbb{R}_+$, so there exists $\varepsilon > 0$ such that $\kappa_1(a+\varepsilon) < 1$. Using Theorem 4.3.7 (with a replaced by $a+\varepsilon$), this implies that $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$.

Now we suppose that $a > 0$ is such that $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$. We will prove that $\kappa_1(a) \leqslant 1$. Let $Z^r_{l,n,k} = (Z^r_{l,n,k}(1), \cdots, Z^r_{l,n,k}(d))$, where $Z^r_{l,n,k}(j)$ denotes the offspring of type j at time $n+k$ of the l-th particle of type r in the generation n. By iteration of (4.2.1), it is easy to see that the MBPRE $(Z_n^i)_{n\geqslant 0}$ satisfies the relation

$$Z_{n+k}^i = \sum_{j=1}^{d} \sum_{l=1}^{Z_n^i(j)} Z_{l,n,k}^j, \quad n,k \geqslant 0.$$

It follows that for all $n,k\geqslant 0$ and $1\leqslant i\leqslant d$,

$$\begin{aligned} W_{n+k}^i &= \sum_{j=1}^{d} \sum_{l=1}^{Z_n^i(j)} \frac{\langle Z_{l,n,k}^{rj}, U_{n+k,\infty}\rangle}{\lambda_{0,n+k-1}U_{0,\infty}(i)} \\ &= \sum_{j=1}^{d} \frac{U_{n,\infty}(j)}{\lambda_{0,n-1}U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(j)} W_{l,n,k}^j, \end{aligned} \tag{4.3.41}$$

where

$$W_{l,n,k}^j := \frac{\langle Z_{l,n,k}^j, U_{n+k,\infty}\rangle}{\lambda_{n,n+k-1}U_{n,\infty}(j)}.$$

Clearly $(W_{l,n,k}^j)_{k\geqslant 0}$ is the fundamental martingale associated to the MBPRE $(Z_{l,n,k}^j)_{k\geqslant 0}$ in the shifted random environment $T^n\xi$, starting with the l-th particle of type j in the generation n. Hence $(W_{l,n,k}^j)_{k\geqslant 0}$ converges a.s. to a random variable $W_{l,n}^j$. Letting $k\to+\infty$ in (4.3.41), we get the following distributional equation on the limit variables W^i: for all $1\leqslant i\leqslant d$ and $n\geqslant 0$,

$$W^i = \sum_{r=1}^{d} \frac{U_{n,\infty}(r)}{\lambda_{0,n-1}U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(r)} W_{l,n}^r, \tag{4.3.42}$$

where for each $n\in\mathbb{N}$, under $\mathbb{P}_\xi$, the random variables $W_{l,n}^r$, indexed by $l\geqslant 0$ and $1\leqslant r\leqslant d$, are independent of each other and independent of Z_n^i, each with the distribution

$$\mathbb{P}_\xi(W_{l,n}^r \in \cdot) = \mathbb{P}_{T^n\xi}(W^r\in\cdot).$$

Therefore, we get from (4.3.42) that for all $1 \leqslant i \leqslant d$ and $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\mathbb{E}_\xi (W^i)^{-a} &= \mathbb{E}_\xi \left(\sum_{r=1}^{d} \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(r)} W_{l,n}^r \right)^{-a} \\
&\geqslant \sum_{r=1}^{d} \left(\frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \right)^{-a} \mathbb{E}_\xi \left[\left(\sum_{l=1}^{Z_n^i(r)} W_{l,n}^r \right)^{-a} \mathbb{1}_{\{Z_n^i = e_r\}} \right] \\
&= (\lambda_{0,n-1} U_{0,\infty}(i))^a \sum_{r=1}^{d} U_{n,\infty}(r)^{-a} \mathbb{E}_{T^n \xi} (W^r)^{-a} \mathbb{P}_\xi (Z_n^i = e_r). \qquad (4.3.43)
\end{aligned}$$

For $n \geqslant 0$, let X_n be the vector in $\mathbb{R}^d$ whose i-th entry is

$$X_n(i) = U_{n,\infty}(i)^{-a} \mathbb{E}_{T^n \xi} (W^i)^{-a}, \quad 1 \leqslant i \leqslant d. \qquad (4.3.44)$$

By **P1** and the definition of the random matrices $P_1(\xi_n)$, we have that for all $n \geqslant 1$ and $1 \leqslant i, r \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\mathbb{P}_\xi (Z_{n+1}^i = e_r) &= \sum_{j=1}^{d} \mathbb{P}_\xi (Z_n^i = e_j) \mathbb{P}_\xi (Z_{n+1}^i = e_r | Z_n^i = e_j) \\
&= \sum_{j=1}^{d} \mathbb{P}_\xi (Z_n^i = e_j) P_1(\xi_n)(j, r).
\end{aligned}$$

By iterating this relation, we get that for all $n \geqslant 1$ and $1 \leqslant i, r \leqslant d$, $\mathbb{P}$-a.s.,

$$\mathbb{P}_\xi (Z_n^i = e_r) = \left[\prod_{k=0}^{n-1} P_1(\xi_k) \right] (i, r). \qquad (4.3.45)$$

Using this and the notation X_n (cf. (4.3.44)), we can re-write (4.3.43) as follows: for all $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$X_0 \geqslant \lambda_{0,n-1}^a \prod_{k=0}^{n-1} P_1(\xi_k) \, X_n. \qquad (4.3.46)$$

By (4.1.5) and (4.3.22), we get that for all $n \geqslant 1$,

$$\lambda_{0,n-1} = \| M_{0,n-1} U_{n,\infty} \| \geqslant \frac{A_2}{dA} \| M_{0,n-1} \mathbf{1} \| \geqslant \frac{A_2}{dA} \| M_{0,n-1} \| \quad \mathbb{P}\text{-a.s.}$$

This, together with (4.3.46), implies that for all $n \geqslant 1$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \|X_0\|_\infty &\geqslant \left(\frac{A_2}{dA}\right)^a \|M_{0,n-1}\|^a \left\| \prod_{k=0}^{n-1} P_1(\xi_k)\ X_n \right\|_\infty \\ &\geqslant \left(\frac{A_2}{dA}\right)^a \|M_{0,n-1}\|^a \left\| \prod_{k=0}^{n-1} P_1(\xi_k) \right\|_\infty \min_{1\leqslant i\leqslant d} X_n(i). \end{aligned} \tag{4.3.47}$$

Notice that $(X_n)_{n\geqslant 0}$ is a stationary ergodic sequence of strictly positive random vectors, and X_n is independent of $\xi_0, \cdots, \xi_{n-1}$. Therefore, taking expectation in (4.3.47), we get that for all $n \geqslant 1$,

$$\mathbb{E}\|X_0\|_\infty \geqslant \left(\frac{A_2}{dA}\right)^a \mathbb{E}\left[\|M_{0,n-1}\|^a \left\| \prod_{k=0}^{n-1} P_1(\xi_k) \right\|_\infty\right] \mathbb{E}\Big(\min_{1\leqslant i\leqslant d} X_0(i)\Big). \tag{4.3.48}$$

Using again (4.3.22), for any $1 \leqslant i \leqslant d$ we have

$$0 < \mathbb{E}X_0(i) = \mathbb{E}[U_{0,\infty}(i)^{-a}\mathbb{E}_\xi(W^i)^{-a}] \leqslant \left(\frac{A_2}{dA}\right)^{-a} \mathbb{E}(W^i)^{-a} < +\infty.$$

Therefore, by (4.3.48) we obtain for all $n \geqslant 1$,

$$\mathbb{E}\|M_{0,n-1}\|^a \left\| \prod_{k=0}^{n-1} P_1(\xi_k) \right\|_\infty \leqslant \left(\frac{dA}{A_2}\right)^a \frac{\mathbb{E}\|X_0\|_\infty}{\mathbb{E}\Big(\min_{1\leqslant i\leqslant d} X_0(i)\Big)}.$$

This implies that

$$\kappa_1(a) = \lim_{n\to+\infty} \left(\mathbb{E}\|M_{0,n-1}\|^a \left\| \prod_{k=0}^{n-1} P_1(\xi_k) \right\|_\infty\right)^{1/n} \leqslant 1,$$

which is the desired result. This concludes the proof of Theorem 4.2.1 and the remark following it. □

Proof of Corollary 4.2.2. First we suppose that $\mathbb{E}P_1(\xi_0)$ is nilpotent. By Lemma 4.3.9, we know that $\kappa_1(a) = 0 < 1$ for all $a \geqslant 0$. Therefore, using Theorem 4.2.1, we get that $\mathbb{E}(W^i)^{-a} < +\infty$ for all $1 \leqslant i \leqslant d$ and $a > 0$.

Now, we assume that $\mathbb{E}P_1(\xi_0)$ is not nilpotent. By Lemma 4.3.9 and the condition $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$, the function κ_1 is continuous and strictly increasing on $\mathbb{R}_+$ with $\kappa_1(0) = \rho\Big(\mathbb{E}P_1(\xi_0)\Big) < 1$ and $\lim_{a\to+\infty} \kappa_1(a) = +\infty$. Therefore, there exists a unique constant $a_0 > 0$ such that $\kappa_1(a_0) = 1$, and we have $\kappa_1(a) < 1$ if $a < a_0$, and $\kappa_1(a) > 1$ if

$a > a_0$. Using again Theorem 4.2.1, this implies that

$$\max_{1\leqslant i\leqslant d} \mathbb{E}(W^i)^{-a} \begin{cases} < +\infty & \text{if } a \in [0, a_0), \\ = +\infty & \text{if } a \in (a_0, +\infty). \end{cases}$$

The proof of Corollary 4.2.2 is complete. □

4.3.5 Proof of Theorem 4.2.3

In this section, we prove Theorem 4.2.3 which gives the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ for a small $a > 0$, under the moment condition **P3**, instead of the boundedness condition **P2** considered in the preceding section. We will prove the following theorem.

Theorem 4.3.10. *Assume conditions **P1**, **P3** and $\gamma > 0$. Then there exist two constants $a > 0$ and $C > 0$ such that for all $1 \leqslant i \leqslant d$, all $t > 0$, and all $x > 0$,*

$$\phi^i(t) \leqslant \frac{C}{t^a}, \tag{4.3.49}$$

$$\mathbb{P}(W^i \leqslant x) \leqslant C x^a \quad \text{and} \quad \mathbb{E}(W^i)^{-a} \leqslant C. \tag{4.3.50}$$

Clearly Theorem 4.2.3 follows from Theorem 4.3.10. We need some previous results to prove Theorem 4.3.10. The first one gives the non-degeneracy of the limits W^i under the conditions of Theorem 4.3.10.

Lemma 4.3.11. *The following implication holds: **P3** ⇒ (4.2.5). Moreover, under the conditions **P3** and $\gamma > 0$, the limit W^i is non-degenerate for each $1 \leqslant i \leqslant d$, and (4.2.6) hold.*

Proof. We first prove the implication **P3** ⇒ (4.2.5). Assume condition **P3**. Then, by Hölder's inequality $\mathbb{E}_\xi|XY| \leqslant (\mathbb{E}_\xi|X|^\alpha)^{1/\alpha}(\mathbb{E}_\xi|Y|^\beta)^{1/\beta}$ with $\alpha = 1/(1-\eta)$ and $\beta = 1/\eta$, we see that for all $1 \leqslant i,j \leqslant d$ we have, $\mathbb{P}$-a.s.,

$$\begin{aligned}
& \mathbb{E}_\xi \left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right) \\
= \; & \mathbb{E}_\xi \left[\left(\frac{Z_1^i(j)}{M_0(i,j)} \right)^{1-\eta} \left(\frac{Z_1^i(j)}{M_0(i,j)} \right)^{\eta} \log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right] \\
\leqslant \; & \left[\mathbb{E}_\xi \left(\frac{Z_1^i(j)}{M_0(i,j)} \right) \right]^{1-\eta} \left\{ \mathbb{E}_\xi \left[\frac{Z_1^i(j)}{M_0(i,j)} \left(\log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right)^{1/\eta} \right] \right\}^{\eta} \\
= \; & \left\{ \mathbb{E}_\xi \left[\frac{Z_1^i(j)}{M_0(i,j)} \left(\log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right)^{1/\eta} \right] \right\}^{\eta}. \qquad (4.3.51)
\end{aligned}$$

We know that there exists a constant $C > 0$ such that $\log^+ x \leqslant C x^{\eta(p-1)}$ for all $x > 0$. Taking expectation in (4.3.51), we obtain that for all $1 \leqslant i,j \leqslant d$,

$$\mathbb{E} \left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)} \right) \leqslant C \mathbb{E} \left[\mathbb{E}_\xi \left(\frac{Z_1^i(j)}{M_0(i,j)} \right)^p \right]^{\eta} < +\infty, \qquad (4.3.52)$$

since $\mathbb{E}\theta_0(p)^\eta < +\infty$ by condition **P3**. Hence the condition (4.2.5) holds.

Recall that by [32, Theorem 2.6 and Corollary 2.8], condition (4.2.5) together with $\gamma > 0$ implies that the limit W^i is non-degenerate for each $1 \leqslant i \leqslant d$, and (4.2.6) hold. Combining this with the implication **P3** $\Rightarrow$ (4.2.5), we conclude the proof of Lemma 4.3.11. □

By the sub-additive ergodic theorem and the fact that $\|M_{0,n}(i,\cdot)\| \geqslant 1$ a.s. for any $1 \leqslant i \leqslant d$ (which follows from **P1**), for all $s < 0$ the limit

$$\kappa(s) := \lim_{n \to +\infty} \left(\mathbb{E} \|M_{0,n-1}\|^s \right)^{1/n} \qquad (4.3.53)$$

exists, with $\kappa(s) = \sup_{n \geqslant 1} \left(\mathbb{E}\|M_{0,n-1}\|^s \right)^{1/n} \leqslant 1$. The proof of Theorem 4.3.10 is based on the following result about the properties of the function κ.

Lemma 4.3.12. *Assume conditions **P1**, **P3** and $\gamma > 0$. Then for all $s < 0$,*

$$\kappa(s) < 1.$$

Moreover, if $\eta > 0$ satisfies ***P3****, then for all $0 \leqslant a < s \leqslant \eta/2$,*

$$\left(\mathbb{E}\Big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^{-s}\Big]\right)^{1/n} \underset{n\to+\infty}{\to} \kappa(a-s) < 1. \tag{4.3.54}$$

Proof. First we prove that $\kappa(s) < 1$ for all $s < 0$. Recall that $\iota(M) = \inf_{\|x\|=1} \|Mx\|$, for $M \in \mathcal{M}_d(\mathbb{R})$, and $\iota\Big(M_{0,n}^T\Big) \geqslant 1$ a.s. by (4.3.10) and **P1**. Since ι is super-multiplicative, the sequence $\mathbb{E}\,\iota\Big(M_{0,n-1}^T\Big)^s$ is sub-multiplicative. So using the sub-additive ergodic theorem we get

$$\lim_{n\to+\infty} \left(\mathbb{E}\,\iota\Big(M_{0,n-1}^T\Big)^s\right)^{1/n} = \inf_{n\geqslant 1} \left(\mathbb{E}\,\iota\Big(M_{0,n-1}^T\Big)^s\right)^{1/n} \leqslant 1, \quad \forall s < 0.$$

By the inequalities $\iota\Big(M^T\Big) \leqslant \|M^T\| \leqslant d\|M\|$, $M \in \mathcal{M}_d(\mathbb{R})$, this implies that for all $s < 0$,

$$\kappa(s) \leqslant \lim_{n\to+\infty} \left(\mathbb{E}\,\iota\Big(M_{0,n-1}^T\Big)^s\right)^{1/n} = \inf_{n\geqslant 1} \left(\mathbb{E}\,\iota\Big(M_{0,n-1}^T\Big)^s\right)^{1/n}.$$

Therefore, for all $s < 0$ and $n \geqslant 1$, we have

$$\kappa(s)^n \leqslant \mathbb{E}\,\iota\Big(M_{0,n-1}^T\Big)^s \leqslant 1. \tag{4.3.55}$$

Clearly by (4.3.55), if $\kappa(s) = 1$, then $\mathbb{E}\,\iota\Big(M_{0,n-1}^T\Big)^s = 1$ for all $n \geqslant 1$, which is equivalent to $\iota\Big(M_{0,n-1}^T\Big) = 1$ a.s. for all $n \geqslant 1$. Consequently, if we show that

$$\lim_{n\to+\infty} \mathbb{P}\Big(\iota\Big(M_{0,n-1}^T\Big) = 1\Big) = 0, \tag{4.3.56}$$

then we will have $\kappa(s) < 1$ for all $s < 0$. So, it remains to prove (4.3.56). Using **P1** and the identity $\iota(M) = \min_{1\leqslant j\leqslant d} \|M(\cdot,j)\|$, $M \in \mathcal{M}_d(\mathbb{R})$, for all $n \geqslant 1$ we have

$$\mathbb{P}\Big(\iota\Big(M_{0,n-1}^T\Big) = 1\Big) \leqslant \sum_{i=1}^d \mathbb{P}(\|M_{0,n-1}(i,\cdot)\| = 1) \leqslant \sum_{i=1}^d \mathbb{P}(\|Z_n^i\| = 1). \tag{4.3.57}$$

By Lemma 4.3.11, the conditions **P3** and $\gamma > 0$ imply that the limits W^i, $1 \leqslant i \leqslant d$, are non-degenerate, and (4.2.6) holds. Combining this with condition **P1**, we obtain that $\lim_{n\to+\infty} \|Z_n^i\| = +\infty$ a.s. Therefore, we obtain that for all $1 \leqslant i \leqslant d$,

$$\lim_{n\to+\infty} \mathbb{P}(\|Z_n^i\| = 1) = 0, \tag{4.3.58}$$

and (4.3.56) follows from (4.3.57). So $\kappa(s) < 1$ for all $s < 0$, which is the desired result.

Now we prove (4.3.54). Let $\eta > 0$ be such that condition **P3** holds, and set $0 < a < s \leqslant \eta/2$. First, for all $n \geqslant 1$ we have

$$\mathbb{E}\Big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^{-s}\Big] \geqslant \mathbb{E}\|M_{0,n-1}\|^{a-s}. \tag{4.3.59}$$

Next, for all $n \geqslant 2$,

$$\begin{aligned}
&\mathbb{E}\Big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^{-s}\Big]\\
=\;&\mathbb{E}\Big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} \Big(\sum_{r=1}^{d} M_0(i,r)M_{1,n-1}(r,j)\Big)^{-s}\Big]\\
\leqslant\;&\mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,r\leqslant d} M_0(i,r)^{-s}\Big]\mathbb{E}\Big[\|M_{1,n-1}\|^a \max_{1\leqslant j\leqslant d} \|M_{1,n-1}(\cdot,j)\|^{-s}\Big]\\
=\;&\mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,r\leqslant d} M_0(i,r)^{-s}\Big]\mathbb{E}\Big[\|M_{0,n-2}\|^a \max_{1\leqslant j\leqslant d} \|M_{0,n-2}(\cdot,j)\|^{-s}\Big]
\end{aligned}$$

It follows that for all $n \geqslant 3$,

$$\begin{aligned}
&\mathbb{E}\Big[\|M_{0,n-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^{-s}\Big]\\
\leqslant\;&\mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}\Big] \times\\
&\mathbb{E}\Big[\|M_{0,n-2}\|^a \max_{1\leqslant j\leqslant d} \Big(\sum_{r=1}^{d} \|M_{0,n-3}(\cdot,r)\|M_{n-2}(r,j)\Big)^{-s}\Big]\\
\leqslant\;&\mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}\Big] \times\\
&\mathbb{E}\Big[\|M_{n-2}\|^a \max_{1\leqslant i,j\leqslant d} M_{n-2}(i,j)^{-s}\Big]\mathbb{E}\Big[\|M_{0,n-3}\|^a (d\|M_{0,n-3}\|)^{-s}\Big]\\
=\;&d^{-s}\Big(\mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}\Big]\Big)^2 \mathbb{E}\|M_{0,n-3}\|^{a-s}.
\end{aligned} \tag{4.3.60}$$

Then, by Hölder's inequality and condition **P3**, since $2a < 2s \leqslant \eta$, we have

$$\begin{aligned}
\mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}\Big] &\leqslant \Big(\mathbb{E}\|M_0\|^{2a}\Big)^{1/2}\Big(\mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-2s}\Big]\Big)^{1/2}\\
&< +\infty.
\end{aligned}$$

Combining this with the inequalities (4.3.59) and (4.3.60), we obtain that for all $n \geqslant 3$,

$$\mathbb{E}\|M_{0,n-1}\|^{a-s} \leqslant \mathbb{E}\Big[\|M_{0,n-1}\|^{a} \max_{1\leqslant i,j\leqslant d} M_{0,n-1}(i,j)^{-s}\Big] \leqslant C\,\mathbb{E}\|M_{0,n-3}\|^{a-s},$$

where $C = d^{-s}\Big(\mathbb{E}\Big[\|M_0\|^a \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{-s}\Big]\Big)^2 < +\infty$. This, together with (4.3.53), implies (4.3.54), which concludes the proof of Lemma 4.3.12. □

Proof of Theorem 4.3.10. The implication (4.3.49) $\Rightarrow$ (4.3.50) holds by Lemma 4.3.2. Therefore, it remains to show (4.3.49) under the given conditions.

Assume condition **P1**, **P3** and $\gamma > 0$. Set $\psi_\xi(t) = (\psi_\xi^1(t), \cdots, \psi_\xi^d(t))$, where for all $t \geqslant 0$, and $1 \leqslant i \leqslant d$,

$$\psi_\xi^i(t) = \phi_\xi^i(tU_{0,\infty}(i)).$$

Let ψ be the function on $\mathbb{R}_+$ defined by

$$\psi(t) = \mathbb{E}\|\psi_\xi(t)\|_\infty, \quad t \geqslant 0.$$

Since ϕ_ξ^i is a decreasing function on $\mathbb{R}_+$ and $0 < U_{0,\infty}(i) < 1$ a.s., for all $t \geqslant 0$ we have

$$\|\phi(t)\|_\infty = \|\mathbb{E}\phi_\xi(t)\|_\infty \leqslant \mathbb{E}\|\phi_\xi(t)\|_\infty \leqslant \mathbb{E}\|\psi_\xi(t)\|_\infty = \psi(t).$$

Therefore, if we prove that there exists a constant $a > 0$ such that

$$\psi(t) = O_{t\to+\infty}(t^{-a}), \tag{4.3.61}$$

then (4.3.49) holds.

Now we prove (4.3.61). As in the proof of Theorem 4.2.1, the argument will still be based on (4.3.21). The idea is to take expectation at both sides of this inequality, and to get an inequality on ψ in order to use Lemma 4.3.8 to conclude. The difficult point is to have a bound in terms of ψ while taking expectation on the right hand side of (4.3.21). This will be done by truncation and iteration.

Recall that $\lambda_{0,n-1} = \|M_{0,n-1}U_{n,\infty}\| \leqslant \|M_{0,n-1}\|$ for all $n \geqslant 1$, and ψ_ξ^i is decreasing on $\mathbb{R}_+$. Therefore, we get that for all $n \geqslant 1$ and $t \geqslant 0$,

$$\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big) \leqslant \psi_{T^n\xi}\Big(\frac{t}{\|M_{0,n-1}\|}\Big) \quad \mathbb{P}\text{-a.s.} \tag{4.3.62}$$

Moreover, by similar arguments as in (4.3.14)-(4.3.16), we have that for all $K > 0$, there exist two constants $t_K > 0$ and $\beta_K \in (0,1)$ such that for all $t \geqslant t_K$, $\|\phi_\xi(t)\|_\infty \leqslant \beta_K$ $\mathbb{P}$-a.s. on the event $\{\max_{1\leqslant i\leqslant d} \mathbb{E}_\xi (W^i)^p \leqslant K^p\}$. Therefore when $t \geqslant \frac{t_K \lambda_{0,n-1}}{\min\limits_{1\leqslant i\leqslant d} U_{n,\infty}(i)}$ and $\max\limits_{1\leqslant i\leqslant d} \mathbb{E}_{T^n\xi}(W^i)^p \leqslant K^p$, we have a.s.

$$\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \leqslant \beta_K.$$

Combining this with (4.3.21) and (4.3.62), we obtain that for all $K > 0$, $n \geqslant 1$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{E}\Big[\|\psi_\xi(t)\|_\infty \mathbb{1}_{\{t \geqslant \frac{t_K\lambda_{0,n-1}}{\min\limits_{1\leqslant i\leqslant d} U_{n,\infty}(i)},\ \max\limits_{1\leqslant i\leqslant d} \mathbb{E}_{T^n\xi}(W^i)^p \leqslant K^p\}}\Big] \\
\leqslant\ & \mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k)+\beta_K Q_1(\xi_k)\Big)\psi_{T^n\xi}\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big\|_\infty\Big] \\
\leqslant\ & \mathbb{E}\Big\{\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k)+\beta_K Q_1(\xi_k)\Big)\Big\|_\infty \mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big\|_\infty\Big|\xi_0,\cdots,\xi_{n-1}\Big]\Big\} \\
=\ & \mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k)+\beta_K Q_1(\xi_k)\Big)\Big\|_\infty \psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big].
\end{aligned}$$

This, together with (4.3.20), implies that for all $K > 0$, $n \geqslant 1$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\psi(t) \\
\leqslant\ & \mathbb{E}\Big[\|\psi_\xi(t)\|_\infty \mathbb{1}_{\{t \geqslant \frac{t_K\lambda_{0,n-1}}{\min_{1\leqslant i\leqslant d} U_{n,\infty}(i)},\ \max\limits_{1\leqslant i\leqslant d} \mathbb{E}_{T^n\xi}(W^i)^p \leqslant K^p\}}\Big] \\
&+\mathbb{E}\Big[\|\psi_\xi(t)\|_\infty \mathbb{1}_{\{\max\limits_{1\leqslant i\leqslant d} \mathbb{E}_{T^n\xi}(W^i)^p > K^p\}}\Big] + \mathbb{E}\Big[\|\psi_\xi(t)\|_\infty \mathbb{1}_{\{t < \frac{t_K\lambda_{0,n-1}}{\min_{1\leqslant i\leqslant d} U_{n,\infty}(i)}\}}\Big] \\
\leqslant\ & \mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k)+\beta_K Q_1(\xi_k)\Big)\Big\|_\infty \psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big] \\
&+\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \mathbb{1}_{\{\max\limits_{1\leqslant i\leqslant d} \mathbb{E}_{T^n\xi}(W^i)^p > K^p\}}\Big] \\
&+\mathbb{P}\Big(\frac{\|M_{0,n-1}\|}{\min_{1\leqslant i\leqslant d} U_{n,\infty}(i)} > \frac{t}{t_K}\Big). \qquad (4.3.63)
\end{aligned}$$

Now we control the second and last terms in (4.3.63).

We first find a bound for the second term in terms of ψ. Using the triangular inequality in L^p and (4.3.8), we get that for all $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\Big(\mathbb{E}_\xi (W^i)^p\Big)^{1/p} \leqslant 1 + \sum_{n=0}^{+\infty} \Big(\mathbb{E}_\xi |W^i_{n+1} - W^i_n|^p\Big)^{1/p}$$
$$\leqslant 1 + B_p \sum_{n=0}^{+\infty} \theta_n(p)^{\frac{1}{p}} \max_{1\leqslant j\leqslant d} M_{0,n-1}(i,j)^{\frac{1-p}{p}}.$$

By the sub-additivity of the function $x \mapsto x^{\frac{\eta}{2}}$ on $\mathbb{R}_+$ (since $0<\eta<1$), this implies that for all $K>0$, $n \geqslant 1$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \mathbb{1}_{\{\max\limits_{1\leqslant i\leqslant d} \mathbb{E}_{T^n\xi}(W^i)^p > K^p\}}\Big] \\
\leqslant\ & \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \max_{1\leqslant i\leqslant d}\Big(\mathbb{E}_{T^n\xi}(W^i)^p\Big)^{\frac{\eta}{2p}}\Big] \\
\leqslant\ & \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty\Big] + \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}} \times \\
& \sum_{k=0}^{+\infty} \mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \theta_{n+k}(p)^{\frac{\eta}{2p}} \max_{1\leqslant i,j\leqslant d} M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big].
\end{aligned} \tag{4.3.64}$$

By (4.3.20) and (4.3.62), we get that for $n,k \geqslant 0$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \leqslant \Big\|\psi_{T^{n+k+1}\xi}\Big(\frac{t}{\lambda_{0,n+k}}\Big)\Big\|_\infty \leqslant \Big\|\psi_{T^{n+k+1}\xi}\Big(\frac{t}{\|M_{0,n+k}\|}\Big)\Big\|_\infty.$$

Combining this with (4.3.64), we obtain that for all $K>0$, $n \geqslant 1$ and $t \geqslant 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\lambda_{0,n-1}}\Big)\Big\|_\infty \mathbb{1}_{\{\max\limits_{1\leqslant i\leqslant d} \mathbb{E}_{T^n\xi}(W^i)^p > K^p\}}\Big] \\
\leqslant\ & \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\Big\|\psi_{T^n\xi}\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big\|_\infty\Big] + \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}} \times \\
& \sum_{k=0}^{+\infty} \mathbb{E}\Big[\Big\|\psi_{T^{n+k+1}\xi}\Big(\frac{t}{\|M_{0,n+k}\|}\Big)\Big\|_\infty \theta_{n+k}(p)^{\frac{\eta}{2p}} \max_{1\leqslant i,j\leqslant d} M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big] \\
\leqslant\ & \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big] + \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}} \times \\
& \sum_{k=0}^{+\infty} \mathbb{E}\Big[\psi\Big(\frac{t}{\|M_{0,n+k}\|}\Big)\theta_{n+k}(p)^{\frac{\eta}{2p}} \max_{1\leqslant i,j\leqslant d} M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big],
\end{aligned} \tag{4.3.65}$$

where, for the last inequality, we have used the fact that $M_{0,n+k}$ and $\theta_{n+k}(p)$ are independent of $T^{n+k+1}\xi$. We have therefore obtained a bound of the second term in (4.3.63) in terms of ψ.

For the last term in (4.3.63), by Markov's inequality we get that for all $K>0$, $n\geqslant 1$ and $t>0$,

$$\mathbb{P}\Big(\frac{\|M_{0,n-1}\|}{\min_{1\leqslant i\leqslant d}U_{n,\infty}(i)}>\frac{t}{t_K}\Big)\leqslant\Big(\frac{t_K}{t}\Big)^{\frac{\eta}{2}}\mathbb{E}\Big(\frac{\|M_{0,n-1}\|^{\frac{\eta}{2}}}{\min_{1\leqslant i\leqslant d}U_{n,\infty}(i)^{\frac{\eta}{2}}}\Big)$$
$$\leqslant\frac{C(n,K)}{t^{\eta/2}},\tag{4.3.66}$$

where $C(n,K):=t_K^{\frac{\eta}{2}}\Big(\mathbb{E}\|M_0\|^{\frac{\eta}{2}}\Big)^n\mathbb{E}\Big(\max_{1\leqslant i\leqslant d}U_{0,\infty}(i)^{-\frac{\eta}{2}}\Big)$. We claim that $C(n,K)<\infty$ because, by (4.1.4), Cauchy-Schwarz's inequality and condition **P3**, we have for all $1\leqslant i\leqslant d$

$$\begin{aligned}\mathbb{E}\Big(U_{0,\infty}(i)^{-\frac{\eta}{2}}\Big)&=\mathbb{E}\Big(\frac{\langle M_0(i,\cdot),U_{1,\infty}\rangle^{-\frac{\eta}{2}}}{\|M_0U_{1,\infty}\|^{-\frac{\eta}{2}}}\Big)\\&\leqslant\mathbb{E}\Big(\frac{\max_{1\leqslant j\leqslant d}M_0(i,j)^{-\frac{\eta}{2}}}{\|M_0\|^{-\frac{\eta}{2}}}\Big)\\&\leqslant\Big(\mathbb{E}\Big[\max_{1\leqslant j\leqslant d}M_0(i,j)^{-\eta}\Big]\Big)^{1/2}\Big(\mathbb{E}\|M_0\|^{\eta}\Big)^{1/2}<+\infty.\end{aligned}\tag{4.3.67}$$

Putting together the inequalities (4.3.63), (4.3.64), (4.3.65) and (4.3.66), we obtain the following inequality on ψ: for all $K>0$, $n\geqslant 1$ and $t>0$,

$$\begin{aligned}&\psi(t)\\\leqslant\;&\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k)+\beta_KQ_1(\xi_k)\Big)\Big\|_\infty\psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big]\\&+\Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}}\sum_{k=0}^{+\infty}\mathbb{E}\Big[\psi\Big(\frac{t}{\|M_{0,n+k}\|}\Big)\theta_{n+k}(p)^{\frac{\eta}{2p}}\max_{1\leqslant i,j\leqslant d}M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]\\&+\frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\Big[\psi\Big(\frac{t}{\|M_{0,n-1}\|}\Big)\Big]+\frac{C(n,K)}{t^{\frac{\eta}{2}}}.\end{aligned}\tag{4.3.68}$$

We will write this inequality in the form (4.3.27) in order to use Lemma 4.3.8. For

$K > 0$ and $n \geqslant 1$, set

$$\alpha_{n,K} := \mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \beta_K Q_1(\xi_k)\Big)\Big\|_\infty\Big] + \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}} \mathbb{E}\Big[\theta_0(p)^{\frac{\eta}{2p}}\Big] \sum_{k=0}^{+\infty} \mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d} M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big] + \frac{1}{K^{\frac{\eta}{2}}}.$$

By condition **P3** and equality (4.3.18), and using (4.3.54) in Lemma 4.3.12 with $a = 0$ and $s = \frac{\eta(p-1)}{2p} \in \big(0, \frac{\eta}{2}\big)$, we get for all $K > 0$ and $n \geqslant 1$,

$$\alpha_{n,K} < +\infty.$$

For any $K > 0$ and $n \geqslant 1$, let $\tilde{A}_{n,K}$ be a positive random variable whose distribution is determined by the following expectation: for all bounded and mesurable function h on $\mathbb{R}_+$,

$$\begin{aligned}
&\mathbb{E}h(\tilde{A}_{n,K}) \\
= \;& \frac{1}{\alpha_{n,K}}\mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \beta_K Q_1(\xi_k)\Big)\Big\|_\infty h\Big(\frac{1}{\|M_{0,n-1}\|}\Big)\Big] \\
&+ \frac{B_p^{\frac{\eta}{2}}}{K^{\frac{\eta}{2}}\alpha_{n,K}} \sum_{k=0}^{+\infty} \mathbb{E}\Big[h\Big(\frac{1}{\|M_{0,n+k}\|}\Big)\theta_{n+k}(p)^{\frac{\eta}{2p}} \max_{1\leqslant i,j\leqslant d} M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big] \\
&+ \frac{1}{K^{\frac{\eta}{2}}\alpha_{n,K}}\mathbb{E}\Big[h\Big(\frac{1}{\|M_{0,n-1}\|}\Big)\Big].
\end{aligned}$$

So we can rewrite (4.3.68) as in the form (4.3.27): for all $K > 0$, $n \geqslant 1$ and $t > 0$,

$$\psi(t) \leqslant \alpha_{n,K}\mathbb{E}\Big[\psi(\tilde{A}_{n,K}t)\Big] + \frac{C(n,K)}{t^{\eta/2}}. \tag{4.3.69}$$

We will prove that there exist $K > 0$, $n \geqslant 1$ and $0 < a < \eta/2$ such that

$$\alpha_{n,K} < 1, \tag{4.3.70}$$

and

$$\alpha_{n,K}\mathbb{E}\tilde{A}_{n,K}^{-a} < 1. \tag{4.3.71}$$

By Lemma 4.3.8, inequality (4.3.69), together with (4.3.70) and (4.3.71), implies (4.3.61), which is the desired result. So it remains to prove (4.3.70) and (4.3.71).

First, we prove (4.3.70). By (4.3.45), for all $1 \leqslant i, j \leqslant d$ we have

$$\mathbb{P}(Z_n^i = e_j) = \mathbb{E}\Big[\prod_{k=0}^{n-1} P_1(\xi_k)\Big](i,j) = \Big(\mathbb{E}P_1(\xi_0)\Big)^n(i,j).$$

This, together with (4.3.58), implies that

$$\rho\Big(\mathbb{E}P_1(\xi_0)\Big) < 1. \tag{4.3.72}$$

Notice that the relation (4.3.36) can be proved by using (4.3.72) instead of the condition $\|\|P_1(\xi_0)\|_\infty\|_{L^\infty} < 1$. Therefore, (4.3.72) implies that for all $K > 0$,

$$\limsup_{n\to+\infty} \mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k) + \beta_K Q_1(\xi_k)\Big)\Big\|_\infty\Big] \leq \beta_K. \tag{4.3.73}$$

By the sub-additivity we have

$$\lim_{n\to+\infty}\left(\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k) + \beta_K Q_1(\xi_k)\Big)\Big\|_\infty\Big]\right)^{1/n} = \inf_{n\geqslant 1}\left(\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k) + \beta_K Q_1(\xi_k)\Big)\Big\|_\infty\Big]\right)^{1/n}.$$

Together with (4.3.73), this implies that for all $K > 0$,

$$\mathbb{E}\Big[\Big\|\prod_{k=0}^{n-1}\Big(P_1(\xi_k) + \beta_K Q_1(\xi_k)\Big)\Big\|_\infty\Big] \underset{n\to+\infty}{\to} 0.$$

It follows that for all $K > 0$,

$$\alpha_{n,K} \underset{n\to+\infty}{\to} \left(\frac{B_p}{K}\right)^{\frac{\eta}{2}} \mathbb{E}\Big[\theta_0(p)^{\frac{\eta}{2p}}\Big] \sum_{k=0}^{+\infty} \mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d} M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big] + \frac{1}{K^{\frac{\eta}{2}}}. \tag{4.3.74}$$

Letting $K \to +\infty$ in (4.3.74), we deduce

$$\lim_{K\to+\infty}\lim_{n\to+\infty} \alpha_{n,K} = 0,$$

so that (4.3.70) holds for K and n sufficiently large.

Now, we prove (4.3.71). For all $K > 0$, $n \geqslant 1$ and $a > 0$, we have

$$\begin{aligned} & \alpha_{n,K}\mathbb{E}\tilde{A}_{n,K}^{-a} \\ = \; & \mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \beta_K Q_1(\xi_k)\Big)\Big\|_\infty \|M_{0,n-1}\|^a\Big] + \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\|M_{0,n-1}\|^a \\ & + \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}} \sum_{k=0}^{+\infty} \mathbb{E}\Big[\|M_{0,n+k}\|^a \theta_{n+k}(p)^{\frac{\eta}{2p}} \max_{1\leqslant i,j\leqslant d} M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]. \end{aligned}$$

By the independence of the environments ξ_n, $n \geqslant 0$, we have

$$\begin{aligned} & \mathbb{E}\Big[\|M_{0,n+k}\|^a \theta_{n+k}(p)^{\frac{\eta}{2p}} \max_{1\leqslant i,j\leqslant d} M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big] \\ & \qquad = \mathbb{E}\Big[\|M_{n+k}\|^a \theta_{n+k}(p)^{\frac{\eta}{2p}}\Big]\, \mathbb{E}\Big[\|M_{0,n-1}\|^a\Big] \times \\ & \qquad\quad \mathbb{E}\Big[\|M_{n,n+k-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{n,n+k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big] \end{aligned}$$

Therefore, by the stationarity, we get that for all $K > 0$, $n \geqslant 1$ and $a > 0$,

$$\begin{aligned} & \alpha_{n,K}\mathbb{E}\tilde{A}_{n,K}^{-a} \\ = \; & \mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \beta_K Q_1(\xi_k)\Big)\Big\|_\infty \|M_{0,n-1}\|^a\Big] + \frac{1}{K^{\frac{\eta}{2}}}\mathbb{E}\|M_{0,n-1}\|^a \\ & + \Big(\frac{B_p}{K}\Big)^{\frac{\eta}{2}} \mathbb{E}\Big[\|M_0\|^a \theta_0(p)^{\frac{\eta}{2p}}\Big] \times \\ & \mathbb{E}\|M_{0,n-1}\|^a \sum_{k=0}^{+\infty} \mathbb{E}\Big[\|M_{0,k-1}\|^a \max_{1\leqslant i,j\leqslant d} M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}}\Big]. \end{aligned} \tag{4.3.75}$$

For any fixed $n \geqslant 1$, we have $\|M_{0,n-1}\|^a \to 1$ $\mathbb{P}$-a.s. as $a \to 0$, and $\|M_{0,n-1}\|^a$ is dominated by $\|M_{0,n-1}\|^{\eta(p-1)/4p}$ when $a \in \Big(0, \frac{\eta(p-1)}{4p}\Big)$. We will apply the dominated convergence theorem in (4.3.75) as $a \to 0$. Notice that, by condition **P3** and (4.3.18) , for all $K > 0$ and $n \geqslant 1$,

$$\begin{aligned} \mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \beta_K Q_1(\xi_k)\Big)\Big\|_\infty \|M_{0,n-1}\|^{\frac{\eta(p-1)}{4p}}\Big] & \leqslant \mathbb{E}\|M_{0,n-1}\|^{\frac{\eta(p-1)}{4p}} \\ & \leqslant \Big(\mathbb{E}\|M_0\|^{\frac{\eta(p-1)}{4p}}\Big)^n < +\infty. \end{aligned}$$

Using (4.3.54) in Lemma 4.3.12, we obtain

$$\sum_{k=0}^{+\infty} \mathbb{E}\Big[\|M_{0,k-1}\|^{\frac{\eta(p-1)}{4p}} \max_{1\leqslant i,j\leqslant d} M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}} \Big] < +\infty.$$

By Hölder's inequality and condition **P3** we have

$$\mathbb{E}\Big[\|M_0\|^{\frac{\eta(p-1)}{4p}} \theta_0(p)^{\frac{\eta}{2p}} \Big] \leqslant \Big(\mathbb{E}\|M_0\|^{\frac{\eta}{4}} \Big)^{\frac{p}{p-1}} \Big(\mathbb{E}\theta_0(p)^{\frac{\eta}{2}} \Big)^{\frac{1}{p}} < +\infty.$$

Therefore, applying the Lebesgue dominated convergence theorem, by letting $a \to 0$ in (4.3.75), we obtain that for all $K > 0$ and $n \geqslant 1$,

$$\lim_{a\to 0} \Big(\alpha_{n,K} \mathbb{E}\tilde{A}_{n,K}^{-a} \Big) = \mathbb{E}\Big[\Big\| \prod_{k=0}^{n-1} \Big(P_1(\xi_k) + \beta_K Q_1(\xi_k) \Big) \Big\|_{\infty} \Big] + \frac{1}{K^{\frac{\eta}{2}}} + \Big(\frac{B_p}{K} \Big)^{\frac{\eta}{2}} \mathbb{E}\Big[\theta_0(p)^{\frac{\eta}{2p}} \Big] \sum_{k=0}^{+\infty} \mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d} M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}} \Big]. \tag{4.3.76}$$

Letting $n \to +\infty$ in (4.3.76), by (4.3.74) it holds that for all $K > 0$,

$$\lim_{n\to+\infty} \lim_{a\to 0} \Big(\alpha_{n,K} \mathbb{E}\tilde{A}_{n,K}^{-a} \Big) = \frac{1}{K^{\frac{\eta}{2}}} + \Big(\frac{B_p}{K} \Big)^{\frac{\eta}{2}} \mathbb{E}\Big[\theta_0(p)^{\frac{\eta}{2p}} \Big] \sum_{k=0}^{+\infty} \mathbb{E}\Big[\max_{1\leqslant i,j\leqslant d} M_{0,k-1}(i,j)^{\frac{\eta(1-p)}{2p}} \Big].$$

Then, letting $K \to +\infty$, we conclude that

$$\lim_{K\to+\infty} \lim_{n\to+\infty} \lim_{a\to 0} \Big(\alpha_{n,K} \mathbb{E}\tilde{A}_{n,K}^{-a} \Big) = 0.$$

This implies (4.3.71) for $K > 0$ and $n \geqslant 1$ sufficiently large, and $a > 0$ small enough. Therefore (4.3.61) holds, and this concludes the proof of Theorem 4.3.10. □

4.4 Central limit theorem for $\log \|Z_n^i\|$

In this section, we prove Theorem 4.2.4, that is, a central limit theorem for the logarithm of the population size $\log \|Z_n^i\|$ for each $1 \leqslant i \leqslant d$. To this end, we will use the following central limit theorem for the norm cocycle $\log \|M_{0,n-1}^T x\|$ established by Hennion [40, Theorem 3].

Lemma 4.4.1. *Assume conditions* (4.2.3) *and* ***P4***. *Then there exists* $\sigma \geqslant 0$ *such that for all* $x \in \mathcal{S}$, *as* $n \to \infty$,

$$\frac{\log \|M_{0,n-1}^T x\| - n\gamma}{\sqrt{n}} \to \mathcal{N}(0, \sigma^2) \quad \textit{in law.}$$

Proof of Theorem 4.2.4. By definition of the martingale (W_n^i), for any $n \geqslant 1$ and $1 \leqslant i \leqslant d$ we have, $\mathbb{P}$-a.s.,

$$\frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \min_{1 \leqslant j \leqslant d} U_{n,\infty}(j) \leqslant W_n^i \leqslant \frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \frac{1}{\min_{1 \leqslant j \leqslant d} U_{n,\infty}(j)}.$$

From this, we obtain the two following inequalities: for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\log \|Z_n^i\| \leqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i - \min_{1 \leqslant j \leqslant d} \log U_{n,\infty}(j), \tag{4.4.1}$$

$$\log \|Z_n^i\| \geqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i + \min_{1 \leqslant j \leqslant d} \log U_{n,\infty}(j). \tag{4.4.2}$$

We know that $(U_{n,\infty})$ is a stationary and ergodic sequence of positive random variables, so for all $1 \leqslant j \leqslant d$ it holds

$$\frac{\log U_{n,\infty}(j)}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0, \tag{4.4.3}$$

where $\xrightarrow{\mathbb{P}}$ denotes the convergence in probability with respect to $\mathbb{P}$. On the other hand, by [32, Theorem 2.6 and Corollary 2.8] we know that under conditions (4.2.5) and $\gamma > 0$ the limit W^i of the martingale (W_n^i) is non-degenerate, and (4.2.6) holds. This, combining with condition **P1**, implies that $W^i > 0$ $\mathbb{P}$-a.s. for each $1 \leqslant i \leqslant d$. Therefore, we obtain that for all $1 \leqslant i \leqslant d$,

$$\frac{\log W_n^i}{\sqrt{n}} \underset{n \to +\infty}{\to} 0 \quad \mathbb{P}\text{-a.s.} \tag{4.4.4}$$

Putting together the relations (4.4.1)-(4.4.4), we get that for all $1 \leqslant i \leqslant d$,

$$\left| \frac{\log \|Z_n^i\| - n\gamma}{\sqrt{n}} - \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sqrt{n}} \right| \xrightarrow{\mathbb{P}} 0. \tag{4.4.5}$$

Using Lemma 4.4.1 for $x = e_i$, we see that

$$\frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sqrt{n}} \to \mathcal{N}(0,\sigma^2) \quad \text{in law.}$$

Combining this with (4.4.5), we conclude the proof of Theorem 4.2.4. □

4.5 Berry-Esseen bound for $\log \|Z_n^i\|$

In this section, we prove Theorem 4.2.5 which gives a Berry-Esseen bound for the logarithm of the population size $\log \|Z_n^i\|$, for any $1 \leqslant i \leqslant d$.

First, we formulate the following lemma giving the convergence in L^1 of $\log W_n^i$ to $\log W^i$ with an exponential rate, for all $1 \leqslant i \leqslant d$.

Lemma 4.5.1. *Assume conditions* ***P1****,* ***P3*** *and* $\gamma > 0$*. Then there exist two constants* $C > 0$ *and* $\delta \in (0,1)$ *such that for all* $n \geqslant 0$ *and* $1 \leqslant i \leqslant d$*,*

$$\mathbb{E}|\log W_n^i - \log W^i| \leqslant C\delta^n.$$

Proof. For any $n \geqslant 0$ and $1 \leqslant i \leqslant d$, set

$$R_n^i := \frac{W_{n+1}^i}{W_n^i} - 1.$$

Then, for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$ we have

$$\log W_{n+1}^i - \log W_n^i = \log(1 + R_n^i). \tag{4.5.1}$$

Let $K \in (0,1)$ be a constant. From (4.5.1) we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\mathbb{E}\Big|\log W_{n+1}^i - \log W_n^i\Big| &= \mathbb{E}\Big|\log(1+R_n^i)\mathbb{1}_{\{R_n^i \geqslant -K\}}\Big| + \mathbb{E}\Big|\log(1+R_n^i)\mathbb{1}_{\{R_n^i < -K\}}\Big| \\ &=: I_1(n) + I_2(n). \end{aligned} \tag{4.5.2}$$

In the following, $C > 0$ will be a constant which may depend on K, p and η, and which may differ from line to line. Now we control the two terms $I_1(n)$ and $I_2(n)$.

Control of $I_1(n)$. Since $\frac{\eta p}{4} \leqslant 1$, the function $x \mapsto |x|^{-\frac{\eta p}{4}} \log(1+x)$ is bounded on

$[-K,+\infty)$, so for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$I_1(n) = \mathbb{E}|\log(1+R_n^i)\mathbb{1}_{\{R_n^i \geqslant -K\}}| \leqslant C\mathbb{E}|R_n^i|^{\frac{\eta p}{4}}. \tag{4.5.3}$$

Control of $I_2(n)$. Applying Theorem 4.2.3, we get that there exists a constant $a>0$ such that $\mathbb{E}(W^i)^{-a} < +\infty$ for any $1 \leqslant i \leqslant d$. Recall that by Lemma 4.3.11 we have the implication **P3** $\Rightarrow$ (4.2.5). So (4.2.5) holds, and using Lemma 4.3.1 with the convex function $x \mapsto x^{-a}$, this implies that for all $1 \leqslant i \leqslant d$,

$$\sup_{n\geqslant 0} \mathbb{E}(W_n^i)^{-a} = \mathbb{E}(W^i)^{-a} < +\infty. \tag{4.5.4}$$

Combining (4.5.1), (4.5.4) and the inequality $|\log(x)|^2 \leqslant C(x+x^{-a})$ for $x>0$, we obtain that for all $1 \leqslant i \leqslant d$,

$$\begin{aligned}\sup_{n\geqslant 0}\Big(\mathbb{E}|\log(1+R_n^i)|^2\Big)^{\frac12} &\leqslant 2\sup_{n\geqslant 0}\Big(\mathbb{E}|\log W_n^i|^2\Big)^{\frac12}\\ &\leqslant C\sup_{n\geqslant 0}\Big(\mathbb{E}W_n^i + \mathbb{E}(W_n^i)^{-a}\Big)^{\frac12}\\ &\leqslant C\Big(1+\mathbb{E}(W^i)^{-a}\Big)^{\frac12} < +\infty. \end{aligned}\tag{4.5.5}$$

Applying Cauchy-Schwarz's inequality, (4.5.5) and Markov's inequality, we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} I_2(n) &= \mathbb{E}|\log(1+R_n^i)\mathbb{1}_{\{R_n^i<-K\}}|\\ &\leqslant \Big(\mathbb{E}|\log(1+R_n^i)|^2\Big)^{\frac12}\Big(\mathbb{E}\mathbb{1}_{\{R_n^i<-K\}}\Big)^{\frac12}\\ &\leqslant \Big[\sup_{k\geqslant 0}\Big(\mathbb{E}|\log(1+R_k^i)|^2\Big)^{\frac12}\Big]\mathbb{P}(|R_n^i|>K)^{\frac12}\\ &\leqslant C(\mathbb{E}|R_n^i|^{\frac{\eta p}{4}})^{\frac12}. \end{aligned}\tag{4.5.6}$$

Together with (4.5.2), (4.5.3) and (4.5.6), this implies that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\mathbb{E}|\log W_{n+1}^i - \log W_n^i| &\leqslant C\mathbb{E}|R_n^i|^{\frac{\eta p}{4}} + C(\mathbb{E}|R_n^i|^{\frac{\eta p}{4}})^{\frac12}\\ &\leqslant C\mathbb{E}\Big[(\mathbb{E}_\xi|R_n^i|^p)^{\frac{\eta}{4}}\Big] + C\Big\{\mathbb{E}\Big[(\mathbb{E}_\xi|R_n^i|^p)^{\frac{\eta}{4}}\Big]\Big\}^{\frac12}, \end{aligned}\tag{4.5.7}$$

since $\frac{\eta}{4} < 1$. Notice that inequality (4.5.7) holds for any $p \in (1,2]$ satisfying condition

P3. In the following, we take $p \in (1,2]$ sufficiently close to 1 such that p verifies **P3** and $p < 1 + a$.

Now we show that there exists a constant $\delta \in (0,1)$ such that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}\Big[(\mathbb{E}_\xi |R_n^i|^p)^{\frac{\eta}{4}}\Big] \leqslant C\delta^{2n}. \tag{4.5.8}$$

By (4.3.5), for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} R_n^i &= \frac{1}{W_n^i} \sum_{r=1}^{d} \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1) \\ &= \sum_{r=1}^{d} \frac{U_{n,\infty}(r)}{\langle Z_n^i, U_{n,\infty}\rangle} \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1). \end{aligned} \tag{4.5.9}$$

Recall that, given the environment ξ, for each $1 \leqslant r \leqslant d$ the random variables $W_{l,n}^r$ indexed by $l \geqslant 1$ are i.i.d., and independent of $\xi_0, \dots, \xi_{n-1}$ and Z_n^i. Using (4.5.9), the convexity of the function $x \mapsto x^p$ on $\mathbb{R}_+$ (together with the fact that $\sum_{r=1}^d U_{n,\infty}(r) = 1$), and Lemma 4.3.4, for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s., we obtain

$$\begin{aligned} \mathbb{E}_\xi |R_n^i|^p &\leqslant \mathbb{E}_\xi \left(\sum_{r=1}^{d} \frac{U_{n,\infty}(r)}{\langle Z_n^i, U_{n,\infty}\rangle} \left| \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1) \right| \right)^p \\ &\leqslant \sum_{r=1}^{d} U_{n,\infty}(r) \mathbb{E}_\xi \left(\frac{1}{\langle Z_n^i, U_{n,\infty}\rangle^p} \left| \sum_{l=1}^{Z_n^i(r)} \left(W_{l,n}^r - 1\right) \right|^p \right) \\ &\leqslant B_p^p \sum_{r=1}^{d} \mathbb{E}_\xi \left[\frac{U_{n,\infty}(r) Z_n^i(r)}{\langle Z_n^i, U_{n,\infty}\rangle^p} \right] \mathbb{E}_\xi |W_{1,n}^r - 1|^p \\ &\leqslant B_p^p \mathbb{E}_\xi \langle Z_n^i, U_{n,\infty}\rangle^{1-p} \max_{1 \leqslant r \leqslant d} \mathbb{E}_\xi |W_{1,n}^r - 1|^p. \end{aligned}$$

Combining this with (4.3.7) and the convexity of $x \mapsto x^{1-p}$, we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \mathbb{E}_\xi |R_n^i|^p &\leqslant B_p^p \mathbb{E}_\xi (W_n^i)^{1-p} \cdot \langle M_{0,n-1}(i,\cdot), U_{n,\infty}\rangle^{1-p} \cdot \theta_n(p) \\ &\leqslant B_p^p \mathbb{E}_\xi (W_n^i)^{1-p}\, \theta_n(p) \max_{1 \leqslant r,j \leqslant d} M_{0,n-1}(r,j)^{1-p}. \end{aligned} \tag{4.5.10}$$

Therefore, by Cauchy-Schwarz's inequality and the independence between $\theta_n(p)$ and

$M_{0,n-1}$, we deduce from (4.5.10) that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\mathbb{E}\Big[(\mathbb{E}_\xi |R_n^i|^p)^{\frac{\eta}{4}}\Big] \\
\leqslant\ & B_p^{\frac{\eta p}{4}} \Big(\mathbb{E}\Big[\mathbb{E}_\xi (W_n^i)^{1-p}\Big]^{\frac{\eta}{2}}\Big)^{\frac{1}{2}} \Big(\mathbb{E}\Big[\theta_n(p)^{\frac{\eta}{2}} \max_{1\leqslant r,j\leqslant d} M_{0,n-1}(r,j)^{\frac{\eta(1-p)}{2}}\Big]\Big)^{\frac{1}{2}} \\
\leqslant\ & B_p^{\frac{\eta p}{4}} \Big(\mathbb{E}(W_n^i)^{1-p}\Big)^{\frac{\eta}{4}} \Big(\mathbb{E}\Big[\theta_0(p)^{\frac{\eta}{2}}\Big]\mathbb{E}\Big[\max_{1\leqslant r,j\leqslant d} M_{0,n-1}(r,j)^{\frac{\eta(1-p)}{2}}\Big]\Big)^{\frac{1}{2}}, \qquad (4.5.11)
\end{aligned}$$

since $\frac{\eta}{2} < 1$. Clearly condition **P3** implies that $\mathbb{E}\Big[\theta_0(p)^{\frac{\eta}{2}}\Big] < +\infty$. Since $0 < p-1 < a$, by Hölder's inequality and (4.5.4), for any $1 \leqslant i \leqslant d$ we have

$$\sup_{n\geqslant 0} \mathbb{E}(W_n^i)^{1-p} \leqslant \sup_{n\geqslant 0} \Big(\mathbb{E}(W_n^i)^{-a}\Big)^{\frac{p-1}{a}} = \Big(\mathbb{E}(W^i)^{-a}\Big)^{\frac{p-1}{a}} < +\infty. \qquad (4.5.12)$$

Moreover, by (4.3.54) (with $a = 0$) it holds that for all $n \geqslant 0$,

$$\mathbb{E}\Big[\max_{1\leqslant r,j\leqslant d} M_{0,n-1}(r,j)^{\frac{\eta(1-p)}{2}}\Big] \leqslant C\delta^{4n}, \qquad (4.5.13)$$

where $\delta > 0$ is a constant such that $\kappa(\frac{\eta(1-p)}{2}) < \delta^4 < 1$. Combining inequalities (4.5.11)-(4.5.13), we get (4.5.8).

Now, from (4.5.7) and (4.5.8) it follows that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}|\log W_{n+1}^i - \log W_n^i| \leqslant C\delta^n.$$

This implies that for all $n \geqslant 0$, $k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
\mathbb{E}|\log W_{n+k}^i - \log W_n^i| &\leqslant \sum_{r=n}^{n+k-1} \mathbb{E}|\log W_{r+1}^i - \log W_r^i| \\
&\leqslant C \sum_{r=n}^{n+k-1} \delta^r \\
&\leqslant C\delta^n. \qquad (4.5.14)
\end{aligned}$$

So $(\log W_n^i)_{n\geqslant 0}$ is a Cauchy sequence in L^1, hence it converges in L^1 to $\log W^i$, for all $1 \leqslant i \leqslant d$. By letting $k \to +\infty$ in (4.5.14), we obtain that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}|\log W^i - \log W_n^i| \leqslant C\delta^n.$$

This concludes the proof of Lemma 4.5.1. □

Now we formulate the Berry-Esseen bound for $\log \|M_{0,n-1}^T y\|$, for any $y \in \mathcal{S}$. This result was established in [75, Theorem 2.1]; it plays a crucial role in proving the Berry-Esseen theorem for $\log \|Z_n^i\|$.

Lemma 4.5.2. *Assume conditions* ***P3*** *and* ***P5****. Then there exists a constant $C > 0$ such that for all $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$,*

$$\left| \mathbb{P}\left(\frac{\log \|M_{0,n-1}^T y\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{C}{\sqrt{n}}.$$

The next lemma gives inequalities about the concentration of the joint law $\Big(\log \|Z_n^i\|, \log \|M_{0,n-1}(i$ for any $1 \leqslant i \leqslant d$. It reveals that $\log \|Z_n^i\|$ and $\log \|M_{0,n-1}(i,\cdot)\|$ behaves similarly with large probability.

Lemma 4.5.3. *Assume conditions* ***P1****,* ***P3****,* ***P5*** *and $\gamma > 0$. Then there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,*

$$\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x \right) \leqslant \frac{C}{\sqrt{n}}, \tag{4.5.15}$$

and

$$\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \right) \leqslant \frac{C}{\sqrt{n}}. \tag{4.5.16}$$

Proof. We will only give a proof of (4.5.15), since the other inequality (4.5.16) can be proved by similar arguments. For $n \geqslant 1$, $0 \leqslant m \leqslant n$, $y \in \mathcal{S}$ and $1 \leqslant i \leqslant d$, set

$$S_{m,n}^y := \frac{\log \|M_{m,n-1}^T y\| - (n-m)\gamma}{\sigma\sqrt{n}} \quad \text{and} \quad L_{m,n}^i := \frac{\log W_m^i}{\sigma\sqrt{n}},$$

where by convention $M_{m,n-1}$ denotes the identity matrix when $m = n$. By (4.4.2) we have that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} &\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x \right) \\ &\qquad\qquad \leqslant \mathbb{P}\left(S_{0,n}^{e_i} + L_{n,n}^i + \min_{1 \leqslant j \leqslant d} \frac{\log U_{n,\infty}(j)}{\sigma\sqrt{n}} \leqslant x, S_{0,n}^{e_i} > x \right). \end{aligned} \tag{4.5.17}$$

In the following we take $m := m(n) = \lfloor \sqrt{n} \rfloor$, where $\lfloor x \rfloor$ is the integer part of x; $C > 0$ will be a constant which may depend on p and η, and which may differ from line to line. By Markov's inequality and Lemma 4.5.1, we get that there exists a constant $\delta \in (0,1)$ such that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\mathbb{P}\Big(|L^i_{n,n} - L^i_{m,n}| > \frac{1}{\sqrt{n}}\Big) &\leqslant \sqrt{n}\,\mathbb{E}\Big|L^i_{n,n} - L^i_{m,n}\Big| \\ &= \frac{1}{\sigma}\,\mathbb{E}|\log W^i_n - \log W^i_m| \\ &\leqslant \frac{1}{\sigma}\,\mathbb{E}|\log W^i_n - \log W^i| + \frac{1}{\sigma}\,\mathbb{E}|\log W^i_m - \log W^i| \\ &\leqslant C(\delta^n + \delta^m).\end{aligned}$$

Since $\delta^n + \delta^m = o(\frac{1}{\sqrt{n}})$ as $n \to +\infty$, this implies that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{P}\Big(|L^i_{n,n} - L^i_{m,n}| > \frac{1}{\sqrt{n}}\Big) \leqslant \frac{C}{\sqrt{n}}.$$

Combining this with (4.5.17), we obtain that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}&\mathbb{P}\Big(\frac{\log \|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\Big) \\ \leqslant\;& \mathbb{P}\Big(S^{e_i}_{0,n} + L^i_{m,n} + \min_{1\leqslant j\leqslant d}\frac{\log U_{n,\infty}(j)}{\sigma\sqrt{n}} \leqslant x + \frac{1}{\sqrt{n}}, S^{e_i}_{0,n} > x\Big) \\ &+ \mathbb{P}\Big(|L^i_{n,n} - L^i_{m,n}| > \frac{1}{\sqrt{n}}\Big) \\ \leqslant\;& \mathbb{P}\Big(S^{e_i}_{0,n} + L^i_{m,n} + \min_{1\leqslant j\leqslant d}\frac{\log U_{n,\infty}(j)}{\sigma\sqrt{n}} \leqslant x + \frac{1}{\sqrt{n}}, S^{e_i}_{0,n} > x\Big) + \frac{C}{\sqrt{n}}.\end{aligned} \tag{4.5.18}$$

Recall that for $y \in \mathcal{S}$ and $M \in \mathcal{G}_+$, we denote by $M \cdot y := \frac{My}{\|My\|}$ the projective action of M on $\mathcal{G}_+$. Then, for $y \in \mathcal{S}$ the process

$$X^y_0 = y, \quad \text{and} \quad X^y_n = M^T_{0,n-1} \cdot y, \quad n \geqslant 1,$$

is a Markov chain on $\mathcal{S}$. Notice that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, we have the decomposition

$$\begin{aligned} S_{0,n}^{e_i} &= \frac{\log \|M_{m+1,n-1}^T(M_{0,m}^T e_i)\| - n\gamma}{\sigma\sqrt{n}} \\ &= \frac{\log \|M_{0,m}^T e_i\| + \log \|M_{m+1,n-1}^T(M_{0,m}^T \cdot e_i)\| - n\gamma}{\sigma\sqrt{n}} \\ &= \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}}. \end{aligned} \tag{4.5.19}$$

Moreover, by (4.1.4) we have that for all $n \geqslant 0$,

$$1 \geqslant \min_{1 \leqslant r \leqslant d} U_{n,\infty}(r) = \min_{1 \leqslant r \leqslant d} \frac{\langle M_n(r,\cdot), U_{n+1,\infty}\rangle}{\|M_n U_{n+1,\infty}\|} \geqslant \min_{1 \leqslant r,j \leqslant d} \frac{M_n(r,j)}{\|M_n\|}. \tag{4.5.20}$$

Therefore, putting together the relations (4.4.1) and (4.5.18)-(4.5.20), we obtain that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} &\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x \right) \\ \leqslant\ &\mathbb{P}\left(\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + B_{m,n}^i \leqslant x + \frac{1}{\sqrt{n}}, \right. \\ &\left. \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} > x \right) + \frac{C}{\sqrt{n}}, \end{aligned} \tag{4.5.21}$$

with

$$\begin{aligned} B_{m,n}^i := \frac{1}{\sigma\sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|} + \frac{1}{\sigma\sqrt{n}} \min_{1 \leqslant r,j \leqslant d} \log \frac{M_m(r,j)}{\|M_m\|} \\ + \min_{1 \leqslant j \leqslant d} \frac{\log U_{n,\infty}(j)}{\sigma\sqrt{n}}. \end{aligned}$$

Denote by $\nu_{m,n}^i$ the joint law of $\left(X_{m+1}^{e_i}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i}, B_{m,n}^i \right)$ on $\mathcal{S} \times \mathbb{R} \times \mathbb{R}$. For $y \in \mathcal{S}$ and $x \in \mathbb{R}$, set

$$G_{m,n}^y(x) = \mathbb{P}(S_{m,n}^y \leqslant x).$$

Since $S_{m+1,n}^y$ is independent of $X_{m+1}^{e_i}$, $S_{0,m+1}^{e_i}$ and $B_{m,n}^i$ for any $y \in \mathcal{S}$, we obtain from

(4.5.21) that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x\right) \\
\leqslant\ & \int \mathbb{P}\left(S_{m+1,n}^y + s + t \leqslant x + \frac{1}{\sqrt{n}}, S_{m+1,n}^y + s > x\right)\nu_{m,n}^i(dy, ds, dt) + \frac{C}{\sqrt{n}} \\
=\ & \int \mathbb{1}_{\{t \leqslant \frac{1}{\sqrt{n}}\}}\left[G_{m+1,n}^y\left(x - s - t + \frac{1}{\sqrt{n}}\right) - G_{m+1,n}^y(x-s)\right]\nu_{m,n}^i(dy, ds, dt) \\
& + \frac{C}{\sqrt{n}}. \qquad (4.5.22)
\end{aligned}$$

The random matrices M_n, $n \geqslant 0$, are i.i.d., so for $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$ we have

$$G_{m+1,n}^y(x) = \mathbb{P}\left(\frac{\log \|M_{0,n-m-1}^T y\| - (n-m-1)\gamma}{\sigma\sqrt{n}} \leqslant x\right) = G_{0,n-m-1}^y(a_n x),$$

with $a_n = \sqrt{\frac{n}{n-m-1}}$. Notice that $a_n = (1 - \frac{m+1}{n})^{-1/2} = 1 + O(\frac{m}{n}) = 1 + O(\frac{1}{\sqrt{n}})$ as $n \to +\infty$. Therefore, applying the Berry-Esseen bound of Lemma 4.5.2, we get that for all $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$,

$$\begin{aligned}
\left|G_{m+1,n}^y(x) - \Phi(a_n x)\right| &= \left|G_{0,n-m-1}^y(a_n x) - \Phi(a_n x)\right| \\
&\leqslant \frac{C}{\sqrt{n-m-1}} \\
&= \frac{C a_n}{\sqrt{n}} \leqslant \frac{C}{\sqrt{n}}. \qquad (4.5.23)
\end{aligned}$$

Moreover, using the mean value theorem on the function $t \mapsto \Phi(tx)$ with $t \geqslant 1$, we obtain that for all $n \geqslant 1$ and $x \in \mathbb{R}$,

$$|\Phi(a_n x) - \Phi(x)| \leqslant |a_n - 1| \sup_{t \geqslant 1} |x\Phi'(tx)| \leqslant \frac{C}{\sqrt{n}} \sup_{z \in \mathbb{R}} |z\Phi'(z)|. \qquad (4.5.24)$$

It is clear that $z \mapsto |z\Phi'(z)|$ is a bounded function on $\mathbb{R}$. Combining this with the inequalities (4.5.22)-(4.5.24), we deduce that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x \Big) \\ \leqslant \ & \int \mathbb{1}_{\{t \leqslant \frac{1}{\sqrt{n}}\}} \Big| \Phi\Big(x - s - t + \frac{1}{\sqrt{n}}\Big) - \Phi(x-s) \Big| \nu_{m,n}^i(dy, ds, dt) + \frac{C}{\sqrt{n}}. \end{aligned} \tag{4.5.25}$$

By the mean value theorem and the fact that $\sup_{x\in\mathbb{R}} |\Phi'(x)| \leqslant 1$, for all $x, z \in \mathbb{R}$ we have

$$|\Phi(x+z) - \Phi(x)| \leqslant |z|.$$

This, together with (4.5.25), implies that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x \Big) \\ \leqslant \ & \int \mathbb{1}_{\{t \leqslant \frac{1}{\sqrt{n}}\}} \Big| \frac{1}{\sqrt{n}} - t \Big| \nu_{m,n}^i(dy, ds, dt) + \frac{C}{\sqrt{n}} \\ \leqslant \ & \int |t| \nu_{m,n}^i(dy, ds, dt) + \frac{C}{\sqrt{n}} \\ = \ & \mathbb{E}|B_{m,n}^i| + \frac{C}{\sqrt{n}}. \end{aligned} \tag{4.5.26}$$

By definition of $B_{m,n}^i$, combining with (4.4.1), (4.4.2) and (4.5.20), we get that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned} \sigma\sqrt{n}|B_{m,n}^i| &\leqslant \Big| \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|} \Big| + \min_{1\leqslant r,j\leqslant d} |\log M_m(r,j)| + |\log \|M_m\|| \\ &\quad + \min_{1\leqslant j\leqslant d} |\log U_{n,\infty}(j)| \\ &\leqslant |\log W_m^i| + |\log \|M_m\|| \\ &\quad + \max_{1\leqslant r,j\leqslant d} \Big(|\log U_{m,\infty}(j)| + |\log M_m(r,j)| + |\log U_{n,\infty}(j)| \Big) \\ &\leqslant |\log W_m^i| + 2|\log \|M_m\|| + |\log \|M_n\|| \\ &\quad + \max_{1\leqslant r,j\leqslant d} \Big(2|\log M_m(r,j)| + |\log M_n(r,j)|| \Big). \end{aligned} \tag{4.5.27}$$

By Lemma 4.5.1 we have $\sup_{n\geqslant 0} \mathbb{E}|\log W_n^i| < +\infty$ for any $1 \leqslant i \leqslant d$. Moreover, from condition **P3** and the inequality $|\log x| \leqslant C(x^\eta + x^{-\eta})$ for $x > 0$, it holds that

$\mathbb{E}|\log \|M_0\|| < +\infty$ and $\mathbb{E}|\log M_0(r,j)| < +\infty$, $1 \leqslant r, j \leqslant d$. Therefore, taking expectation in (4.5.27), this implies that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}|B^i_{m,n}| \leqslant \frac{C}{\sqrt{n}}. \tag{4.5.28}$$

Hence, (4.5.15) follows from (4.5.26) and (4.5.28). This concludes the proof of Lemma 4.5.3. □

Proof of Theorem 4.2.5. For $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$, we write

$$\begin{aligned}
& \mathbb{P}\bigg(\frac{\log \|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \bigg) \\
= \; & \mathbb{P}\bigg(\frac{\log \|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \bigg) \\
& + \mathbb{P}\bigg(\frac{\log \|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x \bigg) \\
= \; & \mathbb{P}\bigg(\frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \bigg) \\
& - \mathbb{P}\bigg(\frac{\log \|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \bigg) \\
& + \mathbb{P}\bigg(\frac{\log \|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} > x \bigg).
\end{aligned}$$

By Lemma 4.5.3, we get that there exists a constant $C > 0$ such that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\left| \mathbb{P}\bigg(\frac{\log \|Z^i_n\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \bigg) - \mathbb{P}\bigg(\frac{\log \|M_{0,n-1}(i,\cdot)\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \bigg) \right| \leqslant \frac{C}{\sqrt{n}}. \tag{4.5.29}$$

Combining (4.5.29) with Lemma 4.5.2, we obtain the Berry-Esseen bound for $\log \|Z^i_n\|$, for any $1 \leqslant i \leqslant d$. This concludes the proof of Theorem 4.2.5. □

Chapter 5
Cramér type moderate deviation expansion for supercritical multi-type branching processes in random environments

Résumé. Soit $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d))$, $n \geqslant 0$, un processus de branchement d-type surcritique dans un environnement aléatoire indépendant et identiquement distribué $\xi = (\xi_0, \xi_1, \ldots)$, commençant avec une particule initiale de type i. On établit un théorème de type Cramér sur les déviations modérées pour $\log \|Z_n^i\|$. Pour cela, on commence par montrer des résultats uniformes pour l'existence des moments harmoniques et une borne de type Berry-Esseen sous de nouvelles mesures bien choisies, en utilisant la théorie de trou spectral des produits de matrices aléatoires et la martingale fondamentale que l'on a trouvé dans un précédent article.

Abstract. Let $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d))$, $n \geqslant 0$, be a supercritical d-type branching process in an independent and identically distributed random environment $\xi = (\xi_0, \xi_1, \cdots)$, starting with one initial particle of type i. We establish a Cramér type moderate deviation expansion for $\log \|Z_n^i\|$. To this end, we first prove uniform results for the existence of harmonic moments and Berry-Esseen type bound under suitably changed measure, using the spectral gap theory on products of random matrices and the fundamental martingale that we found in an earlier paper.

5.1 Introduction

Let $Z_n = (Z_n(1), \cdots, Z_n(d))$, $n \geqslant 0$, be a d-type branching process ($d \geqslant 1$) in an independent and identically distributed (i.i.d.) random environment $\xi = (\xi_0, \xi_1, \cdots)$. Denote by M_n be the $d \times d$ random matrix whose components are

$$M_n(i,j) = \mathbb{E}_\xi[Z_{n+1}(j) \mid Z_n = e_i], \quad 1 \leqslant i, j \leqslant d,$$

where $\mathbb{E}_\xi$ is the conditional expectation given the environment ξ, e_i is the vector with component 1 in the i-th place and 0 elsewhere. So $M_n(i,j)$ is the conditional mean of the number of particles of type j produced by a particle of type i of n-th generation, given the environment. We define the Lyapunov exponent of the sequence (M_n) as

$$\gamma := \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\|,$$

where $M_{0,n-1} = M_0 \cdots M_{n-1}$ is the product matrix, $\|M_{0,n-1}\|$ denotes its its operator norm with respect to the L^1-vectorial norm (see (5.2.2) and (5.2.1)).

The study of the asymptotic behaviour of the branching process (Z_n) is a complex problem which attracts a lot of attention during the last decades. Concerning the critical case $\gamma = 0$ and subcritical case $\gamma < 0$, see for example the works of Peigné, Le Page and Pham [57] , Vatutin and Dyakonova [70], and Vatutin and Wachtel [73], who studied the convergence rate of the survival probability of the branching process. For the supercritical case $\gamma > 0$, in [32], [33] and [34] we established asymptotic properties of Z_n such as Kesten-Stigum type theorem, L^p convergence, harmonic moments and Berry-Esseen type theorem.

In this paper, we continue to consider the supercritical case $\gamma > 0$, for which we will give more results on the asymptotic behaviour of (Z_n). In the sequel, we always assume $\gamma > 0$. Denote by (Z_n^i) the branching process (Z_n) which starts with one initial particle of type i, that is when $Z_0 = e_i$. In [32], under suitable conditions, we established a strong law of large numbers for $\log\|Z_n^i\|$: on the explosion event $\{\|Z_n^i\| \to +\infty\}$, it holds that

$$\lim_{n\to+\infty} \frac{1}{n}\log\|Z_n^i\| = \gamma \quad \text{a.s.} \tag{5.1.1}$$

Then, under additional assumptions, we proved in [34] a central limit theorem (CLT) for $\log\|Z_n^i\|$: there exists $\sigma \geqslant 0$ such that for each $1 \leqslant i \leqslant d$,

$$\frac{\log\|Z_n^i\| - n\gamma}{\sqrt{n}} \to \mathcal{N}(0,\sigma^2) \quad \text{in law,} \tag{5.1.2}$$

where $\mathcal{N}(0,\sigma^2)$ denotes the normal law with mean 0 and variance σ^2. We have also established in [34], under further moment conditions, a Berry-Esseen type theorem for $\log\|Z_n^i\|$, which gives the rate of convergence in the CLT: we showed that for all $n \geqslant 1$,

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{C}{\sqrt{n}}, \tag{5.1.3}$$

where $\sigma^2 = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[(\log \|M_{0,n-1}^T x\| - n\gamma)^2]$ is the asymptotic variance independent of x, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\,dt$ is the standard normal distribution function, and $C > 0$ is a constant.

The objective of this paper is to establish a Cramér type moderate deviation expansion for $\log \|Z_n^i\|$. We will prove (cf. Theorem 2.1) that uniformly in $0 \leqslant x \leqslant o(\sqrt{n})$, as $n \to +\infty$,

$$\frac{\mathbb{P}\Big(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\Big)}{1-\Phi(x)} = \mathrm{e}^{\frac{x^3}{\sqrt{n}}\zeta(\frac{x}{\sqrt{n}})}\left[1 + O\Big(\frac{1+x}{\sqrt{n}}\Big)\right], \tag{5.1.4}$$

where ζ is the Cramér series (for the precise definition see (5.2.12)). Notice that a version of this result has been proved by Grama, Liu and Miqueu in [31, Theorem 1.1] for the single type case $d = 1$. The expansion (5.1.4) is totally new for $d \geqslant 2$.

Now we explain briefly our approach for the proof of (5.1.4). For any $s \in (-\eta, \eta)$ with $\eta > 0$ small, we define a transfer operator P_s (see (5.2.10)) naturally occurring in the products of random matrices. Using the spectral theory on P_s (see e.g. Buraczewski, Damek, Guivarc'h and Mentemeier [13], Guivarc'h and Le Page [36], Xiao, Grama and Liu [75]), we define the changed measure $\mathbb{P}_s^{e_i}$ (see section 5.3) for the branching process (Z_n^i). To prove (5.1.4), we extend the Berry-Esseen bound (5.1.3) for the changed measure $\mathbb{P}_s^{e_i}$ uniformly in $s \in (-\eta, \eta)$ (cf. Theorem 5.4.1 in Section 5.4): for all $n \geqslant 1$ and $x \in \mathbb{R}$,

$$\sup_{s\in(-\eta,\eta)} \left|\mathbb{P}_s^{e_i}\Big(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\Big) - \Phi(x)\right| \leqslant \frac{C}{\sqrt{n}}, \tag{5.1.5}$$

where $\Lambda(s) = \log \kappa(s)$, $\kappa(s)$ is the spectral radius of P_s, $\sigma_s^2 = \Lambda''(s)$, and $C > 0$ is a constant. Then, we combine (5.1.5) with the standard techniques from Petrov [63] to obtain (5.1.4).

The Berry-Esseen bound (5.1.5) plays an important role in our approach. Let us explain its proof. Our method is an adaptation of the arguments that we used in the proof of (5.1.3) in [34]. The fundamental martingale (W_n^i) associated to the process (Z_n^i), defined in [32], will play a central role. For each $n, k \geqslant 0$, let $\rho_{n,n+k}$ be the spectral radius of the product matrix $M_{n,n+k} = M_n \cdots M_{n+k}$. It is well known that, by the Perron-Frobenius theorem, $\rho_{n,n+k}$ is an eigenvalue of $M_{n,n+k}$, and there exists $U_{n,n+k}$ a non negative eigenvector associated to $\rho_{n,n+k}$, with $\|U_{n,n+k}\| = 1$. Using the results of Hennion [40, Lemma 3.3 and Theorem 1], under suitable conditions, for each $n \geqslant 0$ the limit

$$U_{n,\infty} := \lim_{k\to\infty} U_{n,n+k} \tag{5.1.6}$$

exists a.s., with $U_{n,\infty} > 0$ (for a vector or matrix U we write $U > 0$ to mean that each of the components of U is strictly positive) and $\|U_{n,\infty}\| = 1$; in addition, it holds that

$$M_n U_{n+1,\infty} = \lambda_n U_{n,\infty}, \tag{5.1.7}$$

where $\lambda_n, n \geqslant 0$ are positive random scalars called the pseudo-spectral radii of the random matrices (M_n). Iterating (5.1.7), we get

$$M_{n,n+k} U_{n+k+1,\infty} = \lambda_{n,n+k} U_{n,\infty}, \quad n, k \geqslant 0, \tag{5.1.8}$$

where $\lambda_{0,n} := \lambda_0 \cdots \lambda_n$. Then, the martingale (W_n^i) is defined by (see [32]):

$$W_0^i = 1, \quad W_n^i = \frac{\langle Z_n^i, U_{n,\infty}\rangle}{\lambda_{0,n-1} U_{0,\infty}(i)}, \quad n \geqslant 1. \tag{5.1.9}$$

We get from (5.1.9) the two following relations which make the link between $\log \|Z_n^i\|$ and $\log \|M_{0,n-1}(i,\cdot)\|$:

$$\log \|Z_n^i\| \leqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i - \min_{1\leqslant j\leqslant d} \log U_{n,\infty}(j), \tag{5.1.10}$$

$$\log \|Z_n^i\| \geqslant \log \|M_{0,n-1}(i,\cdot)\| + \log W_n^i + \min_{1\leqslant j\leqslant d} \log U_{n,\infty}(j). \tag{5.1.11}$$

We recall that $(U_{n,\infty})$ is a stationary sequence of random variables, and since (W_n^i) is a non-negative martingale, the limit $W^i = \lim_{n\to+\infty} W_n^i$ exists a.s.. It follows that, when W^i is non degenerate, the terms $\log W_n^i$ and $\log U_{n,\infty}(j)$ in (5.1.10) and (5.1.11) will be negligible in the limit properties that we consider. More precisely, we will use the Berry-Esseen bound under the changed measure $\mathbb{P}_s^{e_i}$ proved in Xiao, Grama and Liu [75] to control $\log \|M_{0,n-1}(i,\cdot)\|$; then, by giving a tight control of the quantities $\log W_n^i$ and $\log U_{n,\infty}(j)$ under $\mathbb{P}_s^{e_i}$, we will obtain (5.1.5) from the two inequalities (5.1.10) and (5.1.11). For the term $\log W_n^i$, we will establish a sufficient condition for the existence of harmonic moments of the limit W^i under $\mathbb{P}_s^{e_i}$ uniformly in $s \in (-\eta, \eta)$: we will show that there exists $a > 0$ such that for all $1 \leqslant i \leqslant d$,

$$\sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i} (W^i)^{-a} < +\infty, \tag{5.1.12}$$

(cf. Theorem 5.3.3). The proof of (5.1.12) is one of the key arguments to prove (5.1.5) and (5.1.4).

The rest of the paper is organized as follows. In Section 5.2, we introduce some necessary notation and we formulate the Cramér type moderate deviation expansion for $\log \|Z_n^i\|$. We define in Section 5.3 the changed measure $\mathbb{P}_s^{e_i}$ and we study under $\mathbb{P}_s^{e_i}$ the harmonic moments of W^i. Section 5.4 is devoted to proof of the Berry-Esseen type theorem for $\log \|Z_n^i\|$ under $\mathbb{P}_s^{e_i}$. Finally we prove in Section 5.5 the Cramér type moderate deviation expansion for $\log \|Z_n^i\|$.

5.2 Notation, preliminaries and main results

For $d \geqslant 1$, let $\mathcal{M}_d(\mathbb{R})$ be the set of $d \times d$ matrices with entries in $\mathbb{R}$. The d-dimensional space of vectors $\mathbb{R}^d$ will be equipped with the scalar product and the L^1-norm respectively defined by

$$\langle x, y \rangle := \sum_{i=1}^{d} x(i)\, y(i) \quad \text{and} \quad \|x\| := \sum_{i=1}^{d} |x(i)|, \quad x, y \in \mathbb{R}^d. \tag{5.2.1}$$

We equip the space $\mathcal{M}_d(\mathbb{R})$ with the operator norm with respect to the L^1 vectoriel norm:

$$\|M\| := \sup_{x \in \mathcal{S}} \|Mx\|, \tag{5.2.2}$$

where $\mathcal{S} = \{x \in \mathbb{R}^d : x \geqslant 0, \|x\| = 1\}$ is the intersection of the unit sphere with the positive quadrant. Let $\mathbf{0} = (0, \cdots, 0) \in \mathbb{R}^d$ be the vector with all coordinates equal to 0, and $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^d$ the vector with all coordinates equal to 1. Let $\mathbb{N} = \{0, 1, \dots\}$ be the set of non-negative integers. Set $\mathbb{1}_A$ for the indicator function of an event A. We denote by c, C strictly positive constants which may differ from line to line.

Now we define precisely a multi-type branching process in a random environment (MBPRE). The random environment $\xi = (\xi_n)_{n \geqslant 0}$ is an independent and identically distributed (i.i.d.) sequence of random variables with values in an abstract space $\mathbb{X}$; each realization of ξ_n is associated to d probability generating functions

$$f_n^r(s) = \sum_{k_1,\cdots,k_d=0}^{\infty} p^r_{k_1,\cdots,k_d}(\xi_n) s_1^{k_1}\cdots s_d^{k_d}, \quad s=(s_1,\dots,s_d)\in[0,1]^d,$$

$1 \leqslant r \leqslant d$. A MBPRE $Z_n = (Z_n(1),\cdots,Z_n(d))$, $n \geqslant 0$ in the random environment ξ is a sequence of random vectors in $\mathbb{N}^d$ such that

$$Z_0 \in \mathbb{N}^d \text{ is fixed, and } Z_{n+1} = \sum_{r=1}^{d}\sum_{l=1}^{Z_n(r)} N^r_{l,n} \text{ for } n \geqslant 0, \tag{5.2.3}$$

where, given the environment ξ, $N^r_{l,n} = (N^r_{l,n}(1),\cdots,N^r_{l,n}(d))$ indexed by $l \geqslant 1$, $n \geqslant 0$ and $1 \leqslant r \leqslant d$ are independent random vectors with probability generating function f_n^r. The random variable $N^r_{l,n}(j)$ represents the offspring of type j at time $n+1$ of the l-th particle of type r in generation n, and $Z_n(j)$ denotes the number of particles of type j in generation n. As explained in the introduction, when $Z_0 = e_i$, we write Z_n^i for Z_n, i.e. (Z_n^i) is the MBPRE which starts with one initial particle of type i.

Given the environment ξ, the underlying probability will be dented by $\mathbb{P}_\xi$, which is called the quenched law. Denote by τ the law of the environment ξ. The total probability $\mathbb{P}$, called annealed law, can be defined as $\mathbb{P}(dx,d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$. The expectation with respect to $\mathbb{P}_\xi$ and $\mathbb{P}$ are denoted respectively by $\mathbb{E}_\xi$ and $\mathbb{E}$. With our notation,

$$f_n^r(s) = \mathbb{E}_\xi\left(\prod_{j=1}^{d} s_j^{N^r_{l,n}(j)}\right), \quad s=(s_1,\dots,s_d)\in[0,1]^d,$$

are the quenched probability generating function of $N^r_{l,n}$, which represent the offspring distributions for particles of generation n. For all $n \geqslant 0$, the mean matrix M_n can be expressed in terms of $f_n = (f_n^1,\cdots,f_n^d)$:

$$M_n(i,j) = \frac{\partial f_n^i}{\partial s_j}(\mathbf{1}) = \mathbb{E}_\xi\Big[Z_{n+1}(j)\Big|Z_n = e_i\Big], \quad 1 \leqslant i,j \leqslant d,$$

where $\frac{\partial f}{\partial s_j}(\mathbf{1})$ denotes the left derivative at $\mathbf{1}$ of a d-dimensional probability generating function f with respect to s_j. M_n is the matrix of means of the offspring distributions in the sense that $M_n(i,j)$ represents the conditioned mean of the number of children of type j produced by a particle of type i at time n. The hypothesis that the environment (ξ_n) is i.i.d. implies that $(M_n)_{n\geqslant 0}$ is a sequence of i.i.d. random matrices. We will use the products of these matrices:

$$M_{k,n} := M_k \cdots M_n, \quad 0 \leqslant k \leqslant n.$$

Notice that

$$\mathbb{E}_\xi Z_{n+1}^i(j) = M_{0,n}(i,j), \quad n \geqslant 0, 1 \leqslant i,j \leqslant d. \tag{5.2.4}$$

Throughout the paper, we assume that the matrix M_0 satisfies the first moment condition

$$\mathbb{E} \log^+ \|M_0\| < +\infty. \tag{5.2.5}$$

When (5.2.5) holds, the limit

$$\gamma := \lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|$$

exists and is called the Lyapunov exponent of the sequence of matrices $(M_n)_{n \geqslant 0}$; moreover, a strong law of large numbers has been established in [26]:

$$\lim_{n \to +\infty} \frac{1}{n} \log \|M_{0,n-1}\| = \gamma \quad \mathbb{P}\text{-a.s.}$$

The Lyapunov exponent γ permits a classification of MBPRE's (see e.g. [32]): a MBPRE is subcritical if $\gamma < 0$, critical if $\gamma = 0$, and supercritical if $\gamma > 0$. We always consider the supercritical case, which means $\gamma > 0$.

The goal of the present paper is to establish a Cramér type moderate deviation expansion for $\log \|Z_n^i\|$ in the supercritical case. The asymptotic behaviour of the MBPRE (Z_n^i), when it is supercritical, is determined by that of the product of random matrices $M_{0,n-1}$ and the fundamental martingale (W_n^i) that we mentioned in the introduction. Set $\rho_{n,n+k}$ the spectral radius of $M_{n,n+k}$. By the Perron-Frobenius theorem (see e.g. [7]), we know that $\rho_{n,n+k}$ is a positive eigenvalue of $M_{n,n+k}$, and there exist positive right and left eigenvectors $U_{n,n+k}$ and $V_{n,n+k}$ associated to $\rho_{n,n+k}$ with the normalizations $\|U_{n,n+k}\| = 1$ and $\langle V_{n,n+k}, U_{n,n+k} \rangle = 1$. Let $\mathcal{G}_+^0$ be the set of matrices whose entries are strictly positive. Assuming that M_0 is a.s. allowable in the sense that every row and column contains a strictly positive element, and that

$$\mathbb{P}\Big(\bigcup_{n\geqslant 0}\Big\{M_{0,n}\in\mathcal{G}_+^0\Big\}\Big)>0, \tag{5.2.6}$$

Hennion [40, Lemma 3.3 and Theorem 1] proved that the random vectors $U_{n,\infty}$ and the random scalars λ_n defined by (5.1.6) and (5.1.7) exist. It is easily seen that the relation (5.1.8) holds and that the sequences $(U_{n,\infty})$ and (λ_n) are stationary and ergodic. Under the same conditions, we proved in [32, Theorem 1] that the sequence (W_n^i) defined by (5.1.9) is a non-negative martingale under the probability measures $\mathbb{P}_\xi$ and $\mathbb{P}$, w.r.t. the filtration

$$\mathcal{F}_0=\sigma(\xi),\quad \mathcal{F}_n=\sigma\Big(\xi,N_{l,k}^r(j),0\leqslant k\leqslant n-1,1\leqslant r,j\leqslant d,l\geqslant 1\Big)\text{ for } n\geqslant 1.$$

Define $W^i:=\lim_{n\to+\infty}W_n^i$, the a.s. limit of the martingale (W_n^i).

Under the supercritical condition $\gamma>0$, we proved in [32, Theorem 2.6 and Corollary 2.8] that the condition

$$\mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)}\log^+\frac{Z_1^i(j)}{M_0(i,j)}\right)<+\infty\quad\forall 1\leqslant i,j\leqslant d \tag{5.2.7}$$

is sufficient for the non-degeneracy of each W^i in the sense that $\mathbb{P}(W^i>0)>0$, with

$$\mathbb{E}_\xi W^i=1\quad\text{and}\quad \mathbb{P}_\xi(W^i>0)=\mathbb{P}_\xi\Big(\|Z_n^i\|\underset{n\to+\infty}{\to}+\infty\Big)=1-q^i(\xi)>0\text{ a.s.}, \tag{5.2.8}$$

where $q^i(\xi)$ is the probability of extinction of the process (Z_n^i).

Now we introduce some conditions to formulate the Cramér type moderate deviation expansion for $\log\|Z_n^i\|$. For $n\geqslant 0$, define the vector $p_0(\xi_n)$ whose components are

$$p_0(\xi_n)(i):=f_n^i(\mathbf{0})=\mathbb{P}_\xi(\|Z_1^i\|=0),\quad 1\leqslant i\leqslant d.$$

Throughout the paper, we will assume that each individual of the population gives birth to at least one child :

K1. The vector $p_0(\xi_0)=(f_0^1(\mathbf{0}),\dots,f_0^d(\mathbf{0}))$ satisfies

$$p_0=\mathbf{0}\quad\mathbb{P}\text{-a.s.} \tag{5.2.9}$$

Notice that when **K1** and (5.2.7) hold, we have $q^i(\xi) = 0$ a.s. and $\|Z_n^i\| \to +\infty$ a.s. as $n \to +\infty$ by (5.2.8). For all $n \geqslant 0$ and $p > 1$ denote by

$$\theta_n(p) := \max_{1 \leqslant i,j \leqslant d} \mathbb{E}_\xi \left| \frac{N_{1,n}^i(j)}{M_n(i,j)} - 1 \right|^p.$$

We need the following moment conditions on the offspring distributions :

K2. There exist two constants $p \in (1, 2]$ and $\eta_0 \in (0, 1)$ such that

$$\mathbb{E}\|M_0\|^{\eta_0} < +\infty, \quad \max_{1 \leqslant i,j \leqslant d} \mathbb{E}M_0(i,j)^{-\eta_0} < +\infty \quad \text{and} \quad \mathbb{E}\theta_0(p)^{\eta_0} < +\infty.$$

Clearly, **K2** implies that M_0 is a.s. allowable and (5.2.6) holds. We proved in [34, Lemma 3.11] that **K2** implies (5.2.7). Therefore when $\gamma > 0$ and **K2** holds, each W^i is non-degenerate. By [75, Proposition 3.14], we know that under the condition **K2**, the asymptotic variance

$$\sigma^2 = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}[(\log \|M_{0,n-1}^T x\| - n\gamma)^2]$$

exists uniformly in $x \in \mathcal{S}$, with $0 \leqslant \sigma^2 < +\infty$. We will need the assumption

K3. The asymptotic variance σ^2 satisfies

$$\sigma^2 > 0.$$

For $x \in \mathcal{S}$ and $M \in \mathcal{G}_+^0$, define the projective action of M on $\mathcal{S}$ by $M \cdot x := \frac{Mx}{\|Mx\|}$. Denote by μ the law of M_0, and $\Gamma_\mu = [\operatorname{supp} \mu]$ the semi-group generated by the support of μ. Under **K2**, each $M \in \Gamma_\mu$ is strictly positive, hence by the Perron-Frobenius theorem the spectral radius ρ_M of M is the unique eigenvalue with the largest modulus, and it is simple. Denote by u_M the associated unique right eigenvector with unit norm. Set $V(\Gamma_\mu) = \overline{\{\pm u_M, M \in \Gamma_\mu\}}$, where $\overline{A}$ denotes the closure of the set A. We say that μ is arithmetic if there exist $t > 0$, $\theta \in [0, 2\pi)$ and a function $h : \mathcal{S} \to \mathbb{R}$ such that for all $M \in \Gamma_\mu$ and $x \in V(\Gamma_\mu)$,

$$\exp[it \log \|Mx\| - i\theta + ih(M \cdot x) - ih(x)] = 1.$$

By [14, Lemma 7.2], condition **K3** holds when the probability measure μ is non-arithmetic.

We need some additional notation. Let $\mathcal{C}(\mathcal{S})$ be the space of continuous functions on $\mathcal{S}$ with real values. We equip $\mathcal{C}(\mathcal{S})$ with the L^∞-norm

$$\|\varphi\|_\infty := \sup_{x\in\mathcal{S}} \|\varphi x\|, \quad \varphi \in \mathcal{C}(\mathcal{S}).$$

Under condition **K2**, for any $s \in [-\eta_0, \eta_0]$, define the transfer operator P_s as follows : for all $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s\varphi(x) := \mathbb{E}\Big[\|M_0 x\|^s \varphi(M_0 \cdot x)\Big], \quad x \in \mathcal{S}. \tag{5.2.10}$$

By [13, Proposition 3.1] and [33, Proposition 3.1], under **K2**, for $s \in [-\eta_0, \eta_0]$ the limit

$$\kappa(s) := \lim_{n\to+\infty} \Big(\mathbb{E}\|M_{0,n-1}\|^s\Big)^{1/n} \tag{5.2.11}$$

exists, with $0 < \kappa(s) < +\infty$, and is the spectral radius of P_s. Moreover, by [11, Lemma 10.17] the function $s \mapsto \kappa(s)$ is analytic in $(-\eta, \eta)$, for $\eta > 0$ small enough. Set $\Lambda(s) := \log \kappa(s)$ and $\gamma_k := \Lambda^{(k)}(0)$, $k \geqslant 1$. Then $\gamma_1 = \gamma$ and $\gamma_2 = \sigma^2$ (see [14, Corollary 7.3]).

Denote by ζ the Cramér series associated to Λ (see [19] and [63]):

$$\zeta(t) := \frac{\gamma_3}{6\gamma_2^{3/2}} + \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{24\gamma_2^3} t + \frac{\gamma_5\gamma_2^2 - 10\gamma_4\gamma_3\gamma_2 + 15\gamma_3^3}{120\gamma_2^{9/2}} t^2 + \cdots, \tag{5.2.12}$$

which converges for $|t|$ small enough.

The following theorem gives a Cramér type moderate deviation expansion for $\log \|Z_n^i\|$. Recall that $\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \mathrm{e}^{-t^2/2}\, dt, x \in \mathbb{R}$.

Theorem 5.2.1. *Assume conditions **K1**, **K2**, **K3** and $\gamma > 0$. Then, for $0 \leqslant x \leqslant o(\sqrt{n})$ and any $1 \leqslant i \leqslant d$, as $n \to +\infty$,*

$$\frac{\mathbb{P}\Big(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\Big)}{1 - \Phi(x)} = \mathrm{e}^{\frac{x^3}{\sqrt{n}}\zeta(\frac{x}{\sqrt{n}})} \left[1 + O\Big(\frac{1+x}{\sqrt{n}}\Big)\right], \tag{5.2.13}$$

and

$$\frac{\mathbb{P}\Big(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} < -x\Big)}{\Phi(-x)} = \mathrm{e}^{-\frac{x^3}{\sqrt{n}}\zeta(-\frac{x}{\sqrt{n}})} \left[1 + O\Big(\frac{1+x}{\sqrt{n}}\Big)\right]. \tag{5.2.14}$$

In the single type case $d = 1$, Grama, Liu and Miqueu established a version of this result in [31, Theorem 1.3]. Notice that when $d = 1$, we have $\gamma = \mathbb{E}\log m_0$, $\sigma^2 = \mathbb{E}(\log m_0 - \gamma)^2$, where $m_0 = \mathbb{E}_\xi Z_1$, and the condition **K2** reduces to the following: there exist two constants $p \in (1,2]$ and $\eta_0 \in (0,1)$ such that

$$\mathbb{E}m_0^{\eta_0} < +\infty \quad \text{and} \quad \mathbb{E}\theta_0(p)^{\eta_0} < +\infty, \text{ where } \theta_0(p) = \mathbb{E}_\xi\left|\frac{Z_1}{m_0} - 1\right|^p.$$

From Theorem 5.2.1, we obtain the moderate deviation expansion for $\log\|Z_n^i\|$ for $x = o(n^{1/6})$, as $n \to +\infty$:

Corollary 5.2.2. *Assume the conditions of Theorem 5.2.1. Then, for all $0 \leqslant x \leqslant o(n^{1/6})$ and $1 \leqslant i \leqslant d$, as $n \to +\infty$,*

$$\frac{\mathbb{P}\Big(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\Big)}{1 - \Phi(x)} = 1 + O\Big(\frac{1+x}{\sqrt{n}}\Big), \tag{5.2.15}$$

and

$$\frac{\mathbb{P}\Big(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} < -x\Big)}{\Phi(-x)} = 1 + O\Big(\frac{1+x}{\sqrt{n}}\Big). \tag{5.2.16}$$

5.3 Harmonic moments of W^i

In this section, we prove the existence of harmonic moments of W^i under a new measure $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$ for a $\eta > 0$ small enough.

5.3.1 Definition of the change of measure $\mathbb{P}_s^x$

We define a new probability measure called $\mathbb{P}_s^x$, for $x \in \mathcal{S}$, $s \in (-\eta, \eta)$ with $\eta > 0$ small enough. The construction of $\mathbb{P}_s^x$ is based on several properties of the transfer operator P_s defined by (5.2.10). For $s \in [-\eta_0, \eta_0]$, we introduce the conjugate operator P_s^* on $\mathcal{C}(\mathcal{S})$ defined by:

$$P_s^*\varphi(x) := \mathbb{E}\Big[\|M_0^T x\|^s \varphi(M_0^T \cdot x)\Big], \quad x \in \mathcal{S}, \quad \forall \varphi \in \mathcal{C}(\mathcal{S}). \tag{5.3.1}$$

A lot of results have been established on these operators P_s and P_s^* in recent years; in the following proposition, we list some of them established in [13, Proposition 3.1], [36, Corollary 3.20], [33, Proposition 3.1] and [75, Proposition 3.1], which will be used in the proofs of our results.

Proposition 5.3.1. *Assume condition **K2**. Then, for $\eta > 0$ small enough and $s \in (-\eta, \eta)$, the following assertions hold:*

(1) the spectral radii of P_s and P_s^* are both equal to $\kappa(s)$;

(2) there exists a unique strictly positive function $r_s \in \mathcal{C}(\mathcal{S})$ with norm $\|r_s\|_\infty = 1$ such that

$$P_s r_s = \kappa(s) r_s;$$

(3) there exists a unique strictly positive function $r_s^* \in \mathcal{C}(\mathcal{S})$ with norm $\|r_s^*\|_\infty = 1$ such that

$$P_s^* r_s^* = \kappa(s) r_s^*;$$

(4) $\kappa(0) = 1$ and $r_0 = r_0^* = 1$, where 1 denotes the constant function equal to 1 on $\mathcal{S}$;

(5) the function $s \mapsto \kappa(s)$ is analytic on $(-\eta, \eta)$;

(6) the mappings $s \mapsto r_s$ and $s \mapsto r_s^*$ are analytic on $(-\eta, \eta)$.

For $s \in (-\eta, \eta)$, $x \in \mathcal{S}$ and $A \in \mathcal{G}_+^0$, set

$$q_n^s(x, A) := \frac{\|Ax\|^s r_s(A \cdot x)}{\kappa(s)^n r_s(x)}. \tag{5.3.2}$$

Notice that the family (q_n^s) satisfies the following cocycle property: for any $n, m \geqslant 1$ and $A_1, A_2 \in \mathcal{G}_+^0$,

$$q_n^s(x, A_1) q_m^s(A_1 \cdot x, A_2) = q_{n+m}^s(x, A_2 A_1). \tag{5.3.3}$$

Denote by μ the law of the environment ξ_0 on $\mathbb{X}$. It is clear that for any $x \in \mathcal{S}$, by the assertion (2) of Proposition 5.3.1 and since $\kappa(s)$ and r_s are strictly positive, $q_n^s(x, M_{0,n-1}^T)\mu(d\xi_0) \cdots \mu(d\xi_{n-1})$, $n \geqslant 1$, is a sequence of probability measures which forms a projective system on $\mathbb{X}^{\mathbb{N}}$. Therefore, by the Kolmogorov extension theorem, there is a unique probability measure τ_s^x on $\mathbb{X}^{\mathbb{N}}$ with marginals $q_n^s(x, M_{0,n-1}^T)\mu(d\xi_0) \cdots \mu(d\xi_{n-1})$.

Denote by $\mathbb{P}_s^x(dy, d\xi) = \mathbb{P}_\xi(dy)\tau_s^x(d\xi)$ the corresponding annealed probability, and by $\mathbb{E}_s^x$ the expectation with respect to $\mathbb{P}_s^x$. For $x \in \mathcal{S}$ define the process

$$X_0^x = x, \quad \text{and} \quad X_n^x = M_{0,n-1}^T \cdot x, \quad n \geqslant 1,$$

which forms a Markov chain on $\mathcal{S}$. Then, by definition of $\mathbb{P}_s^x$, for $n \geqslant 1$ and any bounded measurable function h on $\mathbb{X}^n$, we have

$$\mathbb{E}\Big[\frac{\|M_{0,n-1}^T x\|^s r_s(X_n^x)}{\kappa(s)^n r_s(x)} h(\xi_0, \cdots, \xi_{n-1})\Big] = \mathbb{E}_s^x\Big[h(\xi_0, \cdots, \xi_{n-1})\Big]. \tag{5.3.4}$$

5.3.2 Existence of harmonic moments of W^i under $\mathbb{P}_s^{e_i}$

It is clear that, under the new measure $\mathbb{P}_s^{e_i}$, the sequence (W_n^i) is a non negative martingale w.r.t. the filtration $(\mathcal{F}_n)$, hence converges $\mathbb{P}_s^{e_i}$-a.s. to a non negative and finite random variable W^i. Denote by ϕ_ξ^i the quenched Laplace transform of W^i , and by ϕ_s^i its annealed Laplace transform under $\mathbb{P}_s^{e_i}$: for all $t \geqslant 0$, $s \in (-\eta, \eta)$, $\eta > 0$ small, and $1 \leqslant i \leqslant d$,

$$\phi_\xi^i(t) = \mathbb{E}_\xi \, \mathrm{e}^{-tW^i} \quad \text{and} \quad \phi_s^i(t) = \mathbb{E}_s^{e_i} \phi_\xi^i(t) = \mathbb{E}_s^{e_i} \, \mathrm{e}^{-tW^i} .$$

In a last article [34, Theorem 3.10] we established the following result which gives a bound for ϕ_0^i, and the existence of the harmonic moments $\mathbb{E}(W^i)^{-a}$ for all $1 \leqslant i \leqslant d$ and $a > 0$ small enough.

Lemma 5.3.2. *Assume conditions* ***K1****,* ***K2*** *and $\gamma > 0$. Then there exist two constants $a > 0$ and $C > 0$ such that for all $t > 0$, all $x > 0$ and $1 \leqslant i \leqslant d$,*

$$\phi_0^i(t) \leqslant \frac{C}{t^a}, \tag{5.3.5}$$

$$\mathbb{P}(W^i \leqslant x) \leqslant Cx^a \quad \textit{and} \quad \mathbb{E}(W^i)^{-a} \leqslant C. \tag{5.3.6}$$

Now, we prove the corresponding results of Lemma 5.3.2 under the probability measure $\mathbb{P}_s^{e_i}$. The following theorem gives a control of ϕ_s^i uniformly in $s \in (-\eta, \eta)$, and that the harmonic moments $\mathbb{E}_s^{e_i}(W^i)^{-a}$ are uniformly bounded in $s \in (-\eta, \eta)$, for small $\eta > 0$ and $a > 0$.

Theorem 5.3.3. *Assume conditions* ***K1***, ***K2*** *and* $\gamma > 0$. *Then, for* $\eta > 0$ *small enough, there exist two constants* $a > 0$ *and* $C > 0$ *such that for all* $t > 0$, *all* $x > 0$ *and* $1 \leqslant i \leqslant d$,

$$\sup_{s \in (-\eta, \eta)} \phi_s^i(t) \leqslant \frac{C}{t^a}, \tag{5.3.7}$$

$$\sup_{s \in (-\eta, \eta)} \mathbb{P}_s^{e_i}(W^i \leqslant x) \leqslant C x^a \quad \text{and} \quad \sup_{s \in (-\eta, \eta)} \mathbb{E}_s^{e_i}(W^i)^{-a} \leqslant C. \tag{5.3.8}$$

5.3.3 Preliminaries to the proof of Theorem 5.3.3

To prove Theorem 5.3.3, we need some preliminary results. The first lemma below combines two results of a previous article [34, Lemmata 3.1 and 3.11]. It gives a link between the expectation of $\varphi(W_n^i)$ and that of $\varphi(W^i)$ under $\mathbb{P}$ and $\mathbb{P}_\xi$, where φ is a positive convex function on $\mathbb{R}_+$.

Lemma 5.3.4. *Assume* $\gamma > 0$ *and ether condition* ***K2*** *or* (5.2.7). *Then for all* $1 \leqslant i \leqslant d$ *and any convex function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\lim_{n \to +\infty} \mathbb{E}_\xi \varphi(W_n^i) = \sup_{n \geqslant 0} \mathbb{E}_\xi \varphi(W_n^i) = \mathbb{E}_\xi \varphi(W^i), \tag{5.3.9}$$

and

$$\lim_{n \to +\infty} \mathbb{E} \varphi(W_n^i) = \sup_{n \geqslant 0} \mathbb{E} \varphi(W_n^i) = \mathbb{E} \varphi(W^i). \tag{5.3.10}$$

The second lemma is a direct consequence of the Marcinkiewicz-Zygmund inequality in [16, Theorem 1.5], as stated in [60, Lemma 1.4]. It allows us to control the L^p-moments of the martingale (W_n^i) under $\mathbb{P}_\xi$.

Lemma 5.3.5. *Let* $(X_k)_{k \in \mathbb{N}^*}$ *be a sequence of i.i.d. random centered variables. Then for all* $n \in \mathbb{N}^*$ *and* $p > 1$ *:*

$$\mathbb{E}\left| \sum_{k=1}^n X_k \right|^p \leqslant \begin{cases} (B_p)^p \mathbb{E}|X_k|^p n, & \text{if } 1 < p \leqslant 2, \\ (B_p)^p \mathbb{E}|X_k|^p n^{\frac{p}{2}}, & \text{if } p > 2, \end{cases}$$

where $B_p = 2\min\{k^{1/2} : k \in \mathbb{N}, k \geqslant \frac{p}{2}\}$.

The following result gives the convergence in L^α of W_n^i to W^i under $\mathbb{P}_s^{e_i}$ with an exponential speed, uniformly in $s \in (-\eta, \eta)$.

Proposition 5.3.6. *Assume conditions* ***K1***, ***K2*** *and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exist constants $C > 0$, $\delta \in (0,1)$ and $\varepsilon > 0$ such that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,*

$$\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}\left[\left(\mathbb{E}_\xi |W_n^i - W^i|^p\right)^\varepsilon\right] \leqslant C\delta^n. \tag{5.3.11}$$

Moreover, with $\alpha = p\varepsilon$, $W_n^i \underset{n\to+\infty}{\longrightarrow} W^i$ in L^α under $\mathbb{P}_s^{e_i}$ uniformly in $s \in (-\eta,\eta)$, such that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i} |W_n^i - W^i|^\alpha \leqslant C\delta^n. \tag{5.3.12}$$

Proof. By (5.2.3) and (5.1.9), for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, we have

$$\begin{aligned} W_{n+1}^i - W_n^i &= \sum_{j=1}^d \frac{U_{n+1,\infty}(j)}{\lambda_{0,n} U_{0,\infty}(i)} \sum_{r=1}^d \sum_{l=1}^{Z_n^i(r)} N_{l,n}^r(j) - W_n^i \\ &= \sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(r)} \sum_{j=1}^d \frac{U_{n+1,\infty}(j) N_{l,n}^r(j)}{\lambda_n U_{n,\infty}(r)} - W_n^i \\ &= \sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \sum_{l=1}^{Z_n^i(r)} (W_{l,n}^r - 1), \end{aligned} \tag{5.3.13}$$

where

$$W_{l,n}^r := \frac{\langle N_{l,n}^r, U_{n+1,\infty}\rangle}{\lambda_n U_{n,\infty}(r)}.$$

Clearly, given the environment ξ, the random variables $W_{l,n}^r$, $l \geqslant 1$, are i.i.d., and they are independent of $\xi_0, \ldots, \xi_{n-1}$ and Z_n^i. Therefore, applying (5.3.13), the convexity of the function $x \mapsto x^p$ on $\mathbb{R}_+$ (together with the fact that $\sum_{r=1}^d \frac{M_{0,n-1}(i,r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} = 1$ a.s. by

(5.1.8)) and Lemma 5.3.5, we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p \leqslant \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \left| \sum_{l=1}^{Z_n^i(r)} \left(W_{l,n}^r - 1 \right) \right| \right)^p \\
&= \mathbb{E}_\xi \left(\sum_{r=1}^d \frac{M_{0,n-1}(i,r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \frac{1}{M_{0,n-1}(i,r)} \left| \sum_{l=1}^{Z_n^i(r)} \left(W_{l,n}^r - 1 \right) \right| \right)^p \\
&\leqslant \sum_{r=1}^d \frac{M_{0,n-1}(i,r) U_{n,\infty}(r)}{\lambda_{0,n-1} U_{0,\infty}(i)} \frac{1}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi \left| \sum_{l=1}^{Z_n^i(r)} \left(W_{l,n}^r - 1 \right) \right|^p \\
&\leqslant B_p^p \max_{1 \leqslant r \leqslant d} \left\{ \frac{\mathbb{E}_\xi Z_n^i(r)}{M_{0,n-1}(i,r)^p} \mathbb{E}_\xi |W_{1,n}^r - 1|^p \right\} \\
&= B_p^p \max_{1 \leqslant r \leqslant d} \mathbb{E}_\xi |W_{1,n}^r - 1|^p \max_{1 \leqslant j \leqslant d} (M_{0,n-1}(i,j))^{1-p}. \qquad (5.3.14)
\end{aligned}$$

Using again the convexity of $x \mapsto x^p$, the same argument yields, for all $n \geqslant 0$ and $1 \leqslant r \leqslant d$, $\mathbb{P}$-a.s., we have

$$\begin{aligned}
\mathbb{E}_\xi |W_{1,n}^r - 1|^p &= \mathbb{E}_\xi \left| \frac{\langle N_{1,n}^r, U_{n+1,\infty} \rangle}{\lambda_n U_{n,\infty}(r)} - 1 \right|^p \\
&= \mathbb{E}_\xi \left| \sum_{j=1}^d \frac{M_n(r,j) U_{n+1,\infty}(j)}{\lambda_n U_{n,\infty}(r)} \left(\frac{N_{1,n}^r(j)}{M_n(r,j)} - 1 \right) \right|^p \\
&\leqslant \max_{1 \leqslant i,j \leqslant d} \mathbb{E}_\xi \left| \frac{N_{1,n}^i(j)}{M_n(i,j)} - 1 \right|^p = \theta_n(p). \qquad (5.3.15)
\end{aligned}$$

This, together with (5.3.14), implies that for all $n \geqslant 0$ and $1 \leqslant r \leqslant d$, $\mathbb{P}$-a.s.,

$$\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p \leqslant B_p^p \theta_n(p) \max_{1 \leqslant j \leqslant d} M_{0,n-1}(i,j)^{1-p}. \qquad (5.3.16)$$

Let $\varepsilon > 0$, and $\eta > 0$ small enough. Taking the moment of order ε under $\mathbb{P}_s^{e_i}$ in (5.3.16), by (5.3.4) and the fact that $\theta_n(p)$ and $M_{0,n-1}$ depend only on the environments ξ_k for $k \leqslant n$, we obtain that for $n \geqslant 0$, $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\mathbb{E}_s^{e_i} \left[\left(\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p \right)^\varepsilon \right] \\
\leqslant \; & (B_p)^{\varepsilon p} \mathbb{E}_s^{e_i} \left[\theta_n(p)^\varepsilon \max_{1 \leqslant j \leqslant d} M_{0,n-1}(i,j)^{\varepsilon(1-p)} \right] \\
= \; & (B_p)^{\varepsilon p} \mathbb{E} \left[q_{n+1}^s \left(e_i, M_{0,n}^T \right) \theta_n(p)^\varepsilon \max_{1 \leqslant j \leqslant d} M_{0,n-1}(i,j)^{\varepsilon(1-p)} \right]. \qquad (5.3.17)
\end{aligned}$$

By Proposition 5.3.1 we know that $s \mapsto r_s$ is a continuous map on $(-\eta, \eta)$, and that r_s is a strictly positive function in $\mathcal{C}(\mathcal{S})$ with norm $\|r_s\|_\infty = 1$. This implies that

$$D := \sup_{s\in(-\eta,\eta)} \left\{ \frac{\sup_{x\in\mathcal{S}} r_s(x)}{\inf_{x\in\mathcal{S}} r_s(x)} \right\} = \sup_{s\in(-\eta,\eta)} \sup_{x\in\mathcal{S}} r_s^{-1}(x) < +\infty, \tag{5.3.18}$$

where $r_s^{-1}(x) := [r_s(x)]^{-1}$ for $x \in \mathcal{S}$. Moreover, we know by **K1** that $\|M_{0,n}^T\| \geqslant 1$ a.s., and κ is a strictly positive increasing function on $(-\eta, \eta)$, so that $\kappa(s) \geqslant \kappa(-\eta) > 0$ for all $s \in (-\eta, \eta)$. This, together with (5.3.18) and the definition of $q_1^s(x, A)$ (see (5.3.2)), implies that for all $x \in \mathcal{S}$ and $s \in (-\eta, \eta)$,

$$q_1^s(x, M_0^T) \leqslant \frac{D}{\kappa(-\eta)} \|M_0^T x\|^s \leqslant \frac{D}{\kappa(-\eta)} \|M_0^T\|^\eta \leqslant \frac{d^\eta D}{\kappa(-\eta)} \|M_0\|^\eta \quad \mathbb{P}\text{-a.s.} \tag{5.3.19}$$

Therefore, combining the relations (5.3.17), (5.3.3), (5.3.4) and (5.3.18), we get that for $n \geqslant 2$, $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\mathbb{E}_s^{e_i}\Big[\Big(\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p\Big)^\varepsilon\Big] \\
\leqslant\ & (B_p)^{\varepsilon p} \mathbb{E}\Big[q_{n-1}^s\Big(e_i, M_{0,n-2}^T\Big) q_1^s\Big(X_{n-1}^{e_i}, M_{n-1}^T\Big) q_1^s\Big(X_n^{e_i}, M_n^T\Big) \times \\
& \theta_n(p)^\varepsilon \max_{1\leqslant j\leqslant d} M_{n-1}(i,j)^{\varepsilon(1-p)} \|M_{0,n-2}(i,\cdot)\|^{\varepsilon(1-p)}\Big] \\
\leqslant\ & (B_p)^{\varepsilon p} \left(\frac{d^\eta D}{\kappa(-\eta)}\right)^2 \mathbb{E}\Big[\|M_0\|^\eta \theta_0(p)^\varepsilon\Big] \mathbb{E}\Big[\|M_0\|^\eta \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{\varepsilon(1-p)}\Big] \times \\
& \mathbb{E}_s^{e_i}\Big[\|M_{0,n-2}(i,\cdot)\|^{\varepsilon(1-p)}\Big].
\end{aligned} \tag{5.3.20}$$

Now we control the three expectations in the right side of (5.3.20). For the two first expectations, by Cauchy Schwarz's inequality and condition **K2**, for $\eta > 0$ and $\varepsilon > 0$ both sufficiently small such that $\eta \leqslant \frac{\eta_0}{2}$ and $\varepsilon \leqslant \frac{\eta_0}{2}$, we have

$$\mathbb{E}\Big[\|M_0\|^\eta \theta_0(p)^\varepsilon\Big] \leqslant \Big(\mathbb{E}\|M_0\|^{2\eta}\Big)^{\frac{1}{2}} \Big(\mathbb{E}\theta_0(p)^{2\varepsilon}\Big)^{\frac{1}{2}} < +\infty, \tag{5.3.21}$$

and

$$\begin{aligned}
\mathbb{E}\Big[\|M_0\|^\eta \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{\varepsilon(1-p)}\Big] &\leqslant \Big(\mathbb{E}\|M_0\|^{2\eta}\Big)^{\frac{1}{2}} \Big(\mathbb{E} \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{2\varepsilon(1-p)}\Big)^{\frac{1}{2}} \\
&< +\infty.
\end{aligned} \tag{5.3.22}$$

For the third expectation, using again (5.3.4) and (5.3.18), we get that for all $n \geqslant 2$ and $s \in (-\eta, \eta)$,

$$\begin{aligned}
&\mathbb{E}_s^{e_i}\Big[\|M_{0,n-2}(i,\cdot)\|^{\varepsilon(1-p)}\Big] \\
=\ & \mathbb{E}\left[\frac{\|M_{0,n-2}(i,\cdot)\|^{s+\varepsilon(1-p)} r_s(X_{n-1}^{e_i})}{\kappa(s)^{n-1} r_s(e_i)}\right] \\
\leqslant\ & D^2\left(\frac{\kappa(s+\varepsilon(1-p))}{\kappa(s)}\right)^{n-1} \mathbb{E}\left[\frac{\|M_{0,n-2}(i,\cdot)\|^{s+\varepsilon(1-p)} r_{s+\varepsilon(1-p)}(X_{n-1}^{e_i})}{\kappa(s+\varepsilon(1-p))^{n-1} r_{s+\varepsilon(1-p)}(e_i)}\right] \\
=\ & D^2\, \mathrm{e}^{(n-1)[\Lambda(s+\varepsilon(1-p))-\Lambda(s)]},
\end{aligned} \tag{5.3.23}$$

where $\Lambda(s) = \log \kappa(s)$ (and the last equality follows from (5.3.4) with $h = 1$). By Proposition 5.3.1, for $\eta > 0$ small enough, the function Λ is analytic on $(-\eta, \eta)$, with $\Lambda(0) = 0$ and $\Lambda'(0) = \gamma$ by [14, Corollary 7.3]. By hypothesis we have $\gamma > 0$, so Λ is strictly increasing on $[-\eta, \eta]$ for $\eta > 0$ small enough. Therefore, taking $\eta > 0$ and $\varepsilon > 0$ both sufficiently small, since Λ is continuous and strictly increasing on $[-\eta, \eta]$, we obtain

$$\sup_{s\in[-\eta,\eta]} \{\Lambda(s+\varepsilon(1-p)) - \Lambda(s)\} < 0. \tag{5.3.24}$$

It follows that $\delta := \mathrm{e}^{\sup_{s\in(-\eta,\eta)}\{\Lambda(s+\varepsilon(1-p))-\Lambda(s)\}} \in (0,1)$, and we deduce from (5.3.23) that for all $n \geqslant 2$ and $s \in (-\eta, \eta)$,

$$\mathbb{E}_s^{e_i}\Big[\|M_{0,n-2}(i,\cdot)\|^{\varepsilon(1-p)}\Big] \leqslant D^2 \delta^{n-1}. \tag{5.3.25}$$

Now, combining the inequalities (5.3.20), (5.3.21), (5.3.22) and (5.3.25), there exists a constant $C > 0$ such that for all $n \geqslant 2$ and $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}\Big[\Big(\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p\Big)^\varepsilon\Big] \leqslant C\delta^n. \tag{5.3.26}$$

Moreover, by similar calculation as in (5.3.20), for all $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$ we have

$$\mathbb{E}_s^{e_i}\Big[\Big(\mathbb{E}_\xi |W_1^i - W_0^i|^p\Big)^\varepsilon\Big] \leqslant (B_p)^{\varepsilon p} \frac{d^\eta D}{\kappa(-\eta)} \mathbb{E}\Big[\|M_0\|^\eta \theta_0(p)^\varepsilon\Big], \tag{5.3.27}$$

and

$$\mathbb{E}_s^{e_i}\left[\left(\mathbb{E}_\xi|W_2^i - W_1^i|^p\right)^\varepsilon\right] \leqslant (B_p)^{\varepsilon p}\left(\frac{d^\eta D}{\kappa(-\eta)}\right)^2 \mathbb{E}\left[\|M_0\|^\eta \theta_0(p)^\varepsilon\right]\mathbb{E}\left[\|M_0\|^\eta \max_{1\leqslant i,j\leqslant d} M_0(i,j)^{\varepsilon(1-p)}\right]. \quad (5.3.28)$$

Therefore, putting together the inequalities (5.3.21), (5.3.22), (5.3.29), (5.3.27) and (5.3.28), by taking $C > 0$ sufficiently large, it holds that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}\left[\left(\mathbb{E}_\xi|W_{n+1}^i - W_n^i|^p\right)^\varepsilon\right] \leqslant C\delta^n. \quad (5.3.29)$$

By the triangular inequality and the sub-additivity of the function $x \mapsto x^\varepsilon$ on $\mathbb{R}_+$, it follows that for all $n, k \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}\left[\left(\mathbb{E}_\xi|W_{n+k}^i - W_n^i|^p\right)^\varepsilon\right] &\leqslant \sum_{r=n}^{n+k-1} \sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}\left[\left(\mathbb{E}_\xi|W_{r+1}^i - W_r^i|^p\right)^\varepsilon\right] \\ &\leqslant C\sum_{r=n}^{n+k-1}\delta^r \\ &\leqslant \frac{C}{1-\delta}\delta^n.\end{aligned}$$

So, by letting $k \to +\infty$, (5.3.11) holds. Let $\alpha = p\varepsilon > 0$. Using Hölder's inequality with $\varepsilon \in (0,1)$, we obtain from (5.3.11) that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}|W^i - W_n^i|^\alpha \leqslant \sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}\left[\left(\mathbb{E}_\xi|W^i - W_n^i|^p\right)^\varepsilon\right] \leqslant \frac{C}{1-\delta}\delta^n.$$

Therefore, (5.3.12) holds. This concludes the proof of Proposition 5.3.6. □

5.3.4 Proof of Theorem 5.3.3

Now we proceed to prove Theorem 5.3.3.

Proof of Theorem 5.3.3. First, we prove the implication (5.3.7) ⇒ (5.3.8). Assume that $\eta > 0$ and $a > 0$ are constants such that (5.3.7) holds. Let $b \in (0, a)$. We know that for all $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}_s^{e_i}(W^i)^{-b} = \frac{1}{\Gamma(b)} \int_0^{+\infty} \phi_s^i(t) t^{b-1} dt,$$

where Γ is the Gamma function. So, by (5.3.7) we get that for all $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}(W^i)^{-b} \leqslant \frac{C}{\Gamma(b)} \int_0^{+\infty} t^{a-b-1} dt < +\infty. \tag{5.3.30}$$

Moreover, by Markov's inequality we have that for all $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \mathbb{P}_s^{e_i}(W^i \leqslant x) \leqslant x^{-b} \sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}(W^i)^{-b}. \tag{5.3.31}$$

It is clear that (5.3.30) and (5.3.31) imply (5.3.8).

Now we prove (5.3.7), which will conclude the proof of Theorem 5.3.3. Let $\eta > 0$ be small enough, and $\varepsilon \in (0, 1)$. For all $n \geqslant 0$, $s \in (-\eta, \eta)$, $t \geqslant 0$ and $1 \leqslant i \leqslant d$, we have

$$\begin{aligned} \phi_s^i(t) &= \mathbb{E}_s^{e_i}\Big[\mathrm{e}^{-tW^i} \, \mathbb{1}_{\{|W_n^i - W^i| \leqslant \varepsilon^n\}} \Big] + \mathbb{E}_s^{e_i}\Big[\mathrm{e}^{-tW^i} \, \mathbb{1}_{\{|W_n^i - W^i| > \varepsilon^n\}} \Big] \\ &\leqslant \mathbb{E}_s^{e_i}\Big[\mathrm{e}^{-t(W_n^i - \varepsilon^n)} \Big] + \mathbb{E}_s^{e_i}\Big[\mathbb{1}_{\{|W_n^i - W^i| > \varepsilon^n\}} \Big] \\ &= \mathrm{e}^{t\varepsilon^n} \, \mathbb{E}_s^{e_i}\Big[\mathrm{e}^{-tW_n^i} \Big] + \mathbb{P}_s^{e_i}\Big(|W_n^i - W^i| > \varepsilon^n \Big). \end{aligned} \tag{5.3.32}$$

By Markov's inequality and Proposition 5.3.6, for $\eta > 0$ small enough, there exist constants $\delta_0 \in (0, 1)$ and $\alpha > 0$ such that for all $n \geqslant 0$, $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$,

$$\mathbb{P}_s^{e_i}\Big(|W_n^i - W^i| > \varepsilon^n \Big) \leqslant \varepsilon^{-\alpha n} \mathbb{E}_s^{e_i} |W_n^i - W^i|^\alpha \leqslant C \bigg(\frac{\delta_0}{\varepsilon^\alpha} \bigg)^n.$$

Taking $\varepsilon \in (0, 1)$ such that $\varepsilon > \delta_0^{1/\alpha}$, we get that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \mathbb{P}_s^{e_i}\Big(|W_n^i - W^i| > \varepsilon^n \Big) \leqslant C \delta_1^n. \tag{5.3.33}$$

where $\delta_1 := \frac{\delta_0}{\varepsilon^\alpha} \in (0, 1)$.

We now give a bound of $\mathbb{E}_s^{e_i}[\mathrm{e}^{-tW_n^i}]$ uniformly in s. By definition of W_n^i, for all $n \geqslant 1$

and $1 \leqslant i \leqslant d$ we have

$$\frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \min_{1\leqslant r\leqslant d} U_{n,\infty}(r) \leqslant W_n^i \leqslant \frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \frac{1}{\min_{1\leqslant r\leqslant d} U_{n,\infty}(r)}. \tag{5.3.34}$$

Moreover, by (5.1.7) we get that for all $n \geqslant 0$,

$$1 \geqslant \min_{1\leqslant r\leqslant d} U_{n,\infty}(r) = \min_{1\leqslant r\leqslant d} \frac{\langle M_n(r,\cdot), U_{n+1,\infty}\rangle}{\|M_n U_{n+1,\infty}\|} \geqslant \min_{1\leqslant r,j\leqslant d} \frac{M_n(r,j)}{\|M_n\|}. \tag{5.3.35}$$

Combining (5.3.34) and (5.3.35), it holds that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$W_n^i \geqslant \frac{\|Z_n^i\|}{\|M_{0,n-1}(i,\cdot)\|} \min_{1\leqslant r,j\leqslant d} \frac{M_n(r,j)}{\|M_n\|} =: Y_n^i. \tag{5.3.36}$$

The interesting point here is that Y_n^i is independent of the future $(\xi_{n+1}, \xi_{n+2}, \cdots)$. Set $\beta > 0$. Recall that $\Lambda(s) = \log\kappa(s)$. By (5.3.36), (5.3.4) and (5.3.18), we obtain that for all $n \geqslant 0$, $s \in (-\eta, \eta)$, $t \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
\mathbb{E}_s^{e_i}\Big[\mathrm{e}^{-tW_n^i}\Big] &\leqslant \mathbb{E}_s^{e_i}\Big[\mathrm{e}^{-tY_n^i}\Big(\mathbb{1}_{\{\log\|M_{0,n}(i,\cdot)\|\leqslant\beta(n+1)\}} + \mathbb{1}_{\{\log\|M_{0,n}(i,\cdot)\|>\beta(n+1)\}}\Big)\Big] \\
&\leqslant \mathbb{E}\left[\frac{\|M_{0,n}(i,\cdot)\|^s r_s\Big(X_{n+1}^{e_i}\Big)}{\kappa(s)^{n+1} r_s(e_i)} \mathrm{e}^{-tY_n^i}\,\mathbb{1}_{\{\log\|M_{0,n}(i,\cdot)\|\leqslant\beta(n+1)\}}\right] \\
&\quad + \mathbb{P}_s^{e_i}\Big(\log\|M_{0,n}(i,\cdot)\| > \beta(n+1)\Big) \\
&\leqslant D\,\mathrm{e}^{(n+1)[\beta s-\Lambda(s)]}\,\mathbb{E}\Big[\mathrm{e}^{-tY_n^i}\Big] + \mathbb{P}_s^{e_i}\Big(\log\|M_{0,n}(i,\cdot)\| > \beta(n+1)\Big).
\end{aligned} \tag{5.3.37}$$

We have to control all the terms on the right side of (5.3.37).

First, we give a suitable bound of $\mathbb{E}[\mathrm{e}^{-tY_n^i}]$. For all $n \geqslant 0$, $t > 0$ and $1 \leqslant i \leqslant d$, we have

$$\mathbb{E}[\mathrm{e}^{-tY_n^i}] = \int_{u=0}^1 \mathbb{P}\Big(\mathrm{e}^{-tY_n^i} \geqslant u\Big) du = \int_{u=0}^1 \mathbb{P}\Big(Y_n^i \leqslant -\frac{\log u}{t}\Big) du. \tag{5.3.38}$$

We will give a suitable bound of $\mathbb{P}(Y_n^i \leqslant x)$ for any $x > 0$, to obtain the decay rate of $\mathbb{E}[\mathrm{e}^{-tY_n^i}]$. To this end we will estimate the harmonic moments of Y_n^i. By [34, Lemma 3.11], condition **K2** implies (5.2.7). Therefore, applying Lemma 5.3.4 with the convex function $x \mapsto x^{-a}$ on $\mathbb{R}_+$ and Lemma 5.3.2, there exists a constant $a > 0$ such that for all

$1 \leqslant i \leqslant d$,

$$\sup_{n \geqslant 0} \mathbb{E}(W_n^i)^{-a} = \mathbb{E}(W^i)^{-a} < +\infty. \tag{5.3.39}$$

Using (5.3.34), (5.3.35) and Hölder's inequality, we obtain that for all $b > 0$, $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} \mathbb{E}(Y_n^i)^{-b} &\leqslant \mathbb{E}\Big[(W_n^i)^{-b}\|M_n\|^b \max_{1\leqslant r,j\leqslant d} M_n(r,j)^{-b}\Big] \\ &\leqslant \Big(\mathbb{E}(W_n^i)^{-3b}\Big)^{\frac{1}{3}}\Big(\mathbb{E}\|M_0\|^{3b}\Big)^{\frac{1}{3}}\Big(\mathbb{E}\max_{1\leqslant r,j\leqslant d} M_0(r,j)^{-3b}\Big)^{\frac{1}{3}}. \end{aligned} \tag{5.3.40}$$

Applying (5.3.40) with $b = \frac{1}{3}\min\{a, \eta_0\} > 0$, by (5.3.39) and condition **K2** we get that for all $1 \leqslant i \leqslant d$,

$$\sup_{n \geqslant 0} \mathbb{E}(Y_n^i)^{-b} \leqslant C.$$

By Markov's inequality, this implies that for all $x > 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{n \geqslant 0} \mathbb{P}\Big(Y_n^i \leqslant x\Big) \leqslant x^b \sup_{n \geqslant 0} \mathbb{E}(Y_n^i)^{-b} \leqslant Cx^b. \tag{5.3.41}$$

Combining the inequalities (5.3.38) and (5.3.41), we deduce that for all $t > 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{n \geqslant 0} \mathbb{E}[\mathrm{e}^{-tY_n^i}] \leqslant Ct^{-b} \int_{u=0}^{1} (-\log u)^b du \leqslant Ct^{-b}. \tag{5.3.42}$$

We next control the probability term in (5.3.37). Let $q > 0$ be a small constant. By Markov's inequality, (5.3.18) and (5.3.4), for all $n \geqslant 1$, $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$, we have

$$\begin{aligned}
& \mathbb{P}_s^{e_i}\Big(\log\|M_{0,n}(i,\cdot)\| > \beta(n+1)\Big) \\
\leqslant\ & \mathrm{e}^{-\beta q(n+1)}\,\mathbb{E}_s^{e_i}\|M_{0,n}(i,\cdot)\|^q \\
=\ & \mathrm{e}^{-\beta q(n+1)}\,\mathbb{E}\left[\frac{\|M_{0,n}(i,\cdot)\|^{s+q} r_s\Big(X_{n+1}^{e_i}\Big)}{\kappa(s)^{n+1} r_s(e_i)}\right] \\
\leqslant\ & D^2\frac{\kappa(s+q)^{n+1}}{\kappa(s)^{n+1}}\,\mathrm{e}^{-\beta q(n+1)}\,\mathbb{E}\left[\frac{\|M_{0,n}(i,\cdot)\|^{s+q} r_{s+q}\Big(X_{n+1}^{e_i}\Big)}{\kappa(s+q)^{n+1} r_{s+q}(e_i)}\right] \\
=\ & D^2\,\mathrm{e}^{(n+1)(\Lambda(s+q)-\Lambda(s)-\beta q)}\,. \qquad (5.3.43)
\end{aligned}$$

We know by Proposition 5.3.1 that Λ is analytic on $(-\eta,\eta)$ for $\eta>0$ small. From now, we choose $\beta > \sup_{s\in(-\eta,\eta)}\Lambda'(s)$. Then, by the mean value theorem, it holds that for all $s,q\in(-\eta/2,\eta/2)$,

$$\Lambda(s+q)-\Lambda(s)-\beta q = (\Lambda'(c)-\beta)q \leqslant q\sup_{s\in(-\eta,\eta)}\Big\{\Lambda'(s)-\beta\Big\} < 0, \qquad (5.3.44)$$

where c is a point between s and $s+q$. Combining this with (5.3.43), we get that for $\eta>0$ small enough (half of the previous value), there exists a constant $\delta_2\in(0,1)$ such that for all $n\geqslant 1$ and $1\leqslant i\leqslant d$,

$$\sup_{s\in(-\eta,\eta)}\mathbb{P}_s^{e_i}\Big(\log\|M_{0,n}(i,\cdot)\| > \beta(n+1)\Big) \leqslant C\delta_2^n. \qquad (5.3.45)$$

We then deal with the term $\mathrm{e}^{(n+1)[\beta s-\Lambda(s)]}$ in (5.3.37). Using the mean value theorem with $\Lambda(0)=0$, we get that for all $s\in(-\eta,\eta)$, $|\Lambda(s)|=|\Lambda(s)-\Lambda(0)|\leqslant\sup_{c\in(-\eta,\eta)}\Lambda'(c)|s|<\beta|s|$, so that $\beta s-\Lambda(s)\leqslant 2\beta|s|$. It follows that for all $n\geqslant 1$ and $s\in(-\eta,\eta)$,

$$\mathrm{e}^{(n+1)[\beta s-\Lambda(s)]} \leqslant \mathrm{e}^{2\beta|s|(n+1)} \leqslant \mathrm{e}^{2\beta\eta(n+1)}\,. \qquad (5.3.46)$$

Putting together the inequalities (5.3.32), (5.3.33), (5.3.42), (5.3.45) and (5.3.46), we obtain that for all $n\geqslant 1$, $t\geqslant 0$ and $1\leqslant i\leqslant d$,

$$\sup_{s\in(-\eta,\eta)}\phi_s^i(t) \leqslant C\,\mathrm{e}^{t\varepsilon^n}\left(\mathrm{e}^{2\beta\eta(n+1)}\,t^{-b}+\delta_2^n\right)+C\delta_1^n.$$

Taking $\delta = \max\left\{ e^{2\beta\eta}\,\varepsilon^b\delta_1, \delta_2, \delta_1 \right\}$, this implies that for all $n \geqslant 1$, $t \geqslant \varepsilon^{-n}$ and $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \phi_s^i(t) \leqslant C\delta^n, \tag{5.3.47}$$

with $\delta \in (0,1)$ for $\eta > 0$ small enough. Define

$$N(t) := \left\lfloor -\frac{\log t}{\log \varepsilon} \right\rfloor + 1, \quad t \geqslant \frac{1}{\varepsilon}.$$

It is clear that when $t \geqslant \frac{1}{\varepsilon}$, we have $N(t) \geqslant 1$, $t \geqslant \varepsilon^{-N(t)}$ and $N(t) \geqslant -\log t/\log \varepsilon$. Therefore, using the inequality (5.3.47) with $n = N(t)$, we get that for all $t \geqslant \frac{1}{\varepsilon}$ and $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \phi_s^i(t) \leqslant C\delta^{N(t)} \leqslant C\delta^{-\frac{\log t}{\log \varepsilon}} = Ct^{-\frac{\log \delta}{\log \varepsilon}},$$

where $\frac{\log \delta}{\log \varepsilon} > 0$. So (5.3.7) holds. This concludes the proof of Theorem 5.3.3. □

5.4 Berry-Esseen bound for $\log \|Z_n^i\|$ under $\mathbb{P}_s^{e_i}$

In this section, we establish a Berry-Esseen bound for $\log \|Z_n^i\|$, the logarithm of the population size $\|Z_n^i\| = Z_n^i(1) + \cdots + Z_n^i(d)$, under the changed measure $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$.

Recall that by Proposition 5.3.1, under condition **K2**, the function $s \mapsto \Lambda(s) = \log \kappa(s)$ is analytic on $(-\eta,\eta)$, hence $\Lambda'(s)$ and $\sigma_s := \Lambda''(s)$ are well defined and analytic on $s \in (-\eta,\eta)$. From [75, Proposition 3.12], we have a strong law of large numbers for $\log \|M_{0,n-1}^T x\|$ under the changed measure $\mathbb{P}_s^x$: for $s \in (-\eta,\eta)$, and $x \in \mathcal{S}$,

$$\frac{1}{n}\log \|M_{0,n-1}^T x\| \underset{n\to+\infty}{\to} \Lambda'(s) \quad \mathbb{P}_s^x\text{-a.s.}$$

Moreover, by [75, Proposition 3.14], uniformly in $s \in (-\eta,\eta)$ and $x \in \mathcal{S}$, we have

$$\sigma_s^2 = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}_s^x[(\log \|M_{0,n-1}^T x\| - n\gamma)^2] \in [0,\infty).$$

When condition **K3** holds, since the function $s \mapsto \sigma_s$ is continuous on $(-\eta,\eta)$ and $\sigma_0 =$

$\sigma > 0$, it follows that for $\eta > 0$ small enough,

$$\inf_{s\in(-\eta,\eta)} \sigma_s > 0. \tag{5.4.1}$$

Now, we formulate a Berry-Esseen bound for $\log \|Z_n^i\|$ under the changed measure $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$.

Theorem 5.4.1. *Assume conditions **K1**, **K2**, **K3** and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exists a constant $C > 0$ such that for all $n \geqslant 1$ and $x \in \mathbb{R}$,*

$$\sup_{s\in(-\eta,\eta)} \left| \mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{C}{\sqrt{n}}.$$

Notice that when $s = 0$, Theorem 5.4.1 reduces to the Berry-Essen bound (5.1.3) under the initial measure $\mathbb{P}$, which has been proved in a previous article [34, Theorem 2.4].

For the proof of Theorem 5.4.1, we need several preliminary results. We start by the following lemma which gives the convergence in L^1 of $\log W_n^i$ to $\log W^i$ under $\mathbb{P}_s^{e_i}$ with an exponential rate, uniformly in s.

Lemma 5.4.2. *Assume conditions **K1**, **K2** and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exist two constants $C > 0$ and $\delta \in (0, 1)$ such that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,*

$$\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i} |\log W_n^i - \log W^i| \leqslant C\delta^n.$$

Proof. For all $n \geqslant 0$ and $1 \leqslant i \leqslant d$, we have

$$\log W_n^i - \log W^i = \log(1 + R_n^i), \tag{5.4.2}$$

where

$$R_n^i := \frac{W_n^i}{W^i} - 1.$$

Let $\eta > 0$ small, and $K \in (0, 1)$. Then, taking the L^1-norm under the changed measure $\mathbb{P}_s^{e_i}$ in (5.4.2), we get that for all $n \geqslant 0$, $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} \mathbb{E}_s^{e_i}\Big| \log W_n^i - \log W^i \Big| \\ = \mathbb{E}_s^{e_i}\Big| \log(1 + R_n^i)\mathbb{1}_{\{R_n^i \geqslant -K\}} \Big| + \mathbb{E}_s^{e_i}\Big| \log(1 + R_n^i)\mathbb{1}_{\{R_n^i < -K\}} \Big|. \end{aligned} \tag{5.4.3}$$

Let $\varepsilon \in (0,1]$ be small enough. Notice that $x \mapsto x^{-\varepsilon}\log(1+x)$ is a bounded function on $[-K,+\infty)$. So, for all $n \geqslant 0$, $s \in (-\eta,\eta)$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}_s^{e_i}|\log(1+R_n^i)\mathbb{1}_{\{R_n^i \geqslant -K\}}| \leqslant C\mathbb{E}_s^{e_i}|R_n^i|^{\varepsilon}. \tag{5.4.4}$$

On the other hand, by Theorem 5.3.3 there exist constants $\eta > 0$ and $a > 0$ such that $\mathbb{E}_s^{e_i}(W^i)^{-a} \leqslant C$ for any $1 \leqslant i \leqslant d$ and uniformly in $s \in (-\eta,\eta)$. Therefore, using **K2** and Lemma 5.3.4 with the convex function $x \mapsto x^{-a}$, we obtain that for all $s \in (-\eta,\eta)$ and $1 \leqslant i \leqslant d$,

$$\sup_{n\geqslant 0} \mathbb{E}_s^{e_i}(W_n^i)^{-a} \leqslant \mathbb{E}_s^{e_i}\left[\sup_{n\geqslant 0} \mathbb{E}_{\xi}(W_n^i)^{-a}\right] = \mathbb{E}_s^{e_i}(W^i)^{-a} \leqslant C. \tag{5.4.5}$$

We know that $|\log x|^2 \leqslant C(x + x^{-a})$ for all $x > 0$. So from (5.4.2), Fatou's lemma and (5.4.5), we get that for all $s \in (-\eta,\eta)$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\sup_{n\geqslant 0}\left(\mathbb{E}_s^{e_i}|\log(1+R_n^i)|^2\right)^{\frac{1}{2}} &\leqslant \sup_{n\geqslant 0}\left(\mathbb{E}_s^{e_i}|\log W_n^i|^2\right)^{\frac{1}{2}} + \left(\mathbb{E}_s^{e_i}|\log W^i|^2\right)^{\frac{1}{2}} \\ &\leqslant 2\sup_{n\geqslant 0}\left(\mathbb{E}_s^{e_i}|\log W_n^i|^2\right)^{\frac{1}{2}} \\ &\leqslant C\sup_{n\geqslant 0}\left(\mathbb{E}_s^{e_i}W_n^i + \mathbb{E}_s^{e_i}(W_n^i)^{-a}\right)^{\frac{1}{2}} \\ &\leqslant C. \end{aligned}\tag{5.4.6}$$

Therefore, by Cauchy-Schwarz's inequality, (5.4.6) and Markov's inequality, we obtain that for all $n \geqslant 0$, $s \in (-\eta,\eta)$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\mathbb{E}_s^{e_i}|\log(1+R_n^i)\mathbb{1}_{\{R_n^i<-K\}}| &\leqslant \left(\mathbb{E}_s^{e_i}|\log(1+R_n^i)|^2\right)^{\frac{1}{2}}\left(\mathbb{E}_s^{e_i}\mathbb{1}_{\{R_n^i<-K\}}\right)^{\frac{1}{2}} \\ &\leqslant \left[\sup_{k\geqslant 0}\left(\mathbb{E}_s^{e_i}|\log(1+R_k^i)|^2\right)^{\frac{1}{2}}\right]\mathbb{P}_s^{e_i}(|R_n^i| > K)^{\frac{1}{2}} \\ &\leqslant C(\mathbb{E}_s^{e_i}|R_n^i|^{\varepsilon})^{\frac{1}{2}}. \end{aligned}\tag{5.4.7}$$

Putting together the relations (5.4.3), (5.4.4) and (5.4.7), we get that for all $n \geqslant 0$, $s \in (-\eta,\eta)$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}_s^{e_i}|\log W_n^i - \log W^i| \leqslant C\mathbb{E}_s^{e_i}|R_n^i|^{\varepsilon} + C\Big(\mathbb{E}_s^{e_i}|R_n^i|^{\varepsilon}\Big)^{\frac{1}{2}}. \tag{5.4.8}$$

By the definition of R_n^i and Cauchy-Schwarz's inequality, for all $n \geqslant 0$, $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$, we have

$$\mathbb{E}_s^{e_i}|R_n^i|^{\varepsilon} = \mathbb{E}_s^{e_i}\Big[(W^i)^{-\varepsilon}|W_n^i - W^i|^{\varepsilon}\Big] \leqslant \Big(\mathbb{E}_s^{e_i}(W^i)^{-2\varepsilon}\Big)^{\frac{1}{2}}\Big(\mathbb{E}_s^{e_i}|W_n^i - W^i|^{2\varepsilon}\Big)^{\frac{1}{2}}.$$

Therefore, by (5.4.5) and (5.3.12) in Proposition 5.3.6, for $\varepsilon > 0$ and $\eta > 0$ small enough, there exists a constant $\delta \in (0,1)$ such that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}|R_n^i|^{\varepsilon} \leqslant C\delta^{2n}. \tag{5.4.9}$$

Combining (5.4.8) and (5.4.9), we obtain that for all $n \geqslant 0$ and $1 \leqslant i \leqslant d$,

$$\sup_{s\in(-\eta,\eta)} \mathbb{E}_s^{e_i}|\log W_n^i - \log W^i| \leqslant C\delta^n.$$

This concludes the proof of Lemma 5.4.2. □

Now we formulate the Berry-Esseen bound for $\log \|M_{0,n-1}^T y\|$ under the changed measure $\mathbb{P}_s^y$, for any $y \in \mathcal{S}$ and uniformly in $s \in (-\eta, \eta)$. This result was established by Xiao, Grama and Liu in [75, Theorem 5.1], and will play a crucial role in proving Theorem 5.4.1. Recall that $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \mathrm{e}^{-t^2/2}\,dt$ is the standard normal distribution function.

Lemma 5.4.3. *Assume conditions* ***K2*** *and* ***K3****. Then, for $\eta > 0$ small enough, there exists a constant $C > 0$ such that for all $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$,*

$$\sup_{s\in(-\eta,\eta)} \left|\mathbb{P}_s^y\left(\frac{\log \|M_{0,n-1}^T y\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) - \Phi(x)\right| \leqslant \frac{C}{\sqrt{n}}.$$

The next lemma gives, for any $1 \leqslant i \leqslant d$, a control of the joint law $\Big(\log \|Z_n^i\|, \log \|M_{0,n-1}(i,\cdot)\|\Big)$ under $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$.

Lemma 5.4.4. *Assume conditions* ***K1****,* ***K2****,* ***K3*** *and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exists a constant $C > 0$ such that for all $n \geqslant 1$, $s \in (-\eta, \eta)$ and $x \in \mathbb{R}$,*

$$\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x\right) \leqslant \frac{C}{\sqrt{n}}, \tag{5.4.10}$$

and

$$\mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \leqslant \frac{C}{\sqrt{n}}. \tag{5.4.11}$$

Proof. Since the proof of (5.4.11) is similar to that of (5.4.10), we will only prove (5.4.10).

Let $s \in (-\eta, \eta)$, where $\eta > 0$ is small enough such that (5.4.1) holds. For all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$, set

$$F_n^i(x) := \mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x\right).$$

As before, we write that $C > 0$ for a constant independent of s and n, which may differ from line to line. For $0 \leqslant m < n$, $y \in \mathcal{S}$ and $1 \leqslant i \leqslant d$, set

$$S_{m,n}^y := \frac{\log\|M_{m,n-1}^T y\| - (n-m)\Lambda'(s)}{\sigma_s\sqrt{n}} \quad \text{and} \quad L_{m,n}^i := \frac{\log W_m^i}{\sigma_s\sqrt{n}}. \tag{5.4.12}$$

By (5.3.34), for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$, we have

$$F_n^i(x) \leqslant \mathbb{P}_s^{e_i}\left(S_{0,n}^{e_i} + L_{n,n}^i + \min_{1\leqslant r\leqslant d}\frac{\log U_{n,\infty}(r)}{\sigma_s\sqrt{n}} \leqslant x, S_{0,n}^{e_i} > x\right). \tag{5.4.13}$$

Set $m := m(n) = \lfloor\sqrt{n}\rfloor$, where $\lfloor x\rfloor$ is the integer part of x. By Markov's inequality and Lemma 5.4.2, for $\eta > 0$ small enough, there exists a constant $\delta \in (0,1)$ such that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
& \mathbb{P}_s^{e_i}\left(|L_{n,n}^i - L_{m,n}^i| > \frac{1}{\sqrt{n}}\right) \\
\leqslant\; & \sqrt{n}\mathbb{E}_s^{e_i}\left|L_{n,n}^i - L_{m,n}^i\right| \\
=\; & \frac{1}{\sigma_s}\mathbb{E}_s^{e_i}|\log W_n^i - \log W_m^i| \\
\leqslant\; & \frac{1}{\sigma_s}\left(\mathbb{E}_s^{e_i}|\log W_n^i - \log W^i| + \mathbb{E}_s^{e_i}|\log W_m^i - \log W^i|\right) \\
\leqslant\; & \frac{C}{\sigma_s}(\delta^n + \delta^m).
\end{aligned} \tag{5.4.14}$$

Notice that $\delta^n + \delta^m = o(\frac{1}{\sqrt{n}})$ as $n \to +\infty$. Combining this with (5.4.1) and (5.4.14), we get that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{P}_s^{e_i}\left(|L_{n,n}^i - L_{m,n}^i| > \frac{1}{\sqrt{n}}\right) \leqslant \frac{C}{\sqrt{n}}.$$

This, together with (5.4.13), implies that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} F_n^i(x) &\leqslant \mathbb{P}_s^{e_i}\left(S_{0,n}^{e_i} + L_{m,n}^i + \min_{1\leqslant r\leqslant d} \frac{\log U_{n,\infty}(r)}{\sigma_s\sqrt{n}} \leqslant x + \frac{1}{\sqrt{n}}, S_{0,n}^{e_i} > x\right) \\ &\quad + \mathbb{P}_s^{e_i}\left(|L_{n,n}^i - L_{m,n}^i| > \frac{1}{\sqrt{n}}\right) \\ &\leqslant \mathbb{P}_s^{e_i}\left(S_{0,n}^{e_i} + L_{m,n}^i + \min_{1\leqslant r\leqslant d} \frac{\log U_{n,\infty}(r)}{\sigma_s\sqrt{n}} \leqslant x + \frac{1}{\sqrt{n}}, S_{0,n}^{e_i} > x\right) + \frac{C}{\sqrt{n}}. \end{aligned} \tag{5.4.15}$$

For any $n \geqslant 1$ and $1 \leqslant i \leqslant d$, we have the following decomposition:

$$\begin{aligned} S_{0,n}^{e_i} &= \frac{\log \|M_{m+1,n-1}^T(M_{0,m}^T e_i)\| - n\gamma}{\sigma_s\sqrt{n}} \\ &= \frac{\log \|M_{0,m}^T e_i\| + \log \|M_{m+1,n-1}^T(M_{0,m}^T \cdot e_i)\| - n\gamma}{\sigma_s\sqrt{n}} \\ &= \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}}. \end{aligned} \tag{5.4.16}$$

By (5.3.34) and (5.3.35), for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$ we have

$$\begin{cases} L_{m,n}^i \leqslant \frac{1}{\sigma_s\sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|} - \frac{1}{\sigma_s\sqrt{n}} \min_{1\leqslant r,j\leqslant d} \log \frac{M_m(r,j)}{\|M_m\|}, \\ L_{m,n}^i \geqslant \frac{1}{\sigma_s\sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|} + \frac{1}{\sigma_s\sqrt{n}} \min_{1\leqslant r,j\leqslant d} \log \frac{M_m(r,j)}{\|M_m\|}. \end{cases} \tag{5.4.17}$$

Therefore, combining the relations (5.4.15)-(5.4.17), we obtain that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} F_n^i(x) \leqslant \mathbb{P}_s^{e_i}\Bigg(&\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + B_{m,n}^i \leqslant x + \frac{1}{\sqrt{n}}, \\ &\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} > x\Bigg) + \frac{C}{\sqrt{n}}, \end{aligned} \tag{5.4.18}$$

where

$$B_{m,n}^i := \frac{1}{\sigma_s\sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|} + \frac{1}{\sigma_s\sqrt{n}} \min_{1\leqslant r,j\leqslant d} \log \frac{M_m(r,j)}{\|M_m\|} + \frac{1}{\sigma_s\sqrt{n}} \min_{1\leqslant r,j\leqslant d} \log \frac{M_n(r,j)}{\|M_n\|}. \tag{5.4.19}$$

Since $S_{0,m+1}^{e_i}$, $S_{m+1,n}^{X_{m+1}^{e_i}}$ and $B_{m,n}^i$ depend only on the environments $\xi_0, \xi_1, \cdots, \xi_n$ (but not on $\xi_{n+1}, \xi_{n+2}, \cdots$), by (5.4.18) and (5.3.4) we get that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$F_n^i(x) \leqslant \mathbb{E}\left[q_{n+1}^s(e_i, M_{0,n}^T)\mathbb{1}_{\left\{\sqrt{\frac{m+1}{n}}S_{0,m+1}^{e_i}+S_{m+1,n}^{X_{m+1}^{e_i}}+B_{m,n}^i\leqslant x+\frac{1}{\sqrt{n}},\sqrt{\frac{m+1}{n}}S_{0,m+1}^{e_i}+S_{m+1,n}^{X_{m+1}^{e_i}}>x\right\}}\right] + \frac{C}{\sqrt{n}}. \tag{5.4.20}$$

For all $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$, set

$$G_{m,n}^y(x) = \mathbb{P}_s^{e_i}(S_{m,n}^y \leqslant x).$$

For each $n \geqslant 1$, denote by h_n the function on $\mathcal{S} \times \mathbb{R} \times \mathbb{R}$ defined as follows: for all $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$h_n(y,z,t) := \mathbb{P}_s^y\left(S_{m+1,n}^y + z + t \leqslant \frac{1}{\sqrt{n}},\ S_{m+1,n}^y + z > 0\right). \tag{5.4.21}$$

Notice that $X_{m+1}^{e_i}$, $S_{0,m+1}^{e_i}$ and $B_{m,n}^i$ are independent of the environments $\xi_{m+1}\cdots\xi_{n-1}$, so they are independent of $S_{m+1,n}^y$ for any $y \in \mathcal{S}$. Therefore, by (5.4.20) and (5.3.3), we see that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
& F_n^i(x) \\
\leqslant\ & \mathbb{E}\Big[q_{m+1}^s(e_i, M_{0,m}^T) q_{n-m-1}^s(X_{m+1}^{e_i}, M_{m+1,n-1}^T) q_1^s(X_n^{e_i}, M_n^T) \\
& \mathbb{1}_{\left\{ \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + B_{m,n}^i \leqslant x + \frac{1}{\sqrt{n}}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} > x \right\}} \Big] + \frac{C}{\sqrt{n}} \\
\leqslant\ & \mathbb{E}\left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u \in \mathcal{S}} q_1^s(u, M_n^T) h_n\Big(X_{m+1}^{e_i}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} - x, B_{m,n}^i \Big) \right] \\
& + \frac{C}{\sqrt{n}}.
\end{aligned} \tag{5.4.22}$$

Now we give a bound of the function h_n. It is clear that for all $n \geqslant 1$, $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$0 \leqslant h_n(y, z, t) = \left[G_{m+1,n}^y\Big(\frac{1}{\sqrt{n}} - z - t \Big) - G_{m+1,n}^y(-z) \right] \mathbb{1}_{\left\{ t \leqslant \frac{1}{\sqrt{n}} \right\}}. \tag{5.4.23}$$

Since the matrices M_n, $n \geqslant 0$, are i.i.d., for all $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$ we have

$$\begin{aligned}
G_{m+1,n}^y(x) &= \mathbb{P}_s^y\left(\frac{\log \|M_{0,n-m-1}^T y\| - (n-m-1)\Lambda'(s)}{\sigma_s \sqrt{n}} \leqslant x \right) \\
&= G_{0,n-m-1}^y(a_n x),
\end{aligned}$$

where $a_n = \sqrt{\frac{n}{n-m-1}}$. It is clear that $a_n = (1 - \frac{m+1}{n})^{-1/2} = 1 + O(\frac{m}{n}) = 1 + O(\frac{1}{\sqrt{n}})$ as $n \to +\infty$. Therefore, using Lemma 5.4.3 we obtain that, for $\eta > 0$ small enough and all $n \geqslant 1$, $y \in \mathcal{S}$ and $x \in \mathbb{R}$,

$$\begin{aligned}
\left| G_{m+1,n}^y(x) - \Phi(a_n x) \right| &= \left| G_{0,n-m-1}^y(a_n x) - \Phi(a_n x) \right| \\
&\leqslant \frac{C}{\sqrt{n-m-1}} \\
&= \frac{C a_n}{\sqrt{n}} \leqslant \frac{C}{\sqrt{n}}.
\end{aligned} \tag{5.4.24}$$

Moreover, applying the mean value theorem on $t \mapsto \Phi(tx)$, for all $n \geqslant 1$ and $x \in \mathbb{R}$, we have

$$
\begin{aligned}
|\Phi(a_n x) - \Phi(x)| &\leqslant |a_n - 1| \sup_{t \geqslant 1} |x \Phi'(tx)| \\
&\leqslant \frac{C}{\sqrt{n}} \sup_{z \in \mathbb{R}} |z \Phi'(z)| \\
&\leqslant \frac{C}{\sqrt{n}}, \qquad (5.4.25)
\end{aligned}
$$

where we have used the fact that $z \mapsto |z\Phi'(z)|$ is a bounded function on $\mathbb{R}$. Combining the relations (5.4.23)-(5.4.25), we get that for all $n \geqslant 1$, $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$
h_n(y, z, t) \leqslant \left| \Phi\left(\frac{1}{\sqrt{n}} - z - t \right) - \Phi(-z) \right| \mathbb{1}_{\left\{ t \leqslant \frac{1}{\sqrt{n}} \right\}} + \frac{C}{\sqrt{n}}. \qquad (5.4.26)
$$

Using again the mean value theorem, since $\sup_{x \in \mathbb{R}} |\Phi'(x)| \leqslant 1$, for all $x, z \in \mathbb{R}$ we have

$$
|\Phi(x + z) - \Phi(x)| \leqslant |z|. \qquad (5.4.27)
$$

This, together with (5.4.26), implies that for all $n \geqslant 1$, $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$
h_n(y, z, t) \leqslant \left| \frac{1}{\sqrt{n}} + t \right| + \frac{C}{\sqrt{n}} \leqslant |t| + \frac{C}{\sqrt{n}}. \qquad (5.4.28)
$$

By (5.4.22) and (5.4.28), we obtain that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$
\begin{aligned}
F_n^i(x) &\leqslant \mathbb{E}\left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u \in \mathcal{S}} q_1^s(u, M_n^T) \left(\left| B_{m,n}^i \right| + \frac{C}{\sqrt{n}} \right) \right] + \frac{C}{\sqrt{n}} \\
&= \mathbb{E}\left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u \in \mathcal{S}} q_1^s(u, M_n^T) \left| B_{m,n}^i \right| \right] \\
&\quad + \frac{1}{\sqrt{n}} \mathbb{E}\left[\sup_{u \in \mathcal{S}} q_1^s(u, M_0^T) \right] + \frac{C}{\sqrt{n}}. \qquad (5.4.29)
\end{aligned}
$$

Then, combining (5.4.29), (5.3.19) and the definition of $B_{m,n}^i$ (see (5.4.19)), we get that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
F_n^i(x) \leqslant &\frac{C}{\sigma_s\sqrt{n}}\mathbb{E}\left[q_{m+1}^s(e_i, M_{0,m}^T)\|M_n\|^\eta\left|\log\frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|}\right|\right] \\
&+\frac{C}{\sigma_s\sqrt{n}}\mathbb{E}\left[q_{m+1}^s(e_i, M_{0,m}^T)\|M_n\|^\eta \max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_m(r,j)}{\|M_m\|}\right|\right] \\
&+\frac{C}{\sigma_s\sqrt{n}}\mathbb{E}\left[q_{m+1}^s(e_i, M_{0,m}^T)\|M_n\|^\eta \max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_n(r,j)}{\|M_n\|}\right|\right] \\
&+\frac{C}{\sqrt{n}}.
\end{aligned}$$

By condition **K2**, (5.4.1) and (5.4.17), this implies that for all $n \geqslant 1$, $x \in \mathbb{R}$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
F_n^i(x) &\leqslant \frac{C}{\sqrt{n}}\mathbb{E}_s^{e_i}\left|\log\frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|}\right| + \frac{C}{\sqrt{n}}\mathbb{E}_s^{e_i}\left[\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_m(r,j)}{\|M_m\|}\right|\right] \\
&\quad+\frac{C}{\sqrt{n}}\mathbb{E}\left[\|M_0\|^\eta\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_0(r,j)}{\|M_0\|}\right|\right]+\frac{C}{\sqrt{n}} \\
&\leqslant \frac{C}{\sqrt{n}}\mathbb{E}_s^{e_i}\left|\log W_m^i\right| + \frac{C}{\sqrt{n}}\mathbb{E}_s^{e_i}\left[\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_m(r,j)}{\|M_m\|}\right|\right] \\
&\quad+\frac{C}{\sqrt{n}}\mathbb{E}\left[\|M_0\|^\eta\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_0(r,j)}{\|M_0\|}\right|\right]+\frac{C}{\sqrt{n}}. \qquad (5.4.30)
\end{aligned}$$

Now we give a bound of the three expectations in (5.4.30). First, by Lemma 5.4.2 we get that, for $\eta > 0$ small enough and all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}_s^{e_i}\left|\log W_m^i\right| \leqslant C. \qquad (5.4.31)$$

Next, using (5.3.3) and (5.3.19), we obtain that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\mathbb{E}_s^{e_i}\left[\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_m(r,j)}{\|M_m\|}\right|\right] \\
=\ &\mathbb{E}\left[q_m^s(e_i, M_{0,m-1}^T)q_1^s(M_{0,m-1}^T\cdot e_i, M_m^T)\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_m(r,j)}{\|M_m\|}\right|\right] \\
\leqslant\ &C\mathbb{E}\left[q_m^s(e_i, M_{0,m-1}^T)\|M_m\|^\eta\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_m(r,j)}{\|M_m\|}\right|\right] \\
=\ &C\mathbb{E}\left[\|M_0\|^\eta\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_0(r,j)}{\|M_0\|}\right|\right]. \qquad (5.4.32)
\end{aligned}$$

Then, by the inequality $|\log x| \leqslant C(x^\eta + x^{-\eta})$ for $x > 0$ and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\|M_0\|^\eta \max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_0(r,j)}{\|M_0\|}\right|\right] \\
\leqslant\ & C\mathbb{E}\left[\max_{1\leqslant r,j\leqslant d} M_0(r,j)^\eta\right] + C\mathbb{E}\left[\|M_0\|^{2\eta}\max_{1\leqslant r,j\leqslant d} M_0(r,j)^{-\eta}\right] \\
\leqslant\ & C\mathbb{E}\|M_0\|^\eta + C\Big(\mathbb{E}\|M_0\|^{4\eta}\Big)^{\frac{1}{2}}\Big(\mathbb{E}\max_{1\leqslant r,j\leqslant d} M_0(r,j)^{-2\eta}\Big)^{\frac{1}{2}}. \qquad (5.4.33)
\end{aligned}$$

Taking $\eta > 0$ small enough, by **K2** it follows that

$$\mathbb{E}\left[\|M_0\|^\eta \max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_0(r,j)}{\|M_0\|}\right|\right] \leqslant C. \qquad (5.4.34)$$

Therefore, combining the ineqalities (5.4.30)-(5.4.34), we get (5.4.10). This concludes the proof of Lemma 5.4.4. □

Now we shall prove Theorem 5.4.1.

Proof of Theorem 5.4.1. Let $\eta > 0$ be sufficiently small such that (5.4.1) holds. For all $n \geqslant 1$, $s \in (-\eta, \eta)$, $x > 0$ and $1 \leqslant i \leqslant d$, we have

$$\begin{aligned}
&\mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\
=\ & \mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\
&+ \mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x\right) \\
=\ & \mathbb{P}_s^{e_i}\left(\frac{\log\|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\
&- \mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\
&+ \mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x, \frac{\log\|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x\right).
\end{aligned}$$

Therefore, using the Berry-Essen bound in lemma 5.4.3 and Lemma 5.4.4, by taking $\eta > 0$

small enough, we get that for all $n \geqslant 1$, $s \in (-\eta, \eta)$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} & \left|\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) - \Phi(x)\right| \\ \leqslant & \left|\mathbb{P}_s^{e_i}\left(\frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) - \Phi(x)\right| \\ & + \mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\ & + \mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x\right) \\ \leqslant & \frac{C}{\sqrt{n}}. \end{aligned}$$

This concludes the proof of Theorem 5.4.1. □

5.5 Proof of Cramér's moderate deviation expansion

In this section, we prove Theorem 5.2.1. The proof is based on a control of the joint law of $\left(\log \|Z_n^i\|, \log \|M_{0,n-1}(i,\cdot)\|\right)$ under $\mathbb{P}_s^{e_i}$, uniformly in $s \in (-\eta, \eta)$. We already have a control in Lemma 5.4.4. Unfortunately this is not sufficient, and we need additional information. For $0 < x < y$, set $\Phi([x,y]) = \Phi(y) - \Phi(x)$. The first result below about the convergence to the normal distribution is a consequence of Theorem 5.4.1 and Lemma 5.4.4.

Lemma 5.5.1. *Assume conditions* ***K1****,* ***K2****,* ***K3*** *and $\gamma > 0$. Then, for $\eta > 0$ small enough, there exists a constant $C > 0$ such that for all $n \geqslant 1$, $s \in (-\eta, \eta)$, $x > 0$ and $1 \leqslant i \leqslant d$,*

$$\left|\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) - \Phi([0,x])\right| \leqslant \frac{C}{\sqrt{n}}, \qquad (5.5.1)$$

and

$$\left|\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} < 0, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \geqslant -x\right)\right.$$
$$\left. - \Phi([-x,0])\right| \leqslant \frac{C}{\sqrt{n}}. \quad (5.5.2)$$

Proof. Let $\eta > 0$ be small enough such that (5.4.1) holds. For all $n \geqslant 1$, $s \in (-\eta,\eta)$, $x > 0$ and $1 \leqslant i \leqslant d$, we have

$$\begin{aligned}
&\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\
= \ &\mathbb{P}_s^{e_i}\left(0 < \frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\
&+\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\
= \ &\mathbb{P}_s^{e_i}\left(0 < \frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\
&-\mathbb{P}_s^{e_i}\left(0 < \frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x\right) \\
&+\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right).
\end{aligned}$$

Then, applying the Berry-Essen bound in Theorem 5.4.1 and the inequalities in Lemma 5.4.4, when $\eta > 0$ is sufficiently small, we obtain that for all $n \geqslant 1$, $s \in (-\eta,\eta)$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\left|\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) - \Phi([0,x])\right| \\
&\quad\leqslant \left|\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) - \Phi([0,x])\right| \\
&\quad+ \mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x\right) \\
&\quad+ \mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > x, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant x\right) \\
&\quad\leqslant \frac{C}{\sqrt{n}}.
\end{aligned}$$

Therefore, (5.5.1) holds. It is clear that (5.5.2) can be proved by similar calculations. This conclude the proof of Lemma 5.5.1. □

The second result gives a control of the probabilities in Lemma 5.5.1 when $x < 0$, uniformly in $s \in (-\eta, \eta)$.

Lemma 5.5.2. *Assume conditions* ***K1***, ***K2***, ***K3*** *and* $\gamma > 0$. *Then, for* $\eta > 0$ *small enough, there exist constants* $C > 0$, $\alpha > 0$, $\beta > 0$ *and* $\delta \in (0,1)$ *such that for all* $n \geqslant 1$, $s \in (-\eta, \eta)$, $x > 0$ *and* $1 \leqslant i \leqslant d$,

$$\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant -x\right) \leqslant \frac{C}{\sqrt{n}} \mathrm{e}^{-\alpha x\sqrt{n}} + C \min\left\{ \mathrm{e}^{-\beta x\sqrt{n}}, x^{-\frac{1}{2}} n^{-\frac{1}{4}} \delta^{\sqrt{n}} \right\}, \quad (5.5.3)$$

and

$$\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} < 0, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \geqslant x\right) \leqslant \frac{C}{\sqrt{n}} \mathrm{e}^{-\alpha x\sqrt{n}} + C \min\left\{ \mathrm{e}^{-\beta x\sqrt{n}}, x^{-\frac{1}{2}} n^{-\frac{1}{4}} \delta^{\sqrt{n}} \right\}. \quad (5.5.4)$$

Proof. We only prove (5.5.3), since the second assertion (5.5.4) can be proved in the same way.

We use the same notation as in the proof of Lemma 5.5.1. Let $\eta > 0$ be small enough such that (5.4.1) holds. Let $s \in (-\eta, \eta)$. As before, $C > 0$ will be a constant independent of s and n, which may differ from line to line.

By (5.3.34), we get that for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
&\mathbb{P}_s^{e_i}\left(\frac{\log \|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > 0, \frac{\log \|M_{0,n-1}(i,\cdot)\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} \leqslant -x\right) \\
\leqslant\ & \mathbb{P}_s^{e_i}\left(S_{0,n}^{e_i} + L_{n,n}^i - \min_{1\leqslant r\leqslant d} \frac{\log U_{n,\infty}(r)}{\sigma_s\sqrt{n}} > 0,\ S_{0,n} \leqslant -x\right) \\
\leqslant\ & \mathbb{P}_s^{e_i}\left(S_{0,n}^{e_i} + L_{m,n}^i - \min_{1\leqslant r\leqslant d} \frac{\log U_{n,\infty}(r)}{\sigma_s\sqrt{n}} > -\frac{x}{2},\ S_{0,n} \leqslant -x\right) \\
&+ \mathbb{P}_s^{e_i}\left(|L_{n,n}^i - L_{m,n}^i| > \frac{x}{2}\right) \\
=:\ & A_1^i(x,n) + A_2^i(x,n). \qquad (5.5.5)
\end{aligned}$$

Now, we give a bound for the two terms $A_1^i(x,n)$ and $A_2^i(x,n)$.

Control of $A_1^i(x,n)$. Using the relations (5.4.16) and (5.4.17), we obtain that for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} &A_1^i(x,n) \\ &\leqslant \mathbb{P}_s^{e_i}\left(\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + K_{m,n}^i > -\frac{x}{2}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} \leqslant -x\right), \end{aligned}$$

where

$$\begin{aligned} K_{m,n}^i := \frac{1}{\sigma_s\sqrt{n}} \log \frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|} - \frac{1}{\sigma_s\sqrt{n}} \min_{1\leqslant r,j\leqslant d} \log \frac{M_m(r,j)}{\|M_m\|} \\ - \frac{1}{\sigma_s\sqrt{n}} \min_{1\leqslant r,j\leqslant d} \log \frac{M_n(r,j)}{\|M_n\|}. \end{aligned}$$

For each $n \geqslant 1$ and $x > 0$, let $h_{n,x}$ be the function on $\mathcal{S} \times \mathbb{R} \times \mathbb{R}$ defined by: for all $y \in \mathcal{S}$, $z \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$h_{n,x}(y,z,t) := \mathbb{P}_s^y\Big(S_{m+1,n}^y + z + t > -\frac{x}{2}, S_{m+1,n}^y + z \leqslant -x\Big). \tag{5.5.6}$$

By an argument similar to the proof of (5.4.20) and (5.4.22), we obtain that for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} & A_1^i(x,n) \\ \leqslant\; & \mathbb{E}\Big[q_{m+1}^s(e_i, M_{0,m}^T) q_{n-m-1}^s(X_{m+1}^{e_i}, M_{m+1,n-1}^T) q_1^s(X_n^{e_i}, M_n^T) \\ & \mathbb{1}_{\left\{\sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} + K_{m,n}^i > -\frac{x}{2}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i} + S_{m+1,n}^{X_{m+1}^{e_i}} \leqslant -x\right\}}\Big] \\ \leqslant\; & \mathbb{E}\left[q_{m+1}^s(e_i, M_{0,m}^T) \sup_{u\in\mathcal{S}} q_1^s(u, M_n^T) h_{n,x}\left(X_{m+1}^{e_i}, \sqrt{\frac{m+1}{n}} S_{0,m+1}^{e_i}, K_{m,n}^i\right)\right]. \end{aligned} \tag{5.5.7}$$

Using (5.4.24), (5.4.25) and (5.4.27), we have that for all $n \geqslant 1$, $x > 0$, $y \in \mathcal{S}$, $z \in \mathbb{R}$, and

$t \in \mathbb{R}$,

$$\begin{aligned}
0 \leqslant h_{n,x}(y,z,t) &= \Big[G^y_{m+1,n}(-x-z) - G^y_{m+1,n}\Big(-\frac{x}{2} - z - t\Big)\Big] \mathbb{1}_{\{t \geqslant \frac{x}{2}\}} \\
&\leqslant \Big(\Big| t - \frac{x}{2} \Big| + \frac{C}{\sqrt{n}} \Big) \mathbb{1}_{\{t \geqslant \frac{x}{2}\}} \\
&\leqslant \Big(t + \frac{C}{\sqrt{n}} \Big) \mathbb{1}_{\{t \geqslant \frac{x}{2}\}}.
\end{aligned}$$

Combining this with (5.5.7), we get that for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$A_1^i(x,n) \leqslant \mathbb{E}\Big[q^s_{m+1}(e_i, M^T_{0,m}) \sup_{u \in \mathcal{S}} q^s_1(u, M^T_n) \Big(K^i_{m,n} + \frac{C}{\sqrt{n}} \Big) \mathbb{1}_{\{K^i_{m,n} \geqslant \frac{x}{2}\}} \Big]. \tag{5.5.8}$$

We will slightly change the expression of the above expectation in order to facilitate the passage to the expectation with respect to the new measure $\mathbb{P}^{e_i}_s$. For any $n \geqslant 1$ and $1 \leqslant i \leqslant d$, set

$$\begin{aligned}
\tilde{K}^i_{m,n} := &\frac{1}{\sigma_s \sqrt{n}} \log \frac{\|Z^i_m\|}{\|M_{0,m-1}(i,\cdot)\|} - \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leqslant r,j \leqslant d} \log \frac{M_m(r,j)}{\|M_m\|} \\
&- \frac{1}{\sigma_s \sqrt{n}} \min_{1 \leqslant r,j \leqslant d} \log \frac{M_{m+1}(r,j)}{\|M_{m+1}\|}.
\end{aligned}$$

Notice that the expectation in (5.5.8) remains the same if the environment ξ_n is replaced by ξ_{m+1} due to the independence structure. So in (5.5.8) we can replace $(M_n, K^i_{m,n})$ by $(M_{m+1}, \tilde{K}^i_{m,n})$. This, together with (5.3.18), yields that for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
A_1^i(x,n) &\leqslant \mathbb{E}\Big[q^s_{m+1}(e_i, M^T_{0,m}) \sup_{u \in \mathcal{S}} q^s_1(u, M^T_{m+1}) \Big(\tilde{K}^i_{m,n} + \frac{C}{\sqrt{n}} \Big) \mathbb{1}_{\{\tilde{K}^i_{m,n} \geqslant \frac{x}{2}\}} \Big] \\
&\leqslant D^2 \mathbb{E}\Big[q^s_{m+1}(e_i, M^T_{0,m}) q^s_1(X^{e_i}_{m+1}, M^T_{m+1}) \Big(\tilde{K}^i_{m,n} + \frac{C}{\sqrt{n}} \Big) \mathbb{1}_{\{\tilde{K}^i_{m,n} \geqslant \frac{x}{2}\}} \Big] \\
&\leqslant C \mathbb{E}^{e_i}_s \Big[\tilde{K}^i_{m,n} \mathbb{1}_{\{\tilde{K}^i_{m,n} \geqslant \frac{x}{2}\}} \Big] + \frac{C}{\sqrt{n}} \mathbb{P}^{e_i}_s \Big(\tilde{K}^i_{m,n} \geqslant \frac{x}{2} \Big).
\end{aligned} \tag{5.5.9}$$

Let $\varepsilon \in (0,1)$ be arbitrary fixed. By Markov's inequality and (5.3.19), for all $n \geqslant 1$, $x > 0$

and $1 \leqslant i \leqslant d$ we have

$$\begin{aligned}
&\mathbb{P}_s^{e_i}\left(\tilde{K}_{m,n}^i \geqslant \frac{x}{2}\right) \\
= \;& \mathbb{P}_s^{e_i}\left(\mathrm{e}^{\varepsilon\sigma_s\sqrt{n}\tilde{K}_{m,n}^i} \geqslant \mathrm{e}^{\frac{\varepsilon\sigma_s}{2}x\sqrt{n}} \right) \\
\leqslant \;& \mathrm{e}^{-\frac{\varepsilon\sigma_s}{2}x\sqrt{n}}\,\mathbb{E}\Big[q_m^s(e_i, M_{0,m-1}^T) q_1^s(X_m^{e_i}, M_m^T) q_1^s(X_{m+1}^{e_i}, M_{m+1}^T)\,\mathrm{e}^{\varepsilon\sigma_s\sqrt{n}\tilde{K}_{m,n}^i}\Big] \\
\leqslant \;& C\,\mathrm{e}^{-\frac{\varepsilon\sigma_s}{2}x\sqrt{n}}\,\mathbb{E}\left[q_m^s(e_i, M_{0,m-1}^T)\left(\frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|}\right)^{\varepsilon}\right] \times \\
& \mathbb{E}\left[\frac{\|M_m\|^{\eta+\varepsilon}}{\min_{1\leqslant r,j\leqslant d} M_m(r,j)^{\varepsilon}}\right]\mathbb{E}\left[\frac{\|M_{m+1}\|^{\eta+\varepsilon}}{\min_{1\leqslant r,j\leqslant d} M_{m+1}(r,j)^{\varepsilon}}\right] \\
\leqslant \;& C\,\mathrm{e}^{-\frac{\varepsilon\sigma_s}{2}x\sqrt{n}}\,\mathbb{E}_s^{e_i}\left(\frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|}\right)^{\varepsilon}\left(\mathbb{E}\Big[\|M_0\|^{\eta+\varepsilon}\min_{1\leqslant r,j\leqslant d} M_0(r,j)^{-\varepsilon}\Big]\right)^2.
\end{aligned} \tag{5.5.10}$$

Notice that $\mathbb{E}_s^{e_i}\Big[\frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|}\Big] = 1$. Therefore, using Jensen's and Cauchy-Shwartz's inequalities and condition **K2**, by taking $\eta > 0$ and $\varepsilon \in (0,1)$ sufficiently small, we get from (5.5.10) that for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}
\mathbb{P}_s^{e_i}\left(\tilde{K}_{m,n}^i \geqslant \frac{x}{2}\right) &\leqslant C\,\mathrm{e}^{-\frac{\varepsilon\sigma_s}{2}x\sqrt{n}}\left(\mathbb{E}_s^{e_i}\left[\frac{\|Z_m^i\|}{\|M_{0,m-1}(i,\cdot)\|}\right]\right)^{\varepsilon} \times \\
&\quad \left(\mathbb{E}\|M_0\|^{2(\eta+\varepsilon)}\right)^{\frac{1}{2}}\left(\mathbb{E}\Big[\min_{1\leqslant r,j\leqslant d} M_0(r,j)^{-2\varepsilon}\Big]\right)^{\frac{1}{2}} \\
&\leqslant C\,\mathrm{e}^{-\frac{\varepsilon\sigma_s}{2}x\sqrt{n}}.
\end{aligned} \tag{5.5.11}$$

Then, using again Cauchy-Shwartz's inequality and (5.5.11), for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$ we have

$$\begin{aligned}
\mathbb{E}_s^{e_i}\Big[\tilde{K}_{m,n}^i \mathbb{1}_{\{\tilde{K}_{m,n}^i \geqslant \frac{x}{2}\}}\Big] &\leqslant \left(\mathbb{E}_s^{e_i}\Big|\tilde{K}_{m,n}^i\Big|^2\right)^{\frac{1}{2}}\mathbb{P}_s^{e_i}\left(\tilde{K}_{m,n}^i \geqslant \frac{x}{2}\right)^{\frac{1}{2}} \\
&\leqslant C\,\mathrm{e}^{-\frac{\varepsilon\sigma_s}{4}x\sqrt{n}}\left(\mathbb{E}_s^{e_i}\Big|\tilde{K}_{m,n}^i\Big|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.5.12}$$

Then, by the triangular inequality in L^2 under $\mathbb{P}_s^{e_i}$ and (5.4.17), we obtain that for all

$n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\left(\mathbb{E}_s^{e_i}\left|\tilde{K}_{m,n}^i\right|^2\right)^{\frac{1}{2}} \leqslant \frac{1}{\sigma_s\sqrt{n}}\left[\left(\mathbb{E}_s^{e_i}\left|\log W_m^i\right|^2\right)^{\frac{1}{2}} + 2\left(\mathbb{E}_s^{e_i}\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_m(r,j)}{\|M_m\|}\right|^2\right)^{\frac{1}{2}} + \left(\mathbb{E}_s^{e_i}\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_{m+1}(r,j)}{\|M_{m+1}\|}\right|^2\right)^{\frac{1}{2}}\right]. \tag{5.5.13}$$

Notice that we have proved in (5.4.6) that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\left(\mathbb{E}_s^{e_i}\left|\log W_n^i\right|^2\right)^{\frac{1}{2}} \leqslant C. \tag{5.5.14}$$

By an argument similar to the proof of (5.4.32)-(5.4.34) with the inequality $|\log x|^2 \leqslant C(x^\eta + x^{-\eta})$ for $x > 0$, we get that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\left(\mathbb{E}_s^{e_i}\max_{1\leqslant r,j\leqslant d}\left|\log\frac{M_n(r,j)}{\|M_n\|}\right|^2\right)^{\frac{1}{2}} \leqslant C. \tag{5.5.15}$$

Therefore, combining the inequalities (5.5.12)-(5.5.15) and (5.4.1), we see that for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\mathbb{E}_s^{e_i}\left[\tilde{K}_{m,n}^i \mathbb{1}_{\{\tilde{K}_{m,n}^i \geqslant \frac{x}{2}\}}\right] \leqslant \frac{C}{\sqrt{n}}\,\mathrm{e}^{-\frac{\varepsilon\sigma_s}{4}x\sqrt{n}}. \tag{5.5.16}$$

Putting together (5.5.9), (5.5.11), (5.5.16) and (5.4.1), we obtain that, with the real $\alpha = \frac{\varepsilon}{4}\inf_{s\in(-\eta,\eta)}\sigma_s > 0$, for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$A_1^i(x,n) \leqslant \frac{C}{\sqrt{n}}\,\mathrm{e}^{-\alpha x\sqrt{n}}. \tag{5.5.17}$$

Control of $A_2^i(x,n)$. First, by Markov's and Jensen's inequalities, and using (5.4.1) and Lemma 5.4.2, we see that for $\eta > 0$ small enough, there exists a constant $\delta_0 \in (0,1)$

such that for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} A_2^i(x,n) &= \mathbb{P}_s^{e_i}\left(|\log W_n^i - \log W_m^i|^{\frac{1}{2}} > \left(\frac{\sigma_s}{2}\right)^{\frac{1}{2}} x^{\frac{1}{2}} n^{\frac{1}{4}} \right) \\ &\leqslant \left(\frac{\sigma_s}{2}\right)^{-\frac{1}{2}} x^{-\frac{1}{2}} n^{-\frac{1}{4}} \Big(\mathbb{E}_s^{e_i} |\log W_n^i - \log W_m^i| \Big)^{\frac{1}{2}} \\ &\leqslant C x^{-\frac{1}{2}} n^{-\frac{1}{4}} \Big(\mathbb{E}_s^{e_i} |\log W_n^i - \log W^i| + \mathbb{E}_s^{e_i} |\log W_m^i - \log W^i| \Big)^{\frac{1}{2}} \\ &\leqslant C x^{-\frac{1}{2}} n^{-\frac{1}{4}} \delta_0^{\frac{m}{2}}. \end{aligned}$$

Taking $\delta = \delta_0^{\frac{1}{2}} \in (0,1)$, since $m = \lfloor \sqrt{n} \rfloor$, it follows that for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$A_2^i(x,n) \leqslant C x^{-\frac{1}{2}} n^{-\frac{1}{4}} \delta^{\sqrt{n}}. \tag{5.5.18}$$

On the other hand, by Markov's inequality we have that for any $a \in (0,1)$, and for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} A_2^i(x,n) &= \mathbb{P}_s^{e_i}\left(\mathrm{e}^{a|\log W_n^i - \log W_m^i|} > \mathrm{e}^{\frac{a\sigma_s}{2} x\sqrt{n}} \right) \\ &\leqslant \mathrm{e}^{-\frac{a\sigma_s}{2} x\sqrt{n}}\, \mathbb{E}_s^{e_i}\left(\mathrm{e}^{a|\log W_n^i - \log W_m^i|} \right) \\ &\leqslant \mathrm{e}^{-\frac{a\sigma_s}{2} x\sqrt{n}} \left[\mathbb{E}_s^{e_i}\left(\frac{W_n^i}{W_m^i}\right)^a + \mathbb{E}_s^{e_i}\left(\frac{W_m^i}{W_n^i}\right)^a \right]. \end{aligned} \tag{5.5.19}$$

By Cauchy-Schwarz's and Jensen's inequalities, and using (5.4.5), for $\eta > 0$ and $a \in \left(0, \frac{1}{2}\right)$ small enough we get that for all $n, k \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} \mathbb{E}_s^{e_i}\left(\frac{W_n^i}{W_k^i}\right)^a &\leqslant \Big(\mathbb{E}_s^{e_i}(W_n^i)^{2a}\Big)^{\frac{1}{2}} \Big(\mathbb{E}_s^{e_i}(W_k^i)^{-2a}\Big)^{\frac{1}{2}} \\ &\leqslant \Big(\mathbb{E}_s^{e_i} W_n^i\Big)^a \Big(\mathbb{E}_s^{e_i}(W^i)^{-2a}\Big)^{\frac{1}{2}} \\ &\leqslant C. \end{aligned}$$

This, together with (5.5.19) and (5.4.1), implies that with $\beta = \frac{a}{2} \inf_{s\in(-\eta,\eta)} \sigma_s > 0$, for all $n \geqslant 1$, $x > 0$ and $1 \leqslant i \leqslant d$,

$$A_2^i(x,n) \leqslant C\,\mathrm{e}^{-\beta x\sqrt{n}}. \tag{5.5.20}$$

Combining (5.5.5), (5.5.17) and (5.5.20), we get (5.5.3). This concludes the proof of Lemma 5.5.2. □

Now we proceed to the proof of Theorem 5.2.1. It is based on the control of the joint law of $\big(\log \|Z_n^i\|, \log \|M_{0,n-1}(i,\cdot)\|\big)$ in Lemmas 5.5.1 and 5.5.2, together with standard techniques from Petrov [63].

Proof of Theorem 5.2.1. Notice that, when $x \in (0,1]$, Theorem 5.2.1 is a direct consequence of the Berry-Esseen bound for $\log \|Z_n^i\|$ (Theorem 5.4.1 with $s = 0$). So, it remains to prove Theorem 5.2.1 for $x \geqslant 1$ such that $x = o(\sqrt{n})$, as $n \to +\infty$.

We first prove (5.2.13). Let $\eta > 0$ be a small constant. Using the changed measure $\mathbb{P}_s^{e_i}$, for all $n \geqslant 1$, $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$ we have

$$\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\right) = r_s(e_i)\kappa(s)^n \mathbb{E}_s^{e_i}\left[r_s^{-1}(X_n^{e_i})\, \mathrm{e}^{-s\log\|M_{0,n-1}(i,\cdot)\|}\, \mathbb{1}_{\{\log\|Z_n^i\| - n\gamma > \sigma x\sqrt{n}\}}\right]. \quad (5.5.21)$$

Since $\Lambda = \log \kappa$, we get from (5.5.21), (5.4.12) and (5.4.1) that, for $\eta > 0$ small enough and all $n \geqslant 1$, $s \in (-\eta, \eta)$ and $1 \leqslant i \leqslant d$,

$$\mathbb{P}\left(\frac{\log \|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\right) = r_s(e_i)\, \mathrm{e}^{-n[s\Lambda'(s) - \Lambda(s)]} \times \mathbb{E}_s^{e_i}\left[r_s^{-1}(X_n^{e_i})\, \mathrm{e}^{-s\sigma_s\sqrt{n}S_{0,n}^{e_i}}\, \mathbb{1}_{\left\{\frac{\log\|Z_n^i\| - n\Lambda'(s)}{\sigma_s\sqrt{n}} > \frac{\sqrt{n}[\gamma - \Lambda'(s)]}{\sigma_s} + \frac{\sigma x}{\sigma_s}\right\}}\right]. \quad (5.5.22)$$

Recall that, by Proposition 5.3.1, the function Λ is analytic on $(-\eta, \eta)$ for $\eta > 0$ small enough, so that $\Lambda(s) = \sum_{k=1}^{+\infty} \frac{\gamma_k}{k!} s^k$ for $s \in (-\eta, \eta)$, where $\gamma_k := \Lambda^{(k)}(0)$, $k \geqslant 1$. From [63], we know that for $x = o(\sqrt{n})$ as $n \to +\infty$, $x \geqslant 1$, and $n \geqslant 1$ sufficiently large, the equation

$$\sqrt{n}[\Lambda'(s) - \gamma] = \sigma x, \quad (5.5.23)$$

has a unique root $s(x,n) \in (0, \eta)$ which has the expression

$$s(x,n) = \frac{t}{\sqrt{\gamma_2}} - \frac{\gamma_3}{2\gamma_2^2}t^2 - \frac{\gamma_4\gamma_2 - 3\gamma_3^2}{6\gamma_2^{7/2}}t^3 + \cdots, \quad \text{with } t = \frac{x}{\sqrt{n}}. \quad (5.5.24)$$

From now, let $s = s(x,n)$. Again from [63], we have the equality:

$$s\Lambda'(s) - \Lambda(s) = \frac{x^2}{2n} - \frac{x^3}{n^{3/2}}\zeta\Big(\frac{x}{\sqrt{n}}\Big), \tag{5.5.25}$$

where $\zeta(t)$ is the Cramér series defined in (5.2.12) (entirely determined by the function Λ), which converges for $|t|$ small enough. Therefore, combining (5.5.22), (5.5.23) and (5.5.25), we get that for $n \geqslant 1$ large enough and $1 \leqslant i \leqslant d$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\Big) \\ = \;& r_s(e_i)\,\mathrm{e}^{-\frac{x^2}{2}+\frac{x^3}{\sqrt{n}}\zeta(\frac{x}{\sqrt{n}})}\,\mathbb{E}_s^{e_i}\Big[r_s^{-1}(X_n^{e_i})\,\mathrm{e}^{-s\sigma_s\sqrt{n}S_{0,n}^{e_i}}\,\mathbb{1}_{\Big\{\frac{\log\|Z_n^i\|-n\Lambda'(s)}{\sigma_s\sqrt{n}}>0\Big\}}\Big] \\ = \;& \mathrm{e}^{-\frac{x^2}{2}+\frac{x^3}{\sqrt{n}}\zeta(\frac{x}{\sqrt{n}})}\,J^i(n), \end{aligned} \tag{5.5.26}$$

where

$$J^i(n) := r_s(e_i)\mathbb{E}_s^{e_i}\Big[r_s^{-1}(X_n^{e_i})\,\mathrm{e}^{-s\sigma_s\sqrt{n}S_{0,n}^{e_i}}\,\mathbb{1}_{\Big\{\frac{\log\|Z_n^i\|-n\Lambda'(s)}{\sigma_s\sqrt{n}}>0\Big\}}\Big].$$

By Proposition 5.3.1, for $\eta > 0$ small enough the map $s \mapsto r_s$ is analytic on $(-\eta,\eta)$, with $r_0 = 1$. Since $s = O\big(\frac{x}{\sqrt{n}}\big)$ as $n \to +\infty$ by (5.5.24), we obtain

$$\|r_s - 1\|_\infty \leqslant C|s| \leqslant \frac{Cx}{\sqrt{n}}.$$

This, together with (5.3.18), implies that for all $y_1, y_2 \in \mathcal{S}$,

$$\begin{aligned} \left|\frac{r_s(y_1)}{r_s(y_2)} - 1\right| &\leqslant r_s^{-1}(y_2)\Big(|r_s(y_1)-1| + |r_s(y_2)-1|\Big) \\ &\leqslant 2D\|r_s - 1\|_\infty \\ &\leqslant \frac{Cx}{\sqrt{n}}. \end{aligned} \tag{5.5.27}$$

Therefore, using (5.5.27) and the definition of $J^i(n)$, we deduce that for all $1 \leqslant i \leqslant d$, as $n \to +\infty$,

$$J^i(n) = J_1^i(n)\Big[1 + O\Big(\frac{x}{\sqrt{n}}\Big)\Big], \tag{5.5.28}$$

where

$$J_1^i(n) := \mathbb{E}_s^{e_i}\left[\mathrm{e}^{-s\sigma_s\sqrt{n}S_{0,n}^{e_i}} \mathbb{1}_{\left\{\frac{\log\|Z_n^i\|-n\Lambda'(s)}{\sigma_s\sqrt{n}}>0\right\}}\right].$$

Then, using Fubini's theorem and the integration by parts, we obtain the following decomposition for $J_1^i(n)$:

$$\begin{aligned} J_1^i(n) &= \mathbb{E}_s^{e_i}\left[\int_{\mathbb{R}} s\sigma_s\sqrt{n}\,\mathrm{e}^{-s\sigma_s\sqrt{n}u}\,\mathbb{1}_{\left\{\frac{\log\|Z_n^i\|-n\Lambda'(s)}{\sigma_s\sqrt{n}}>0, S_{0,n}^{e_i}\leqslant u\right\}}du\right] \\ &= s\sigma_s\sqrt{n}\int_{\mathbb{R}} \mathrm{e}^{-s\sigma_s\sqrt{n}u}\,\mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\|-n\Lambda'(s)}{\sigma_s\sqrt{n}}>0, S_{0,n}^{e_i}\leqslant u\right)du \\ &= s\sigma_s\sqrt{n}\int_0^{+\infty} \mathrm{e}^{-s\sigma_s\sqrt{n}u}\,\mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\|-n\Lambda'(s)}{\sigma_s\sqrt{n}}>0, S_{0,n}^{e_i}\leqslant u\right)du \\ &\quad + s\sigma_s\sqrt{n}\int_{-\infty}^{0} \mathrm{e}^{-s\sigma_s\sqrt{n}u}\,\mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\|-n\Lambda'(s)}{\sigma_s\sqrt{n}}>0, S_{0,n}^{e_i}\leqslant u\right)du \\ &=: J_2^i(n)+J_3^i(n). \end{aligned} \tag{5.5.29}$$

Control of $J_2^i(n)$. For any $n \geqslant 1$, set

$$I(n) := s\sigma_s\sqrt{n}\int_0^{+\infty} \mathrm{e}^{-s\sigma_s\sqrt{n}u}\,\Phi([0,u])du. \tag{5.5.30}$$

By (5.5.1) in Lemma 5.5.1, we get that for all $n \geqslant 1$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned} |J_2^i(n)-I(n)| &\leqslant s\sigma_s\sqrt{n}\int_0^{+\infty} \mathrm{e}^{-s\sigma_s\sqrt{n}u} \\ &\quad \times\left|\mathbb{P}_s^{e_i}\left(\frac{\log\|Z_n^i\|-n\Lambda'(s)}{\sigma_s\sqrt{n}}>0, S_{0,n}^{e_i}\leqslant u\right)-\Phi([0,u])\right|du \\ &\leqslant \frac{C}{\sqrt{n}}s\sigma_s\sqrt{n}\int_0^{+\infty}\mathrm{e}^{-s\sigma_s\sqrt{n}u}\,du \\ &= \frac{C}{\sqrt{n}}. \end{aligned} \tag{5.5.31}$$

So, it remains to estimate $I(n)$. Applying an integration by parts, for all $n \geqslant 1$ we have

$$I(n) = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} \mathrm{e}^{-s\sigma_s\sqrt{n}u-\frac{u^2}{2}}\,du. \tag{5.5.32}$$

Since Λ is analytic on $(-\eta,\eta)$ with $\Lambda'(0)=\gamma$ and $\sigma_s^2=\Lambda''(s)>0$ by (5.4.1), by Taylor's formula we have $\Lambda'(s)-\gamma = s\sigma^2 + O(s^2)$ and $\sigma_s^2 = \sigma^2 + O(s)$. Using (5.5.23) and the fact that $s = O\Big(\frac{x}{\sqrt{n}}\Big)$, we obtain

$$s\sigma_s = \frac{\Lambda'(s)-\gamma}{\sigma} + o(s) = \frac{x}{\sqrt{n}} + o\bigg(\frac{x}{\sqrt{n}}\bigg). \tag{5.5.33}$$

Using standard methods from Petrov [63], from (5.5.32) and (5.5.33) we get

$$I(n) = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} \mathrm{e}^{-xu-\frac{u^2}{2}}\,du + O\bigg(\frac{1}{\sqrt{n}}\bigg).$$

By a simple calculation,

$$\frac{1}{\sqrt{2\pi}}\int_0^{+\infty} \mathrm{e}^{-xu-\frac{u^2}{2}}\,du = \frac{\mathrm{e}^{\frac{x^2}{2}}}{\sqrt{2\pi}}\int_x^{+\infty} \mathrm{e}^{-\frac{u^2}{2}}\,du = \mathrm{e}^{\frac{x^2}{2}}(1-\Phi(x)).$$

So we deduce that, as $n\to+\infty$,

$$I(n) = \mathrm{e}^{\frac{x^2}{2}}(1-\Phi(x)) + O\bigg(\frac{x}{\sqrt{n}}\bigg). \tag{5.5.34}$$

Therefore, from (5.5.31), we get that, as $n\to+\infty$,

$$J_2^i(n) = \mathrm{e}^{\frac{x^2}{2}}(1-\Phi(x)) + O\bigg(\frac{1+x}{\sqrt{n}}\bigg). \tag{5.5.35}$$

Control of $J_3^i(n)$. By the definition of $J_3^i(n)$ (see (5.5.29)) and the bound (5.5.3) in Lemma 5.5.2, there exist some constants $\alpha>0$, $\beta>0$ and $\delta\in(0,1)$ such that for all $n\geqslant 1$ and $1\leqslant i\leqslant d$,

$$\begin{aligned} J_3^i(n) &\leqslant Cs\sigma_s\int_{-\infty}^0 \mathrm{e}^{(s\sigma_s-\alpha)\sqrt{n}|u|}\,du \\ &\quad + Cs\sigma_s\sqrt{n}\int_{-\infty}^0 \mathrm{e}^{s\sigma_s\sqrt{n}|u|}\min\Big\{\mathrm{e}^{-\beta\sqrt{n}|u|}, |u|^{-\frac12}n^{-\frac14}\delta^{\sqrt{n}}\Big\}du \\ &\leqslant Cs\sigma_s\int_{-\infty}^0 \mathrm{e}^{(s\sigma_s-\alpha)\sqrt{n}|u|}\,du + Cs\sigma_s\sqrt{n}\int_{-\infty}^{-1}\mathrm{e}^{(s\sigma_s-\beta)\sqrt{n}|u|}\,du \\ &\quad + Cs\sigma_s n^{1/4}\delta^{\sqrt{n}}\int_{-1}^0 \frac{\mathrm{e}^{s\sigma_s\sqrt{n}|u|}}{\sqrt{|u|}}du. \end{aligned} \tag{5.5.36}$$

Let $\varepsilon\in\Big(0,\min\{\alpha,\beta\}\Big)$. By (5.5.33) we have $s\sigma_s\to 0$ as $n\to+\infty$, hence $s\sigma_s\leqslant\varepsilon$ for n

large enough. Implementing this in (5.5.36), we obtain that for n sufficiently large,

$$\begin{aligned} J_3^i(n) &\leqslant C\int_{-\infty}^0 \mathrm{e}^{(\varepsilon-\alpha)\sqrt{n}|u|}\,du + C\sqrt{n}\int_{-\infty}^{-1} \mathrm{e}^{(\varepsilon-\beta)\sqrt{n}|u|}\,du \\ &\quad + Cn^{1/4}\delta^{\sqrt{n}}\int_{-1}^0 \frac{\mathrm{e}^{\varepsilon\sqrt{n}|u|}}{\sqrt{|u|}}du \\ &\leqslant C\left(\frac{1}{(\alpha-\varepsilon)\sqrt{n}} + \frac{\mathrm{e}^{(\varepsilon-\beta)\sqrt{n}}}{(\beta-\varepsilon)} + n^{1/4}(\delta\,\mathrm{e}^{\varepsilon})^{\sqrt{n}}\int_{-1}^0 \frac{du}{\sqrt{|u|}}\right). \end{aligned}$$

Taking ε small enough such that $\delta\,\mathrm{e}^{\varepsilon} < 1$, it follows that for n sufficiently large,

$$J_3^i(n) \leqslant \frac{C}{\sqrt{n}}. \tag{5.5.37}$$

Now, combining (5.5.29), (5.5.35) and (5.5.37) we get that, as $n \to +\infty$,

$$J_1^i(n) = \mathrm{e}^{\frac{x^2}{2}}(1-\Phi(x)) + O\left(\frac{1+x}{\sqrt{n}}\right).$$

Therefore, using (5.5.26) and (5.5.28), we obtain that, as $n \to +\infty$,

$$\begin{aligned} &\mathbb{P}\left(\frac{\log\|Z_n^i\| - n\gamma}{\sigma\sqrt{n}} > x\right) \\ = \;& \mathrm{e}^{-\frac{x^2}{2}+\frac{x^3}{\sqrt{n}}\zeta(\frac{x}{\sqrt{n}})}\left[\mathrm{e}^{\frac{x^2}{2}}(1-\Phi(x)) + O\left(\frac{1+x}{\sqrt{n}}\right)\right]\left(1+O\left(\frac{x}{\sqrt{n}}\right)\right) \\ = \;& \mathrm{e}^{\frac{x^3}{\sqrt{n}}\zeta(\frac{x}{\sqrt{n}})}(1-\Phi(x))\left[1+O\left(\frac{1+x}{\sqrt{n}}\right)\right]. \end{aligned}$$

This concludes the proof of (5.2.13).

The proof of (5.2.14) is very similar to that of (5.2.13). We just need to consider the unique root of the equation $\sqrt{n}[\Lambda'(s)-\gamma] = -\sigma x$ instead of (5.5.23), and apply the inequality (5.5.2) instead of (5.5.1), and (5.5.4) instead of (5.5.3). This ends the proof of Theorem 5.2.1. □

CONCLUSION

Les processus de branchements multi-type en environnement aléatoire sont un modèle mathématique très étudié de nos jour, ayant de nombreuses applications notamment en biologie cellulaire et dynamique de population. Sa complexité et sa richesse ont amené de plus en plus de chercheurs à s'investir sur ce sujet. La présente thèse généralise des résultats fondamentaux obtenus pour un processus de branchement multi-type sans environnement aléatoire (modèle de Galton-Watson), et pour un processus de branchement uni-type en environnement aléatoire, en régime surcritique. Tout particulièrement, la construction de la martingale fondamentale (W_n^i) du processus (Z_n^i) a permit d'établir de nombreux résultats sur le comportement asymptotique de (Z_n^i) tel que le théorème de type "Kesten-Stigum".

Une première perspective de recherche serait de relaxer la condition de Furstenberg-Kesten (les rapports $\frac{M_0(i,j)}{M_0(k,r)}$ minorés et majorés par des constantes) du théorème de type "Kesten-Stigum" donnant la convergence en probabilité des composantes normalisée $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$; la condition $M_0 > 0$ p.s. est peut-être suffisante. D'autre part, on pourrait essayer de renforcer la convergence en probabilité en convergence presque-sûre, pourquoi pas sous la condition de Furstenberg-Kesten pour commencer. Rappelons que ces résultats existent déjà pour les processus de Galton-Watson.

Dans cette thèse, la martingale fondamentale (W_n^i) a permis d'établir beaucoup de théorèmes limites pour le processus (Z_n^i). On a vu notamment que l'existence des moments harmoniques des limites W^i impliquaient des résultats asymptotiques pour $\log Z_n^i$ tels que des bornes de type Berry-Esseen et des déviations modérées de type Cramèr. On pourrait envisager d'utiliser la martingale (W_n^i) pour d'autres études asymptotiques. Par exemple, il serait intéressant de chercher à obtenir d'autres théorèmes de grandes déviations, comme établir l'asymptotique exacte de $\mathbb{P}(\log Z_n^i \geqslant nq)$ pour $q > 0$, le cas uni-type étant traité dans [29].

L'étude de l'exposant critique pour l'existence des moments harmoniques de W^i peut être une autre perspective de recherche. On l'a identifié sous certaines conditions, mais il reste à voir si les moments harmoniques existent pour cet exposant critique. Le problème reste ouvert pour les processus de branchements multi-type en environnement aléatoires.

Un des problèmes les plus complexes sur le sujet des processus de branchements en environnement aléatoire reste la détermination d'un équivalent de la probabilité $\mathbb{P}(Z_n^i = z | Z_0^i = k)$ quand $n \to +\infty$ pour tout $z, k \in \mathbb{N}^d$. L'étude dans le cas uni-type pose déjà de grosses difficultés.

Les applications des résultats de cette thèse sont potentiellement nombreuses, tout spécialement pour faire des études statistiques de ces processus de branchements en biologie, physique nucléaire etc... Pour une étude statistique de l'évolution d'une population, on peut donner la taille asymptotique de la population totale à partir des moyennes de productions, et obtenir des vitesses de convergence de certains estimateurs.

BIBLIOGRAPHY

[1] Abraham R. and Delmas J.F., "Critical multi-type Galton-Watson trees conditioned to be large", *in*: *Journal of Theoretical Probability* 31 (2018), pp. 757–788.

[2] Afanasyev V.I., "On the maximum of a subcritical branching process in a random environment", *in*: *Stochastic Processes and their Applications* 93(1) (2001), pp. 87–107.

[3] Afanasyev V.I. et al., "Conditional limit theorems for intermediately subcritical branching processes in random environment", *in*: *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* 50(2) (2014), pp. 602–627.

[4] Afanasyev V.I. et al., "Criticality for branching processes in random environment", *in*: *The Annals of Probability* 33(2) (2005), pp. 645–673.

[5] Athreya K.B. and Karlin S., "On branching processes with random environments I:Extinction probabilities", *in*: *The Annals of Mathematical Statistics* 42(5) (1971), pp. 1499–1520.

[6] Athreya K.B. and Karlin S., "Branching Processes with random environments II:Limit theorems", *in*: *The Annals of Mathematical Statistics* 42(6) (1971), pp. 1843–1858.

[7] Athreya K.B. and Ney P.E., *Branching Processes*, New York, 1972.

[8] Bansaye V., "Cell contamination and branching processes in a random environment with immigration", *in*: *Advances in Applied Probability* 41 (2009), pp. 1059–1081.

[9] Bansaye V. and Berestycki J., "Large deviations for branching processes in random environment", *in*: *Markov Processes And Related Fields* 15(4) (2009), pp. 493–524.

[10] Bansaye V. and Vatutin V., "On the survival probability for a class of subcritical branching processes in random environment", *in*: *Bernoulli* 23(1) (2017), pp. 58–88.

[11] Benoist Y. and Quint J. F., *Limit theorems for a supercritical branching process with immigration in a random environment*, 2016.

[12] Biggins J.D., Cohn H., and Nerman O., "Multi-type branching in varying environment", *in*: *Stochastic Processes and their Applications* 83 (1999), pp. 357–400.

[13] Buraczewski D. et al., "On multidimensional Mandelbrot cascades", *in*: *Journal of Difference Equations and Applications* 20(11) (2014), pp. 1523–1567.

[14] Buraczewski D. and Mentemeier S., “Precise large deviation results for products of random matrices”, *in*: *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques* 52(3) (2016), pp. 1474–1513.

[15] Chaumont L. and Liu R., “Coding multitype forests: application to the law of the total population of branching forests”, *in*: *Transactions of the American Mathematical Society* 368(4) (2016), pp. 2723–2747.

[16] Chow Y.S. and Teicher H., *Probability Theory : Independence, Interchangeability, Martingales*, 2012.

[17] Cohn H., “On the growth of the multitype supercritical branching process in a random environment”, *in*: *The Annals of Probability* 17(3) (1989), pp. 1118–1123.

[18] Cornfeld I.P., Fomin S.V., and Sinai Y., *Ergodic Theory*, 1982.

[19] Cramér H., “Sur un nouveau théorème-limite de la théorie des probabilités”, *in*: *Actual. Sci. Ind.* 736 (1938), pp. 5–23.

[20] Damek E., Gantert N., and Kolesko K., “Absolute continuity of the martingale limit in branching processes in random environment”, *in*: *Electronic Communications in Probability* 24(42) (2019), pp. 1–13.

[21] Durett R., *Theory and Examples*, 1991.

[22] Dyakonova E.E., “Asymptotics behaviour of the probability of non-extinction for a multi-type branching process in a random environment”, *in*: *Discrete Mathematics and Applications* 9(2) (1999), pp. 119–136.

[23] Dyakonova E.E., “Multitype Galton-Watson branching processes in a Markov random environment”, *in*: *Theory of Probability and Its Applications* 56(3) (2012), pp. 119–136.

[24] Dyakonova E. E., “Limit theorem for a multitype critical branching process evolving in a random medium”, *in*: *Discrete Mathematics and Applications* 25(3) (2015), pp. 137–147.

[25] Dyakonova E. et al., “Branching processes in a random environment with immigration stopped at zero”, *in*: *Journal of Applied Probability* 57(1) (2020), pp. 237–249.

[26] Furstenberg H. and Kesten H., “Products of random matrices”, *in*: *The Annals of Mathematical Statistics* 31(2) (1960), pp. 457–469.

[27] Geiger J. and Kersting G., "The survival probability of a critical branching process in a random environment", *in*: *Theory of Probability and Its Applications* 45 (2000), pp. 517–525.

[28] Geiger J., Kersting G., and Vatutin V., "Limit theorems for subcritical branching processes in random environment", *in*: *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* 39(4) (2003), pp. 593–620.

[29] Grama I., Liu Q., and Miqueu E., "Harmonic moments and large deviations for a supercritical branching process in a random environment", *in*: *Electronic Journal of Probability* 22(99) (2017), pp. 1–23.

[30] Grama I., Liu Q., and Miqueu E., "Asymptotic of the distribution and harmonic moments for a supercritical branching process in a random environment", *in*: *Submitted* ().

[31] Grama I., Liu Q., and Miqueu E., "Berry-Esseen's bound and Cramér's large deviation expansion for a supercritical branching process in a random environment", *in*: *Stochastic Processes and their Applications* 127 (2017), pp. 1255–1281.

[32] Grama I., Liu Q., and Pin E., "A Kesten-Stigum type theorem for a supercritical multi-type branching process in a random environment", *in*: *hal-02878026* (2020).

[33] Grama I., Liu Q., and Pin E., "Convergence in L^p for a supercritical multi-type branching process in a random environment", *in*: *hal-02934079* (2020).

[34] Grama I., Liu Q., and Pin E., "Berry-Esseen's bound and harmonic moments for supercritical multi-type branching processes in random environments", *in*: *hal-02911865* (2020).

[35] Grama I., Liu Q., and Pin E., "Cramér type moderate deviation expansion for supercritical multi-type branching processes in random environments", *in*: *hal-02934081* (2020).

[36] Guivarc'h Y. and Le Page É., "Spectral gap properties for linear random walks and Pareto's asymptotics for affine stochastic recursions", *in*: *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques* 52(2) (2016), pp. 503–574.

[37] Guivarc'h Y., Le Page E., and Liu Q., "Limit theorems for subcritical branching processes in random environment", *in*: *Comptes Rendus de l'Académie des Sciences de Paris, Series I* 337 (2003), pp. 603–608.

[38] Guivarc'h Y. and Liu Q., "Propriétés asymptotiques des processus de branchement en environnement aléatoire", *in*: *Comptes Rendus de l'Académie des sciences* I 332 (2001), pp. 339–344.

[39] Harris T.E., *The Theory of Branching Processes*, 1963.

[40] Hennion H., "Limit Theorems for Products of positive Random Matrices", *in*: *The Annals of Probability* 25(4) (1997), pp. 1545–1587.

[41] Hess C., Seri R., and Choirat C., "Ergodic theorems for extended real-valued random variables", *in*: *Stochastic Processes and their Applications* 120 (2010), pp. 1908–1919.

[42] Hong W., Liu M., and Vatutin V., "Limit theorems for supercritical MBPRE with linear fractional offspring distributions", *in*: *Markov Processes And Related Fields* 25 (2019), pp. 1–31.

[43] Huang C. and Liu Q., "Moments, moderate and large deviations for a branching process in a random environment", *in*: *Stochastic Processes and their Applications* 122(2) (2012), pp. 522–545.

[44] Huang C. and Liu Q., "Convergence in L_p and its exponential rate for a branching process in a random environment", *in*: *Electronic Journal of Probability* 19(104) (2014), pp. 1–22.

[45] Jones O.D., "On the convergence of multitype branching processes with varying environments", *in*: *Annals of Applied Probability* 7(3) (1997), pp. 772–801.

[46] Kaplan N., "Some results about multidimensional branching processes with random environments", *in*: *The Annals of Probability* 2(3) (1974), pp. 441–455.

[47] Kesten H., Kozlov M.V., and Spitzer F., "A limit law for random walk in a random environment", *in*: *Compositio Mathematica* 30 (1975), pp. 145–168.

[48] Kesten H. and Spitzer F., "Convergence in Distribution of products of Random Matrices", *in*: *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 67 (1984), pp. 363–386.

[49] Kesten H. and Stigum B.P., "A Limit Theorem for Multidimensional Galton-Watson Processes", *in*: *The Annals of Mathematical Statistics* 37(5) (1966), pp. 1211–1223.

[50] Kersting G. and Vatutin V., *Discrete time branching processes in random environment*, 2017.

[51] Kingman J.F.C., "Subadditive ergodic theory", *in*: *The Annals of Probability* 1 (1973), pp. 883–899.

[52] Kozlov M. V., "On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment", *in*: *Theory of Probability and Its Applications* 21(4) (1976), pp. 742–751.

[53] Kozlov M. V., "On large deviations of strictly subcritical branching processes in a random environment with a geometric distribution of descendants", *in*: *The Annals of Probability* 54(3) (2010), pp. 424–446.

[54] Kurtz T. et al., "A conceptual proof of the Kesten-Stigum Theorem for multi-type branching processes, in *Classical and Modern Branching Processes*", *in*: *Ed. K. B. Athreya, P. Jagers, Springer* (1997), pp. 181–185.

[55] Kyprianou A.E. and Sani A.R., "Martingale convergence and the functional equation in the multi-type branching random walk", *in*: *Bernoulli* 7(4) (2001), pp. 593–604.

[56] Le Page É., "Théorèmes limites pour les produits de matrices aléatoires", *in*: *In Probability measures on groups, Springer Berlin Heidelberg* (1982), pp. 258–303.

[57] Le Page E., Peigné M., and Pham C., "The survival probability of a critical multi-type branching process in i.i.d. random environment", *in*: *The Annals of Probability* 46(5) (2018), pp. 2946–2972.

[58] Liu Q., "Asymptotic properties of supercritical age-dependent branching processes and homogeneous branching random walks", *in*: *Stochastic Processes and their Applications* 82 (1999), pp. 61–87.

[59] Liu Q., "Asymptotic properties and absolute continuity of laws stable by random weighted mean", *in*: *Stochastic Processes and their Applications* 95 (2001), pp. 83–107.

[60] Liu Q., "Local dimensions of the branching measure on a Galton–Watson tree", *in*: *Annales de l'Institut Henri Poincaré, Probabilité s et Statistiques* 37(2) (2001), pp. 195–222.

[61] Liang X. and Liu Q., "Weighted Moments of the Limit of a Branching Process in a Random Environment", *in*: *Proceedings of the Steklov Institute of Mathematics* 282 (2013), pp. 127–145.

[62] Lyons R., Permantle R., and Peres Y., "Conceptual Proofs of L Log L Criteria for Mean Behavior of Branching Processes", *in*: *The Annals of Probability* 23(3) (1995), pp. 1125–1138.

[63] Petrov V.V., *Sums of independent random variables*, 1975.

[64] Seneta E., *Non-negative Matrices and Markov Chains*, 1981.

[65] Smith W.L. and Wilkinson W., "On branching processes in random environments", *in*: *The Annals of Mathematical Statistics* 40(3) (1969), pp. 814–827.

[66] Tanny D., "A Zero-One Law for Stationary Sequences", *in*: *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 30 (1974), pp. 139–148.

[67] Tanny D., "Normalizing constants for branching processes in random environments", *in*: *Stochastic Processes and their Applications* 6 (1978), pp. 201–211.

[68] Tanny D., "On multitype branching processes in a random environment", *in*: *Advances in Applied Probability* 13(3) (1981), pp. 464–497.

[69] Tanny D., "A Necessary and sufficient condition for a branching process in a random environment to grow like the product of its means", *in*: *Stochastic Processes and their Applications* 28 (1988), pp. 123–139.

[70] Vatutin V.A. and Dyakonova E. E., "Multitype branching processes in a random environment: nonextinction probability in the critical case", *in*: *Theory of Probability and Its Applications* 62(4) (2018), pp. 506–521.

[71] Vatutin V.A. and Dyakonova E.E., "Survival probability for a class of multitype subcritical branching processes in random environment", *in*: *(Russian) Mat. Zametki* 107(2) (2020), pp. 163–177.

[72] Vatutin V.A. and Dyakonova E.E., "Decomposable branching processes with two types of particles", *in*: *Discrete Mathematics and Applications* 28(2) (2018), pp. 119–130.

[73] Vatutin V.A. and Wachtel V., "Multi-type subcritical branching processes in a random environment", *in*: *Advances in Applied Probability* 50(A) (2018), pp. 281–289.

[74] Wang Y. and Liu Q., "Limit theorems for a supercritical branching process with immigration in a random environment", *in*: *Science China Mathematics* 60(12) (2017), pp. 2481–2502.

[75] Xiao H., Grama I., and Liu Q., "Berry-Esseen bound and precise moderate deviations for products of random matrices", *in*: *arxiv:1907.02438v1 [math.PR]* (2019).

Titre : Théorèmes limites pour un processus de branchement multi-type dans un environnement aléatoire

Mot clés : Procéssus de branchement multi-type, environnement aléatoire, produits de matrices aléatoires, martingale

Résumé : Le sujet de cette thèse est l'étude d'un processus de branchement d-type $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d)$ dans un environnement aléatoire ξ partant d'un individu de type i, et en régime surcritique. Beaucoup d'études sont faites actuellement sur les processus de branchements, du fait d'un très large domaine d'applications comme en biologie, physique nucléaire ou dynamique de population. De nombreux théorèmes limites sont bien connus pour le modèle uni-type ($d = 1$) et celui de Galton-Watson (où l'environnement est déterministe). L'objectif de cette thèse est de généraliser ces résultats au modèle multi-type en environnement aléatoire, en régime surcritique. Le premier résultat établi a été la découverte de la martingale fondamentale (W_n^i) associée à (Z_n^i), qui converge presque sûrement vers W^i. C'est principalement grâce à cette martingale qu'on a pu démontrer des théorèmes limites sur (Z_n^i). On a établit un théorème de type Kesten-Stigum, donnant la convergence en probabilité de la population normalisée $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$ vers W^i pour tout j, où $\mathbb{E}_\xi Z_n^i(j)$ est l'espérance conditionnelle de $Z_n^i(j)$ sachant l'environnement ξ. De plus on a donné des conditions suffisantes sous lesquelles W^i est non-dégénérée. On a ensuite déterminé une condition nécessaire et suffisante pour la convergence dans L^p de $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$ et de la martingale W_n^i, avec une vitesse exponentielle. On s'est enfin intéressé au comportement asymptotique de $\log \|Z_n^i\|$, où $\| \cdot \|$ est une norme vectorielle ; on a établi notamment une loi des grands nombres et un théorème central limite. De plus, en étudiant l'existence des moments harmoniques de W^i, on a prouvé un théorème de type Berry-Esseen et des déviations modérées de type Cramér pour $\log \|Z_n^i\|$.

Title: Limit Theorem for a multi-type branching process in a random environment

Keywords: Multi-type branching process, random environment, products of random matrices, martingale

Abstract: The subject of this thesis is the study of a d-type branching process $Z_n^i = (Z_n^i(1), \cdots, Z_n^i(d)$ in a random environment ξ starting with one individual of type i, and in the supercritical regime. Actually, a lot of studies are made on the branching processes, because of a large field of applications as in biology, nuclear physic or population dynamics. Several limit theorems are well known for the uni-type model ($d = 1$) and the Galton-Watson's model (where the environment is deterministic). The objective of this thesis is to generalize these results for the multi-type model in random environment, in the supercritical regime. The first established result was the discovery of the fundamental martingale (W_n^i) associated to (Z_n^i), which converges almost surely to W^i. It is mainly because of this martingale that we could prove limit theorems on (Z_n^i). We established a Kesten-Stigum type theorem, giving the convergence in probability of the normalized population size

$Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$ to W^i for all j, where $\mathbb{E}_\xi Z_n^i(j)$ is the conditional expectation of $Z_n^i(j)$ given the environment ξ. Moreover we gave sufficient conditions under which W^i is non-degenerate. Afterwards, we determined a necessary and sufficient condition for the convergence in L^p of $Z_n^i(j)/\mathbb{E}_\xi Z_n^i(j)$ and the martingale W_n^i, with an exponential speed. Finally, we were interested in the asymptotic behaviour of $\log \|Z_n^i\|$, where $\| \cdot \|$ is a vector-norm; specially we established a law of large numbers and a central limit theorem. Furthermore, by studying the existence of the harmonic moments of W^i, we proved a Berry-Esseen type theorem and a Cramér type moderate deviation expansion for $\log \|Z_n^i\|$.